1976-1977

David J. Lutzer

A survey of Dugundji Extension Theory

In: Zdeněk Frolík (ed.): Seminar Uniform Spaces. , 1978. pp. 24–42.

Persistent URL: http://dml.cz/dmlcz/703163

A survey of Dugundji Extension Theory

by

D.J. Lutzer

This is the third lecture which I presented in the spring of 1977 while a visitor to ČSAV. This lecture is entirely independent of the first two. In this lecture, $\mathbb{R}$ denotes the usual space of real numbers.

Everyone knows the classical Tietze-Urysohn extension theorem:

A. Theorem: Let A be a closed subspace of a normal space X and suppose $f : A \longrightarrow \mathbb{R}$ is a continuous bounded, real-valued function. Then there is a continuous real-valued $F : X \longrightarrow \mathbb{R}$ which extends f and has the same bounds as f.

If we let $C^*(A)$ denote the set of all continuous, bounded, real--valued functions on A and define $C^*(X)$ analogously, then the Axiom of Choice yields the following reformulation of Theorem A:

A'. <u>Theorem:</u> Let A be a closed subspace of a normal space X. Then there is a function $\eta : C^*(A) \longrightarrow C^*(X)$ having the property that if $f \in C^*(A)$, then $\eta(f)$ extends f and has the same bounds as f.

It is well known that the sets $C^*(A)$ and $C^*(X)$ have several natural structures, namely:

(1) vector space structure;

(2) topological structure (I will consider three topologies - the sup-norm topology, the compact-open topology, and the pointwise convergence topology);

(3) partial order structure (where $C^*(A)$ and $C^*(X)$ carry the pointwise partial ordering).

The fundamental problem of Dugundji Extension Theory is to determine whether (and when) the function η in Theorem A' can be forced to respect one or more of these additional structures. (Hereafter, η will be called an <u>extender</u> from $C^*(A)$ to $C^*(X)$.) In recent years the problem has been subdivided further: first, how can Theorem A' be sharpened for arbitrary normal spaces (see Theorems B and D, below) and second, what reasonable additional structures on A or X yield well-behaved extenders (see Theorems H and I, below) ? The first half of today's lecture will concentrate on what can and cannot be done in arbitrary normal spaces and the second half will consider normal spaces having additional structure.

First, consider vector space structure. One easily proves

B. <u>Theorem:</u> Let A be a closed subspace of a normal space X. Then there is a linear extender $\eta : C^*(A) \to C^*(X)$.

<u>Proof:</u> Let B be a basis for the vector space $C^*(A)$ and for each $f \in B$ choose an extension $\hat{f} \in C^*(X)$ of f. Define $\eta(f) = \hat{f}$ and extend η linearly over $C^*(A)$.

You will note that the linear extender η of Theorem B is not guaranteed to preserve bounds of functions (compare to Theorem A'). Indeed, it is known that for some normal spaces (and even for some compact Hausdorff spaces) linear extenders which preserve bounds cannot exist. Perhaps the first such example was given by Arens [A] in 1952. Today there are more "modern" approaches which yield a sharper result (see Theorem G) but the Arens example has a particular aesthetic value which other approaches may lack.

C. Example: There is a compact Hausdorff space X containing a closed subspace A such that no linear, bound-preserving extender from $C^*(A)$ to $C^*(X)$ can exist. Let D be a discrete space having $\mathrm{card}(D) = \omega_1$. Let A be the one-point compactification of D. Then (by the usual Tychonoff embedding technique) A can be embedded in the space $X = S^{\omega_1}$, the product of ω_1 copies of the unit circle S. If λ denotes normalized one-dimensional Lebesgue measure on S, then the Kolmogorov product theorem yields a complete measure μ on X whose domain includes all open sets. (The only properties of μ which I will use are that μ is a non-negative measure such that every non-void open set has positive measure, and $\mu(X) = 1$.) Suppose there is a bound-preserving linear extender η : $C^*(A) \to C^*(X)$. Define an "integral" I on $C^*(A)$ by the rule that

$$I(f) = \int_X \eta(f)\,d\mu .$$

Because I is a non-negative linear functional on $C^*(A)$, the Riesz-representation theorem yields a non-negative finite Borel measure ν on A such that if $f \in C^*(A)$ then $I(f) = \int_A f\,d\nu$.

For each $d \in D \subset A$, let ξ_d be the characteristic function of the set $\{d\}$, i.e., $\xi_d : A \to \{0,1\}$ has $\xi_d(x) = 1$ if and only if $x = d$. Each ξ_d is continuous and

$$\nu(\{d\}) = \int_A \xi_d\,d\nu = I(\xi_d) = \int_X \eta(\xi_d)\,d\mu .$$

Because η preserves bounds $\eta(\xi_d) \geq 0$ and $\eta(\xi_d)(d) = \xi_d(d) = 1$. Since $\eta(\xi_d) \in C^*(X)$, $\eta(\xi_d)$ is positive on an open set so that

$\int_X \eta(\xi_d) d\mu > 0$, i.e., $\nu(\{d\}) > 0$ for each of the uncountably many points $d \in D$. But that is impossible because $\nu(A) = 1$. $\square$.

I will have more to say about linear extenders later, but first let me continue to talk about what happens in normal spaces with no additional structure. Consider the topological structure of $C^*(A)$ and $C^*(X)$. Can one always find continuous extenders, provided one does not require linearity ? The answer depends entirely upon which of the three topologies the function spaces carry. Consider first the sup-norm topology, i.e., where $C^*(A)$ is metrized by defining

$$d(f,g) = \sup\{|f(a) - g(a)| \,|\, a \in A\}.$$

A recent result due to van Douwen, Przymusiński and myself [DLP] asserts

D. <u>Theorem</u>: Let A be a closed subspace of the normal space X, and let $C^*(A)$ and $C^*(X)$ carry the sup-norm topology. Then there is a continuous extender $\eta : C^*(A) \to C^*(X)$ such that $\eta(f)$ has the same bounds as does f .

One of the interesting things about Theorem D is that we have three completely separate proofs of the theorem: one is analytic, another depends on the topological theorem that the product of a metric space and a compact space is normal, and the third is selection theoretic. The first two are given in [DLP] and the third appears in [L_2]; I will outline the first and third today. Before presenting the proofs, let me observe that if there is a continuous extender $e : C^*(A) \to C^*(X)$ then a continuous and bound-preserving extender η can be defined by

$$\eta(f)(x) = \max\{I(f), \min\{e(f)(x), S(f)\}\}$$

where $I(f) = \inf\{f(a) | a \in A\}$ and $S(f) = \sup\{f(a) | a \in A\}$. Thus it suffices to find a continuous extender.

First proof of Theorem D: Recall the Bartle-Graves Theorem [BG] : If E and F are Banach spaces, and if $R : E \to F$ is a continuous linear surjection, then there is a continuous function $e : F \to E$ having $e(f) \in R^{-1}\{f\}$ for each $f \in F$. That theorem can be applied to the Banach spaces $E = C^*(X)$ and $F = C^*(A)$, where $R : E \to F$ is defined by $R(g) = g|A$ for each $g \in C^*(X)$, to obtain the desired continuous extender.

Second proof of Theorem D: Recall Michael's first selection theorem [M_1] : Let Y be a paracompact space and let B be a Banach space. For each $y \in Y$ let $E(y)$ be a nonempty, closed, convex subset of B and suppose the association $y \to E(y)$ is lower semi-continuous (see [M_1] for definitions). Then there is a continuous function $e : Y \to B$ such that $e(y) \in E(y)$ for each $y \in Y$. To apply that theorem, let $Y = C^*(A)$ and $B = C^*(X)$. For each $f \in X$ let $E(f) = \{F \in C^*(X) \mid F$ extends $f\}$. The Tietze-Urysohn theorem (Theorem A) shows that the association $f \to E(f)$ is lower semicontinuous so that Michael's theorem yields the required continuous extender. $\square$.

If one wants to use topologies on $C^*(A)$ and $C^*(X)$ other than the sup-norm topology, the situation becomes much more complex, and counter examples outnumber theorems. Consider the topology of pointwise convergence on $C^*(A)$. An analyst would describe it to you by telling you which nets converge. For me it is more convenient to describe basic neighborhoods. Fix $f \in C^*(A)$, a finite subset $S \subset A$ and a real number $\varepsilon > 0$. I define $O(f,S,\varepsilon) = \{g \in C^*(A) \mid$ for each $a \in S$, $|g(a) - f(a)| < \varepsilon\}$, and the collection $\{O(f,S,\varepsilon) \mid \varepsilon > 0$ and $S \subset A$ is finite$\}$ is a system of basic neighborhoods of f in the topology of pointwise-convergence. The space $C^*(X)$ is topologized in an analogous way.

There is a single space, today called the Michael line (see [M_2]), which is a rich source of counterexamples in extension theory. The Michael line M is obtained by retopologizing the set of real numbers, using the collection $\{U \cup V \mid U$ is open in $\mathbb{R}$ and $V \subset \mathbb{P}\}$ as the new topology, where $\mathbb{R}$ is the usual space of real numbers and where $\mathbb{P}$ is the set of irrational numbers. Observe that the set Q of rational numbers is a closed subspace of M and that, as a subspace of M, Q inherits its usual topology. Examples involving M usually involve Baire category arguments; our next example uses a very simple argument of that sort.

E. Example: Equip both $C^*(Q)$ and $C^*(M)$ with the topology of pointwise convergence. Then there is no continuous extender from $C^*(A)$ to $C^*(X)$. For suppose such an extender η: $C^*(A) \to C^*(X)$ exists. (We do not assume η is linear.) Let θ_Q and θ_M denote the zero-functions on Q and M respectively. Replacing the given extender η by the function $\eta'(f) = \eta(f) - \eta(\theta_Q)$ if necessary, I may assume that $\eta(\theta_Q) = \theta_M$.

For each finite set $S \subset Q$ and each integer $n \geq 1$, consider the set $O(\theta_Q, S, 1/n)$ and define

$$R(S,n) = \{x \in \mathbb{R} \mid \text{if } f \in O(\theta_Q, S, 1/n) \text{ then } |\eta(f)(x)| < 1\}.$$

Because η is continuous and has $\eta(\theta_Q) = \theta_M$,

$$\mathbb{R} = \bigcup\{R(S,n) \mid S \subset Q \text{ is finite and } n \geq 1\}.$$

That union is countable and $\mathbb{R}$ is a complete metric space so that for some S_0 and n_0, the set $cl_{\mathbb{R}}(R(S_0, n_0))$ contains an open interval (a,b). Choose a rational number $q_0 \in (a,b) - S_0$. Because S_0 is finite, there is a continuous bounded function $g : M \to \mathbb{R}$ such that $g(q_o) = 2$ and $g(x) = 0$ for each $x \in S_o$. Then $g \in O(\theta_Q, S_o, 1/n_o)$. Because $q_o \in (a,b)$ I may choose a sequence $\langle x_k \rangle$ from $R(S_o, n_o)$ which converges (in the topology of $\mathbb{R}$) to q_o; however, since points of Q have the same basic neighborhood system in both $\mathbb{R}$ and M, $\langle x_k \rangle$ converges to q_o in M. But then $\langle \eta(g)(x_k) \rangle \longrightarrow \eta(g)(q_o)$ since $\eta(g) \in C^*(M)$, even though $|\eta(g)(x_k)| < 1$ for each $k \geq 1$ while $\eta(g)(q_o) = g(q_o) = 2$. That contradiction completes the proof. □ .

The non-existence of continuous extenders from $C^*(Q)$ to $C^*(M)$ where both function spaces carry the compact-open topology, is established in [HLZ_2]. Another method for showing that continuous extenders cannot always be found relies on the next theorem, also appearin in [HLZ_2]; however the theorem is not particularly useful in studying the space M.

F. Theorem: Suppose X is a Hausdorff k-space and that for each closed $A \subset X$ it is possible to find a continuous extender $\eta : C^*(A) \longrightarrow C^*(X)$, where both function spaces carry the compact-open topology. Then X is collectionwise normal.

Recall that a space X is <u>collectionwise normal</u> if, corresponding to any discrete collection Ⓓ of closed subsets of X, there is a disjoint collection $\{U(D) \mid D \in$ Ⓓ$\}$ of open sets having $D \subset U(D)$ fo each $D \in$ Ⓓ. The definition of k-space is given in [K]; if you aren' familiar with it, replace "k-space" by "locally compact space" or by "first-countable space" in the statement of the theorem.

There is one final question which one could ask about extenders in arbitrary normal spaces. We have seen that linear extenders, and continuous extenders (with respect to the sup-norm topology), always exist, but that linear, bound-preserving extenders may fail to exist. Of course, linear, bound-preserving extenders would be very special cases of linear and continuous extenders. (Recall that for a linear transformation $T : E \rightarrow F$, where E and F are normed linear spaces, continuity of T is equivalent to the assertion that the number $\sup\{\|T(x)\| \mid x \in E \text{ and } \|x\| \leq 1\}$ exists and is finite, and that number is called the <u>operator norm of T</u>.) Obviously, then, each bound-preserving linear extender is a continuous linear transformation of operator norm 1 and an isometry. With these notions we are able to sharpen Arens' Example C, above, by proving a theorem. In order to understand the statement of the theorem, recall that for a space Y, $d(Y)$ is the least cardinality of a dense subset of Y while $c(Y)$ is defined to be $\sup\{\text{card}(\mathcal{D}) \mid \mathcal{D}$ is a disjoint collection of open subsets of $Y\}$. The following theorem appears in [DLP].

G. <u>Theorem:</u> Suppose X is a completely regular space having a closed subset A such that $c(A) > d(X)$. Then there is no linear, continuous extender from $C^*(A)$ to $C^*(X)$.

<u>Proof</u>: Suppose there is a linear, continuous extender $\eta : C^*(A) \rightarrow C^*(X)$. Then there is a positive integer m such that if $f \in C^*(A)$ has $\|f\| = \sup\{|f(a)| \mid a \in A\}$ at most 1, then $\|\eta(f)\| \leq m$.

Let $\mathcal{V}$ be a disjoint collection of relatively open subsets of A having $\text{card}(\mathcal{V}) > d(X)$. For each $V \in \mathcal{V}$ choose $a(V) \in V$ and a continuous $g_V : A \rightarrow [0,1]$ having $g_V(a(V)) = 1$ and $g_V[A-V] = \{0\}$. Define a set U_V by $U_V = \{x \in X \mid \eta(g_V)(x) > 1/2\}$. Each U_V is a non-void open subset of X so that, because $\text{card}(\{U_V \mid V \in \mathcal{V}\}) > d(X)$, some point $x_0 \in X$ belongs to infinitely many distinct sets U_V. Choose $V_1, V_2, \ldots, V_{2m}$ in $\mathcal{V}$ such that $x_0 \in \bigcap\{U_i \mid 1 \leq i \leq 2m\}$ where we have written U_i for U_{V_i}. Let

$f = \sum_{i=1}^{2m} g_{V_i}$. Because the sets are pairwise disjoint, $\|f\| = 1$.
Hence, $m \geq \|\eta(f)\| = \|\sum_{i=1}^{2m} \eta(g_{V_i})\| \geq \sum_{i=1}^{2m} \eta(g_{V_i})(x_0) > (2m)(1/2) > m$,

and that contradiction completes the proof. □ .

Note that in Example C, the closed subset A had $c(A) = \omega_1$ while the compact Hausdorff space X had $d(X) = \omega_0$. Similarly, if $X = \beta N$, the Čech-Stone compactification of the set N of natural numbers, and if $A = \beta N - N$, then $c(A) = c > \omega_0 = d(X)$. (The original study of extenders in βN appears in [GS].)

I have spoken about what can and cannot be done in arbitrary normal spaces long enough. Let me now turn to positive results which are available if one looks at normal spaces with extra structure. You probably think it odd that Dugundji's name has not yet been mentioned in a lecture on Dugundji Extension Theory, and I will now remedy that omission. The next theorem is the bounded, real-valued case of a theorem which grew out of work by Borsuk, Kakutani, Dugundji, Arens and Michael. It is the model for all extension theorems. (See [Du],[A], [M_3])

H. Theorem: Suppose A is a closed subspace of a metrizable space X
Then there is a function $\eta : C^*(A) \longrightarrow C^*(X)$ such that

(1) η is an extender;

(2) η is linear;

(3) if $f \in C^*(A)$ then the range of $\eta(f)$ is contained in the convex hull of the range of f;

(4) η is continuous provided both function spaces carry the sup-norm topology, the compact-open topology, or

the topology of pointwise convergence.

The proof of Theorem H is far too delicate to present here; it uses, in a crucial way, Arthur Stone's great theorem that each metric space is paracompact. Speaking loosely, the idea in the proof is to set up a "machine" which acts in exactly the same way on all members of $C^*(A)$. In building this "machine", one considers only a special covering of the space $X-A$, and a partition of unity subordinate to it. One does not even mention a member of $C^*(A)$ until after the "machine" is complete. (Compare that with the usual proofs of the Tietze-Urysohn theorem [K] in which each member $f \in C^*(A)$ is used to build its own extension.)

Since Theorem H appeared in the early 1950's there have been several attempts to generalize it by relaxing the hypothesis that X is metrizable. The most successful generalization was given by Borges in [B_1] where Theorem H was proved for stratifiable spaces instead of metric spaces. Borges defined a space X to be <u>stratifiable</u> if, corresponding to each open set U of X, there is a sequence $\langle S_n(U) \rangle$ of open sets satisfying:

(1) $S_n(U) \subset cl(S_n(U)) \subset U$;

(2) $U = \bigcup\{S_n(U) \mid n \geq 1\}$,

(3) if $U \subset V$ are open sets then $S_n(U) \subset S_n(V)$ for each $n \geq 1$.

It is not hard to see that each metric space is stratifiable, and it is known that there are mathematically important spaces which are stratifiable but not metrizable (e.g., arbitrary CW-complexes). Borges generalized his theorem even further in [B_2].

There are other classes of spaces, perhaps best known for pathological counterexamples, in which some version of Theorem H can be established. For example, consider the generalized ordered spaces, where by a generalized ordered space I mean any topological space which can be topologically embedded in a linearly ordered topological space (i.e., a linearly ordered set endowed with the usual order topology). The following theorem is proved in [HL_1] .

I. Theorem: Let A be a closed subspace of a generalized ordered space X. Then there is a linear extender $\eta : C^*(A) \to C^*(X)$ having the property that for each $f \in C^*(A)$, the range of $\eta(f)$ is contained in the closed convex hull of the range of f. In particular, η is a linear extender with operator norm 1.

The proof of Theorem I is too involved to present here, but the idea is easily understood. For simplicity, consider the case where A is a closed subset of a linearly ordered topological space X. As in the case of the real line, the set $X-A$ is the union of a disjoint family of order-convex sets, called convex components of $X-A$. Let I be a convex component of $X-A$. If I has two end points each belonging to A then we already know the value of each $f \in C^*(A)$ at these end points and our extension machine attempts to draw a straight line, or some analogue thereof, joining the values of f at the end points. The hard case occurs when one or both ends of I are Dedekind cuts in X, and in that case we make use of Banach limits to choose values for the extenssion of f over I. Once again, note that the extension process is defined by the spaces A and X and only after our "machine" is constructed do we apply it to particular members of $C^*(A)$.

We have already seen one strange generalized ordered space, namely the Michael line. A second well-known example is the Sorgenfrey line; see [L_1] for the general theory of such spaces.

A second pathological class for which Theorem H is known to hold is the class of retractifiable spaces. In his dissertation [D_1], van Douwen defined a space X to be retractifiable if and only if each closed $A \subset X$ is a retract of X, i.e., there is a continuous function $r : X \to A$ such that $r(a) = a$ for each $a \in A$. It is obvious how to build an extension machine for such spaces: one defines $\eta : C^*(A) \to C^*(X)$ by $\eta(f) = f \circ r$. Retractifiable spaces must be extremely pathological, e.g., in terms of connectedness: it is known that a metric space is retractifiable if and only if it is strongly zero dimensional.

Until recently there was an outstanding open question in Dugundji Extension Theory which motivated most of the research in the area. After seeing Arens' example (and its relatives), Michael observed that none of the counterexamples were perfect (= each closed set is G_δ), and asked whether each perfect, normal space must satisfy some version of Theorem H. Ten years later, Borges reposed the question in [B_1], asking whether each perfect, paracompact space satisfies Theorem H. Finally, in 1968, after R.W. Heath proved that every stratifiable space has a σ-discrete network, Michael asked whether each paracompact space with a σ-discrete network must satisfy Theorem H. (Recall that a network for a space X is a collection $\mathcal{N}$ of closed sets such that if $x \in U$, where U is open, then some $N \in \mathcal{N}$ has $x \in N \subset U$.) These problems were not solved until 1974, at which time Heath and I answered Michael's original question in the negative and, independently, van Douwen administered the coup de grâce to all three. Each solution was based on an answer to the following question: Suppose that for each closed $A \subset X$ there is a linear, continuous extender from $C^*(A)$ to $C^*(X)$; then what special properties must X have ? Heath and I proved that if one can always obtain extenders of operator norm < 2, then X must be hereditarily collectionwise normal [HL_2] and van Douwen [D_1, D_2] sharpened that result to allow extenders of operator norm < 3. Thus it follows that Bing's Example H [Bi] is a perfectly normal space for which Theorem H, above, cannot be proved. Van Douwen [D_2] went on to describe a perfect, paracompact space with a countable network which does not admit continuous extenders of any operator norm. His space is constructed inductively. The first step in the construction is easy to describe and important in its own right. The points of the space X consist of all points on the x-axis in $\mathbb{R}^2$, together with all points (x,y) in $\mathbb{R}^2$ having both coordinates rational and positive. Any point of the form $(m/n, 1/n)$ has its usual Euclidean neighborhoods. Any point (x,y), with $y > 0$, which is not of that form is isolated. Points on the x-axis have "butterfly-shaped" neighborhoods. More precisely, given $(a,0) \in X$ and $n \geq 1$, the set

$$B(n,(a,0)) = \left\{(x,y) \in X \,\middle|\, |a-x| < \frac{1}{n} \text{ and } \frac{y}{x-a} < \frac{1}{n}\right\} \cup \{(a,0)\},$$

is a basic neighborhood of $(a,0)$. With that topology, the set A of all non-isolated points of X is a closed subspace of X and there is no linear extender $\eta : C^*(A) \longrightarrow C^*(X)$ of operator norm < 3, even though X is paracompact, perfect, and has a countable network. The non-existence of such an extender follows from a Baire Category argument plus the following result of van Douwen [D_1, D_2].

J. <u>Theorem:</u> Suppose that A is a closed subset of a space $(X, \textcircled{T})$, and let $\textcircled{T}_A$ denote the relative topology on A. If there is a linear extender from $C^*(A)$ to $C^*(X)$ having operator norm < 3, then there is a function $\kappa : \textcircled{T}_A \longrightarrow \textcircled{T}$ such that

1) if $U \in \textcircled{T}_A$ then $U = \kappa(U) \cap A$

2) if U and V are disjoint members of $\textcircled{T}_A$ then

$\kappa(U) \cap \kappa(V) = \emptyset$.

So far today I have talked about when and how one can make extenders respect the linear and topological structures of function spaces. In the few remaining minutes, let me talk about partial orderings. In 1970, Zenor introduced the term "monotone normality" to describe a notion appearing in [B_1] . In a space X, let Ⓟ $= \{(A,U) \mid A$ is closed, U is open and $A \subset U\}$. Partially order Ⓟ by defining $(A_1,U_1) < (A_2,U_2)$ if $A_1 \subset A_2$ and $U_1 \subset U_2$. It is well-known that X is normal if and only if for each $(A,U) \in$ Ⓟ there is an open set G with $A \subset G \subset cl(G) \subset U$. A space (X, Ⓣ) is monotonically normal if there is a function G : Ⓟ ⟶ Ⓣ satisfying:

(1) if $(A,U) \in$ Ⓟ then $A \subset G(A,U) \subset cl(G(A,U)) \subset U$;

(2) if $(A_i,U_i) \in$ Ⓟ with $(A_1,U_1) < (A_2,U_2)$ then $G(A_1,U_1) \subset G(A_2,U_2)$.

Monotonically normal spaces satisfy a monotonic extension theorem [HLZ_1] , namely,

K. <u>Theorem:</u> Let A be a closed subspace of a monotonically normal space X. Then there is an extender $\eta : C^*(A) \to C^*(X)$ having the property that if two members f and g of $C^*(A)$ satisfy $f \leq g$, then $\eta(f) \leq \eta(g)$ in $C^*(X)$.

This monotonic extension property is a very strong one and not all normal spaces have the property as may be seen from the next theorem, due independently to van Douwen and myself. One proof appears in [BL, Thm 4.6.7] .

L. Theorem: Suppose that for each closed subspace A of X there is an extender $\eta : C^*(A) \to C^*(X)$ such that if $f \leq g$ in $C^*(A)$ then $\eta(f) \leq \eta(g)$ in $C^*(X)$. Then X is collectionwise normal.

In our paper [HLZ_1] we had asked whether the existence of monotonic extenders characterized monotonically normal spaces. Van Douwen answered that question in the negative by constructing a countable regular retractifiable space which is not monotonically normal [D_1].

I would like to take this chance to add a more philosophical paragraph at the end of my lecture. There has been a movement in American topology during the past five years whose motivation may be summed up as follows. If a topological space $(X, \textcircled{T})$ carries intrinsic structures (such as the various structures on $C^*(X)$, or the containment partial ordering of $\textcircled{T}$, or the partial ordering of the collection $\textcircled{P}$ in the definition of monotone normality), then it is of interest to determine what happens if usual topological notions are tied to this intrinsic structure. In addition to the ideas mentioned in this lecture, you can get a deeper idea of what we have in mind by looking at Zenor's papers [Z_1],[Z_2] and their sequel, written by Gruenhage [G]. (See also [BL].)

Let me close this lecture by listing a few open questions associated with Dugundji Extension Theory.

(1) <u>Dugundji Extension Theory and Moore spaces.[HL_2].</u> I refer you to the lectures of Mike Reed for the definition of a Moore space. The general question in this area asks whether a Moore space X must be metrizable provided that for each closed $A \subset X$ there is a continuous linear extender $\eta : C^*(A) \to C^*(X)$, where both

function spaces carry sup-norm topology. Certain things are known about this question.

a) If one can always obtain linear extenders of operator norm < 3 then van Douwen's theorem [D_2] forces X to be collectionwise normal and hence metrizable.

b) If we consider only those Moore spaces X which satisfy the Countable Chain Condition (i.e., which do not contain any uncountable pairwise disjoint collection of open sets), then X is metrizable if and only if for each closed $A \subset X$, some continuous linear extender from $C^*(A)$ to $C^*(X)$ exists.

c) For any Moore space X, if continuous linear extenders (without any norm restriction) can be found for each closed $A \subset X$, then X is normal and metacompact. That makes me wonder if a consistency result might be hiding here. Indeed, it will be fruitful to determine whether or not continuous linear extenders can be found in Heath's "V-space" [H], which may be described as follows. Assume Martin's Axiom plus the negation of the Continuum Hypothesis. Fix any set $S \subset \mathbb{R}$ having $\mathrm{card}(S) = \omega_1$; then S is a Q-set. The points of Heath's space X are all points $(x,y) \in \mathbb{R}^2$ having $y > 0$ together with all points $(x,0)$ where $x \in S$. Any point (x,y) with $y > 0$ is isolated, and basic neighborhoods of a point $(a,0)$, for $a \in S$, are sets of the form

$$B(a,n) = \{(a,0)\} \cup \{(x,y) \in X \mid y = x-a \text{ and } |x-a| < \tfrac{1}{n}\}.$$

It is known that X is a normal, metacompact Moore space, but whether X admits enough continuous linear extenders is unknown.

(2) <u>Is the Dugundji Extension property hereditary</u> ? More precisely, suppose a space X is known to satisfy the conclusions of either

Theorem H or Theorem I. Must every subspace of X satisfy the same conclusions ? If not, for which subspaces does the conclusion hold. (I seem to recall that the existence of norm 1 linear extenders is inherited by open F_σ -subspaces.)

(3) Uniformly continuous extenders. We saw in Theorem D that if A is any closed subspace of a normal space X, then there is a continuous (not necessarily linear) extender $\eta : C^*(A) \to C^*(X)$, both function spaces being topologized by the sup-norm metric. This third question asks whether η can be taken to be uniformly continuous with respect to the sup-metric. This is intimately related to another question: can one always obtain an extender η which has a Lipschitz constant. We know [DLP] that this Lipschitz constant cannot be < 2 (in general) but we don't know about constants belonging to $[2, +\infty)$.

References

[A] R. Arens, Extension of functions on fully normal spaces, Pacific J.Math. 2(1952), 11-22.

[B_1] C.R. Borges, On stratifiable spaces, Pacific J.Math. 17 (1966), 1-16.

[B_2] C.R. Borges, Absolute extensor spaces, a correction and an answer, Pacific J.Math. 50(1974), 29-30.

[BG] R. Bartle and L. Graves, Mappings between function spaces, Trans.Amer.Math.Soc. 72(1952), 400-413.

[Bi] R.H. Bing, Metrization of topological spaces, Canad.J.Math. 3(1951), 175-186.

[BL] D. Burke and D.J. Lutzer, Recent advances in the theory of generalized metric spaces, Proceedings of the Memphis

State University Topology Conference, Marcel Dekker Lecture Notes in Pure and Applied Mathematics, vol. 24, 1-70.

[D_1] E.van Douwen, Simultaneous extension of continuous functions, Doctoral Thesis, Free University of Amsterdam, 1975.

[D_2] E.van Douwen, Simultaneous linear extension of continuous functions, Gen.Top.Appl. 5(1975), 297-319.

[DLP] E.van Douwen, D.J. Lutzer and T. Przymusiński, Some extensions of the Tietze-Urysohn theorem, Amer.Math.Monthly, to appear.

[Du] J. Dugundji, An extension of Tietze's theorem, Pacific J.Math. 1(1951), 353-367.

[G] G. Gruenhage, Continuously perfectly normal spaces and some generalizations, preprint.

[GS] K. Gęba and Z. Semadeni, Spaces of continuous functions V, Studia Math. 19(1960), 303-320.

[H] R. Heath, Screenability, pointwise paracompactness and metrization of Moore spaces, Canad.J.Math. 16(1964), 763-770.

[HL_1] R. Heath and D.J. Lutzer, Dugundji Extension for linearly ordered spaces, Pacific J.Math. 55(1974), 419-425.

[HL_2] R. Heath and D.J. Lutzer, The Dugundji extension theorem and collectionwise normality, Bulletin Polish.Acad.Sci. 22 (1974), 827-829.

[HLZ_1] R. Heath and D.J. Lutzer and P. Zenor, Monotonically normal spaces, Trans.Amer.Math.Soc. 178(1973), 481-493.

[HLZ_2] R. Heath and D.J. Lutzer and P. Zenor, On continuous extenders, Studies in Topology, Academic Press, 1975, 203-214.

K] J. Kelley, General Topology, Van Nostrand, 1955.

L_1] D.J. Lutzer, On generalized ordered spaces, Dissertationes Math. 89(1972).

L_2] D.J. Lutzer, A selection-theoretic approach to certain extension theorems, Set Theoretic Topology, Academic Press 1977, 269-275.

[M_1] E. Michael, Continuous selections I, Ann. of Math. 63(1956), 361-382.

[M_2] E.Michael, The product of a normal space and a metric space need not be normal, Bulletin Amer.Math.Soc. 69(1963),375

[M_3] E. Michael, Some extension theorems for continuous functions, Pacific J.Math. 3(1953), 789-806.

[Z_1] P. Zenor, A metrization theorem, Collog.Math., to appear.

[Z_2] P. Zenor, Some continuous separation axioms, Fund.Math., to appear.