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A refinement of the category Unif of (Hausdorff) unifor spaces is a concrete category $\mathcal{K}$ having for the objects ll uniform spaces and for the morphisms mappings satisfying Unif $(X,Y) \subset \mathcal{K}(X,Y)$ for all spaces X, Y. The notion of the refinement of a category was introduced by Z. Frolík in [F_1]. The freckle refinement defined and examined here has for the orphisms the so called freckle continuous ($\mathcal{F}$ continuous) mappings $f: X \to Y$ determined by the following property: If $\{x_\alpha \mid \alpha \in I\}$ is set of points of X which is not uniformly discrete in X then $\{f(x_\alpha) \mid \alpha \in I\}$ is not uniformly discrete as well.

The first part of this note brings some properties and examples concerning the corresponding freckle structure ($\mathcal{F}$ structure) of a unifor space (e.g. the connection $\mathcal{F}$ fine spaces - selective ultrafilters and $\mathcal{F}$ structures - product of ultrafilters). The second part is an examination of the plus and minus functors associated with $\mathcal{F}$ and with a similar refinement $\mathcal{F}^2$ (in the sense of the definitions stated by Z. Frolík in [F_2]). It is shown that $\mathcal{F}_+$ is t distal functor, $\mathcal{F}_-$ is the identity and both $\mathcal{F}^2_+$, $\mathcal{F}^2_-$ are identities.

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§ 1. Basics. Examples of freckle fine spaces. Frec - le structure of a space.

The words space and discrete ean Hausdorff uniform space and uniformly discrete.

Definition 1.1: A space X i called freckle fin ($\mathcal{F}$

fine) if $\mathcal{F}(X,Y)$ = Unif (X,Y) for any space Y. Similarly, X is freckle coarse ($\mathcal{F}$ coarse) if $\mathcal{F}(Y,X)$ = Unif (Y,X) for any space Y.

Proposition 1.1: The singleton space $\{x\}$ is the only $\mathcal{F}$ coarse space. The $\mathcal{F}$ fine spaces form a coreflective subcategory of Unif (see [Vi]).

Proof is easy - the $\mathcal{F}$ fine spaces are closed under formation of sums and quotients in Unif.

Proposition 1.2: Let X be $\mathcal{F}$ fine.

(i) Any subspace of X is $\mathcal{F}$ fine.

(ii) The space $p^{\alpha}X$ is $\mathcal{F}$ fine for any cardinal reflexion p^{α}.

(iii) If X is precompact then $X = p^{0}\hat{X}$ where $\hat{X}$ is endowed with the discrete uniformity.

(iv) If $D_1 = \{x_{\alpha} \mid \alpha \in I\}$, $D_2 = \{y_{\alpha} \mid \alpha \in I\}$ are discrete in X then so is $D_1 \cup D_2$.

Proof: The statement (i) is evident. For the part (ii) recall that $p^{\alpha}X$ has for a base uniform covers of the cardinality smaller than $\aleph_{\alpha}$. Let $f: p^{\alpha}X \to Y$ be $\mathcal{F}$ continuous. Then $p^{\alpha}Y = Y$ because the contrary case implies that there is a discrete set in Y of the cardinality $\aleph_{\alpha}$ and so f is not $\mathcal{F}$ continuous. The statement (iii) follows from the fact that any mapping between precompact spaces is $\mathcal{F}$ continuous. Finally, put $A = \bigcup\{x_{\alpha}\}$, $\alpha \in I$ for a discrete set $\{x_{\alpha} \mid \alpha \in I\}$. Then $\{x_{\alpha} \mid \alpha \in I\} \cup \{X - A\}$ is uniform cover of X since it is refined by the meet of a cover realizing the discreteness of $\{x_{\alpha} \mid \alpha \in I\}$ and the cover $\{A, X - A\}$.

Starting with X discrete, Proposition 1.2 (ii) gives some examples of $\mathcal{F}$ fine spaces. We shall show once more example of a different kind. It bases on a special property of ultrafilters on the countable set.

Definition1.2: Let N be countable set. An ultrafilter F on N is said to be selective if, for any partition $\{P_{\alpha} \mid \alpha \in I\}$ of N, either some P_{α} belongs to F or there is a set $S \in F$ such that card $(S \cap P_{\alpha}) = 1$ for any $\alpha \in I$.

Remark 1.1: One can prove under continuous hypothesis that there is a selective ultrafilter on N.

Definition 1.3: Let F be a filter on X. The symbol X_F denotes the uniform space on X having for a base the covers $\{\{x\} \mid x \in X\} \cup S$, $S \in F$.

Proposition 1.3: Let F be an ultrafilter on N. Then N_F is $\mathcal{F}$ fine iff $\mathcal{F}$ is selective.

Proof: Let F fail to be selective. Then there is a partition $\{P_\alpha \mid \alpha \in I\}$ of N such that no P_α belongs to F and, for every choice of $x_\alpha \in P_\alpha$, the set $\{x_\alpha \mid \alpha \in I\}$ does not belong to F. Let N' be a space which has for a subbase the covers of N_F plus the cover $\{P_\alpha \mid \alpha \in I\}$. Then N' and N_F have the same discrete sets but N is finer than N_F. So, N_F is not $\mathcal{F}$ fine.

The proof of the second implication bases on the technique used in the paper [PR]. Suppose F is selective. Let $f: N_F \to M$ be an $\mathcal{F}$ continuous mapping onto a countable metric space M. Take a uniform cover $\mathcal{Z}$ of M. As M is countable $\mathcal{Z}$ can be refined by a point-finite uniform cover $\mathcal{Y} = \{Y_n \mid n \in N\}$ (see [V]). We have to show that $\mathcal{X} = \{X_n \mid n \in N\}$ where $X_n = f^{-1}(Y_n)$ belongs to N_F.

Let $\mathcal{Y}' = \{Y_n' \mid n \in N\}$ be a star-refinement of $\mathcal{Y}$ and let $\mathcal{X}' = \{X_n' \mid n \in N\}$ be a cover with $X_n' = f^{-1}(Y_n')$. Define a partition $\{P_n \mid n \in N\}$ as follows: $P_n = \{x \in N \mid St(x, \mathcal{X}) \subset X_n$ and n is the first such number$\}$. If some P_n belongs to F then $\mathcal{X}$ belongs to N_F. If it is not the case then there is a set $S \in F$ such that card $(P_n \cap S) = 1$ for all $n \in N$. As $\mathcal{X}$ is point-finite, $St(x, \mathcal{X}) \cap S$ is finite. This fact and the selectivity of F makes it possible to find a set $S' \in F$ such that $St(x, \mathcal{X}') \cap St(y, \mathcal{X}') = \emptyset$ for all distinct points $x, y \in S'$. Using again a star-refinement to $\mathcal{Y}'$ we obtain that $f(S')$ is discrete in M but S' is not discrete in N_F - a contradiction. The proof is complete.

Let us define a freckle structure ($\mathcal{F}$ structure) on a set as a set of all discrete sets of points in a uniformity. We shall show that the $\mathcal{F}$ structures are not naturally associated with uniformities (in contrast to proximity and distal structures, see [KP]).

Proposition 1.4: Let $\mathcal{S}$ be the $\mathcal{F}$ structure on an infinite set such that $\mathcal{S}$ consists of all finite sets. Then there is no coarsest uniformity among those which induce $\mathcal{S}$. Proof is easy.

On the other hand, there is an $\mathcal{F}$ structure without the finest uniformity inducing it as follows from the following observation (compare with proximity structures, see [K] and [D]).

Proposition 1.5: There are two uniformities U_1, U_2 inducing the same nondiscrete $\mathcal{F}$ structure but the greatest lower bound $U_1 \wedge U_2$ is discrete.

Proof: First an easy lemma:

Lemma 1.1: Let F be an ultrafilter on X. Then any nondiscrete space finer than X_F has the same $\mathcal{F}$ structure as X_F.

For the proof of the lemma, suppose Z is a discrete set in a finer space Y. Then $X - Z \in F$ and therefore Z is discrete in X_F.

Let us continue the proof of Proposition 1.4. Take two free ultrafilters F_1, F_2 on N and form the Katětov product $F_1 \times F_2$. Recall that $F_1 \times F_2$ is the ultrafilter on $N \times N$ such that $S \in F_1 \times F_2$ iff there exists a set $S_1 \subset F_1$ with $S \supset \bigcup_{x \in S_1} (\{x\} \times S_x)$ where $S_x \in F_2$ for any $x \in S_1$. Let U_1 be a uniformity having for a base the covers of $N \times N_{F_1 \times F_2}$ plus the cover $\{\{x\} \times N\} \mid x \in N\}$ and let U_2 arise in the same way by adjoining the cover $\{N \times \{x\} \mid x \in N\}$. Then U_1, U_2 have the same nondiscrete $\mathcal{F}$ structures (Lemma 1.1) but $U_1 \wedge U_2$ is discrete.

Remark 1.2: The fact that N_F for F selectite is an atom in the lattice of uniformities suggests the question whether any atom is $\mathcal{F}$-fine. The answer is in the negative as there are two noncomparable spaces with the same $\mathcal{F}$ structure and so, according to Lemma 1.1, two atoms with the same $\mathcal{F}$ structure. Therefore one of those is not $\mathcal{F}$ fine.

Let us denote by $p^o \wedge \mathcal{F}$ the refinement of Unif having for the morphisms the mappings which are simultaneously proximally and freckle continuous. Of course, any distally continuous mapping is $p^o \wedge \mathcal{F}$ continuous because, by the definition, a mapping $f: X \longrightarrow Y$ is distally continuous if $\{f^{-1}(Y_\alpha) \mid \alpha \in I\}$ is discrete whenever $\{Y_\alpha \mid \alpha \in I\}$ is. The following example shows that the refinement $p^o \wedge \mathcal{F}$ is strictly finer than the distal one (even in the sense of the fine spacea). So it follows e.g. that there is a smaller coreflective category than the distally fine spaces which contains the metric and the precompact spaces.

Example: Let X be a space which has for a base the countable partitions of X with at most finitely many classes of the same cardinality as X. Let card $X > 2^{\aleph_0}$. Then X is distally fine but not $p^o \wedge \mathcal{F}$ fine.

Proof: Suppose that $f: X \to M$ is a distally continuous mapping onto the metric space M. Since M has at most countable discrete sets then M is separable and card $M \leq 2^{\aleph_0}$. Take a cover $\mathcal{X} \in M$. We may and shall assume that $\mathcal{X} = \{X_n \mid n \in N\}$ is countable and point-finite. Let $\mathcal{P} = \{P_n \mid n \in N\}$ be the partition of M constructing by the procedure $P_1 = X_1$, $P_2 = X_2 - X_1$, $P_3 = X_3 - (X_1 \cup X_2)$ etc. Then $\mathcal{P} \prec \mathcal{X}$ and it suffices to prove that card $f^{-1}(P_n) =$ card X for at most finitely many $n \in N$. Suppose it is not the case. Then there is a discrete set $\{p_n \mid n \in N\}$ in M such that card $f^{-1}\{p_n\} =$ card X (it follows from point-finiteness of $\mathcal{X}$ and from card $X > 2^{\aleph_0}$). So $\{f^{-1}\{p_n\} \mid n \in N\}$ is not discrete in X and it is a contradiction.

Finally, the space X is not $p^o \wedge \mathcal{F}$ fine because the space X' having for a base all countable partitions of X is strictly finer than X but $p^o \wedge \mathcal{F}$ isomorphic to X.

§ 2. Plus and minus functors

First state the basic definition of this paragraph.

Definition 2.1: Let $\mathcal{G}$ be a refinement of Unif. Denote by Inv $\mathcal{G}$ the class of all concrete functors F: : Unif $\longrightarrow$ Unif such that F preserves $\mathcal{G}$ structure, i.e. FX is isomorphic to X in $\mathcal{G}$. Further, denote by $Inv_+\mathcal{G}$ ($Inv_-\mathcal{G}$) the positive (negative) functors in Inv $\mathcal{G}$,i.e. the functors in Inv $\mathcal{G}$ such that FX is coarser (finer) than X. The coarsest element of $Inv_+\mathcal{G}$, if it exists, is denoted by $\mathcal{G}_+$ and the finest element of $Inv_-\mathcal{G}$ is denoted by $\mathcal{G}_-$.

Theorem 2.1: The functor $\mathcal{F}_+$ is the distal functor. It means, $\mathcal{F}_+$ X has for a base the finite dimensional covers of X.

Proof: Observe that the distal functor D belongs to $Inv_+\mathcal{F}$ because X and DX have the same discrete sets (see [KP] and [W]). We shall prove that any functor F $\in$ $\in$ $Inv_+\mathcal{F}$ is finer than D. The space DX, for any X, is projectively generated by all uniformly continuous mappings of X into the hedgehog H(A), card A = card X (see [F_1]). Recall that the space H(A), for any set A, is a metric space of non negative real-valued functions f such that $f(\alpha) > 0$ for at most one $\alpha \in A$ and $f(\alpha) \leq 1$ for all $\alpha \in$ A. The distance in H(A) is given by ℓ_1-norm.

The functor F in question preserves mappings and so it suffices to prove that F is constant on all hedgehogs. We shall prove it in the following lemmas.

Lemma 2.1: $F\langle 0,1\rangle = \langle 0,1\rangle$.

Proof is evident.

Lemma 2.2: If F H(A) = H(A) then F H(B) = H(B) for all sets B with card B $\leq$ card A.

Proof: There exist mappings i: H(B) $\longrightarrow$ H(A), j: H(A) $\longrightarrow$ H(B) such that ji = $id_{H(B)}$. So F i is an embedding.

It follows from Lemma 2.2 that it suffices to examine the hedgehogs over the sets with great cardinalities. The idea of the following lemma is due to Z. Frolík.

Lemma 2.3: Let A have a sequentially regular cardinality. Put $I_\varepsilon^\alpha = \{f \in H(A) \mid \varepsilon \leq f(\alpha) \leq 1\}$. Then there is an ε, $1 > \varepsilon > 0$ such that the family $\{I^{\alpha}_{\varepsilon} \mid \alpha \in A\}$ is discrete in F H(A).

Proof: The set $\{I^{\alpha}_{1} \mid \alpha \in A\}$ is discrete in F H(A). So, for any $\alpha \in A$ there is an $n(\alpha) \in N$ such that $\{I^{\alpha}_{\frac{1}{n(\alpha)}} \mid \alpha \in A\}$ is a discrete family in F H(A). As card A is sequentially regular then there is a set B, $B \subset A$, card B = card A and a number $n \in N$ such that $\{I^{\beta}_{\frac{1}{n}} \mid \beta \in B\}$ is discrete in F H(A). The proof now follows via the natural isomorphism between H(A) and H(B).

Lemma 2.4: Suppose that $0 < \varepsilon < 1$ and that card A is sequentially regular. Then the spaces F H(A) and H(A) induce the same uniformities on the set $\cup I^{\alpha}_{\varepsilon}$, $\alpha \in A$ and the same topological neighbourhoods of the point 0. So, F H(A) = H(A).

Proof: The complement of the ε-neighbournood $\mathcal{O}_{\varepsilon}$ of 0 in H(A) is the joint of a discrete family of closed sets in F H(A) and so it is closed in F H(A). Therefore $\mathcal{O}_{\varepsilon}$ is open in F H(A). The remaining parts of Lemma 2.4 are easy.

Theorem 2.2: The functor $\mathcal{F}_{-}$ is the identity on Unif.

Proof: It holds the following statement: For any space X there is a space $\tilde{X}$ such that

1. X is a quotient space of $\tilde{X}$ in Unif
2. The space $\tilde{X}$ is $\mathcal{F}$ minimal, it means, if $\tilde{X}$ and Y have the same $\mathcal{F}$ structures then Y is not finer than $\tilde{X}$.

Using this, the proof follows in the following way. Let $F \in \mathrm{Inv}_{-}\mathcal{F}$ and let h: $\tilde{X} \longrightarrow X$ be the quotient mapping. Since $F\tilde{X} = \tilde{X}$ and Fh: $F\tilde{X} \longrightarrow F X$ is uniformly continuous then F X = X because h was a quotient mapping. So F is the identity.

It remains to prove the starting statement. First the construction of X for a space X (see [Č], p. 699, [I], p. 52 for the introduction and [H] for further interesting investigations). Let $\mathcal{X}$ be a cover of X and let D (or E) be a discrete (or indiscrete, resp.) two-point space with points c, d. Put $X_{\mathcal{X}} = \sum \{ E_{\langle x,y\rangle} \mid x \neq y,\ y \in St(x,\mathcal{X}) \} + \sum \{ D_{\langle x,y\rangle} \mid x \neq y,\ y \notin St(x,\mathcal{X}) \}$ for $E_{\langle x,y\rangle} = E$, $D_{\langle x,y\rangle} = D$. Finally put $\tilde{X} = \bigwedge X_{\mathcal{X}}$, $\mathcal{X} \in X$ and define $h: \tilde{X} \to X$ such that $h(c) = x$, $h(d) = y$ for h partialized on $D_{\langle x,y\rangle}$ or $E_{\langle x,y\rangle}$. Then h is a quotient mapping.

Take a uniformity U strictly finer than the uniformity $\tilde{U}$ of $\tilde{X}$. Let $\mathcal{Y} \in U - \tilde{U}$. By the construction, for any cover $\mathcal{X} \in \tilde{U}$ there is a two-point set which is discrete of order $\mathcal{Y}$ but not of order $\mathcal{X}$. Then the join of these two-point sets taken over all $\mathcal{X} \in \tilde{U}$ is discrete of order $\mathcal{Y}$ but it is not discrete in $\tilde{U}$. So, $\tilde{X}$ is $\mathcal{F}$ minimal.

Definition 2.2: Let us denote by $\mathcal{F}^2$ the refinement having for morphisms the mappings $f: X \to Y$ such that $f \times f: X \times X \to Y \times Y$ is $\mathcal{F}$ continuous.

Theorem 2.3: Both $\mathcal{F}^2_+$ and $\mathcal{F}^2_-$ are identities.

Proof: Evidently $\mathcal{F}^2_+ = \mathrm{Id}$ because $\mathcal{F}^2 \subset \mathcal{F}$. We shall prove that $\mathcal{F}^2_+ = \mathrm{Id}$. In fact, we shall prove that $\mathrm{Inv}_+ \mathcal{F}^2 = \{\mathrm{Id}\}$.

Lemma 2.5: Let U_1, U_2 be two uniformities on a set. For any cover $\mathcal{X}$ of X put $T_{\mathcal{X}} = \{(x,y) \in X \times X \mid y \notin St(x,\mathcal{X})\}$. If for any $\mathcal{X} \in U_1$ there is a cover $\mathcal{Y} \in U_2$ such that $T_{\mathcal{X}} \subset T_{\mathcal{Y}}$ then U_2 is finer than U_1.

Proof: Let $\mathcal{X}^*$ be a star refinement of $\mathcal{X}$ and let $T_{\mathcal{X}^*} \subset T_{\mathcal{Y}}$ for a cover $\mathcal{Y} \in U_2$. It is easy to check that $\mathcal{Y} \prec \mathcal{X}$.

Lemma 2.6: Let $F \in \mathrm{Inv}_+ \mathcal{F}^2$ and let the space X have a discrete set D with the same cardinality as X. Then $F X = X$.

Proof: Suppose F X is strictly coarser than X for a space X in question. Take a cover $\mathcal{X} \in X - F X$. Then card T = card D and so there is a bijection $\varphi: D \to T_{\mathcal{X}}$.

Put $M = \bigcup_{d \in D} \{\langle d, \varphi(d)_1\rangle, \langle d, \varphi(d)_2\rangle\}$ where $\varphi(d)_1$, $\varphi(d)_2$ mean the first and the second coordinate of $\varphi(d)$. Then M is discrete in $X \times X$ but not in $F\,X \times F\,X$ because there is no cover $\mathcal{Y} \in F\,X$ with $T_{\mathcal{X}} \subset T_{\mathcal{Y}}$.

Now, the proof can be completed as follows. Let $F \in \in \mathrm{Inv}_{+} \mathcal{F}^{2}$. For any space Y form a space X on the set $\sum_{\beta \leq \alpha} Y_\beta$ where α = card Y and $Y_\beta = Y$ for all $\beta \leq \alpha$. The covers $\mathcal{X}_{\mathcal{Y}}$ forming a base of the space X are indexed by covers $\mathcal{Y} \in Y$ such that $\mathcal{X}_{\mathcal{Y}} = \sum_{\beta \leq \alpha} \mathcal{Y}_\beta$, $\mathcal{Y}_\beta = \mathcal{Y}$. According to Lemma 2.6, $F\,X = X$. Moreover, we have uniformly continuous mappings $j: Y \to X$, $k: X \to Y$ such that $k\,j = \mathrm{id}_Y$. Hence $F\,j$ is an embedding and $F\,Y = = Y$.

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