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Equi-reflective Subcategories of Uniform Spaces

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The title of the article refers to the full epi-reflective subcategories $\mathcal{R}$ of separated uniform spaces which possess the following strong extension property: for each uniform space X and equi-unif. cont. family $(f_i) : X \to R$, $R \in \mathcal{R}$, the family of extensions $(\widehat{f_i})$: $X_{\mathcal{R}} \to R$ to the reflection $X_{\mathcal{R}}$ is also equi-unif. cont. In this note we give various characterizations and examples of such sub-categories, related to the commuting of the reflection operator with the formation of finite products, Ascoli-like conditions, and the formation of function spaces. The complete details will appear in $[R]_2$.

In the following $U(X,Y)$ denotes the family of uniformly continuous mappings equipped with the uniformity of uniform convergence and $X * Y$ denotes the semi-uniform product of X and Y defined in [I]. The following is the main result in $[R]_2$.

Theorem: Let $\mathcal{R}$ be a non-trivial epi-reflective subcategory of separated uniform spaces. The following statements are equivalent:

(1) $\mathcal{R}$ is an equi-reflective subcategory.

(2) $U(\mathcal{D}, R) \in \mathcal{R}$ for each $R \in \mathcal{R}$ and uniformly discrete space $\mathcal{D}$.

(3) $U(X,R) \in \mathcal{R}$ for each $R \in \mathcal{R}$ and uniform space X .

(4) (a) If $\mathcal{F} \subset U(R,R')$ is equi-unif. cont. and point-wise closed, $R,R' \in \mathcal{R}$, then $\mathcal{F} \in \mathcal{R}$.

(b) $\mathcal{R}$ contains each uniformly discrete space.

(5) If $\mathcal{F} \subset U(X,R)$ is equi-unif. cont. and point-wise closed for each $R \in \mathcal{R}$ and uniform space X , then $\mathcal{F} \in \mathcal{R}$.

(6) (a) The natural uniform mapping $(X * R)_{\mathcal{R}} \to X_{\mathcal{R}} * R$ is a uni-

form isomorphism, for each $R \in \mathcal{R}$ and uniform space X.

(b) $\mathcal{R}$ contains each uniformly discrete space.

⑦ The natural uniform mapping $(X*Y)_{\mathcal{R}} \to X_{\mathcal{R}}*Y_{\mathcal{R}}$ is a uniform isomorphism, for each pair X and Y of uniform spaces

⑧ There exists a natural uniform mapping $X_{\mathcal{R}}*Y_{\mathcal{R}} \to (X*Y)_{\mathcal{R}}$, for each pair X and Y of uniform spaces.

We remark that conditions ④(a) and ⑥(a) are equivalent in general, and ④(b) is a necessary condition for equi-reflectivity, since the family of precompact spaces satisfies ④(a) and is not equi-reflective.

Corollary: ① Given a non-empty class $\mathcal{S}$ of uniform spaces, there exists a smallest equi-reflective subcategory $\mathcal{R}$ containing $\mathcal{S}$:

$\mathcal{R}$ = epi-reflective hull $\{U(D,S) \mid S \in \mathcal{S}, D$ uniformly discrete$\}$.

② Let C be an epi-reflective subcategory of uniform spaces. Then there exists a largest (non-trivial) equi-reflective subcategory $\mathcal{R}$ contained in C if and only if C contains each uniformly discrete space. In this case

$\mathcal{R} = \{R \in C \mid U(D,R) \in C$ for each uniformly discrete $D\}$

Examples: (1) Each reflective subcategory containing all complete uniform spaces is equi-reflective. Hence by part ⑦ of the Theorem the associated reflection operator commutes with the formation of finite products: $(\prod X_i)_{\mathcal{R}} = \prod (X_i)_{\mathcal{R}}$. Furthermore, if $\mathcal{R}$ is an equi-reflective subcategory and $[0,1] \in \mathcal{R}$, then $\mathcal{R}$ contains each complete uniform space (for each complete uniform space is a closed subspace of a product of spaces of the form $U(D,[0,1])$, D uniformly discrete). (2) The classes of uniformly zero-dimensional and complete uniformly zero-dimensional spaces are equi-reflective. (3) The class of uniform spaces having totally disconnected topology is equi-reflective. (4) The precompact reflection is not distinguished among cardinal reflections by satisfying ⑥(a) - assuming (GCH), $\mathcal{R}_\alpha = \{X \mid X$ contains no uniformly discrete subset of power $\alpha\}$ satisies ⑥(a) if and only if $\beta<\alpha$ implies $2^\beta<\alpha$ (a proof may be odeled on the proof of ([K] ,3.2)).

omments: (1) Example (1) raises the (unsolved) problem of whether he operator associated with any reflective subcategory containing ll complete spaces commutes with the formation of infinite products.

A more thorough investigation of conditions ④(a) and ⑥(a) hould probably be made, for Ascoli type theorems are interesting in any analytic settings where the objects under consideration are not iscrete. Such an investigation would involve the redefining of equireflectivity to avoid the necessity of condition ④(b) - perhaps by ssuming that the equi-unif. cont. families under consideration must e point-wise (or uniformly) bounded. Then conditions ② and ③ ould have to be modified for bounded families. Most likely, these ssumptions would also involve the consideration of the reflective ubcategory of complete locally convex linear topological spaces. (3) ithout the insistence on fullness and epimorphic reflection mappings, have no idea whether each equi-reflective subcategory must contain l uniformly discrete spaces. (4) In conclusion I should mention at the dual of the concept of equi-reflectivity has been studied r coreflective subcategories and is equivalent to finite productity. The reader is referred to [HR] for further details.

References

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