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Universal metric spaces

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Cornet-sp ces given in [P] ar xamples of uniform spa ces of rbitrarily larg point charac er. We will not repeat a full definition of these spaces; we only mention the following: to each regul r cardinals α, β, a m trizable uniform space $U(\alpha,\beta)$ is given such that point-character of $U(\alpha,\beta)$ is greater than m (m is ny regular cardinal $< \beta$) and each uniform space X whose point-character is not greater than β and cov ring ch racter is less or equal to α is homeomorphic to s me subspace of a suitable cartesian product of $U(\alpha,\beta)$. However, there is an aesthetical defect in the last assertion: it is not true in gener l that $U(\alpha,\beta)$ satisfies the same condition on ch racters as an embedded X, usually point-character of $U(\alpha,\beta)$ is greater than β and covering character is greater than α (exceptions are e.g. couples where $\beta = \omega_0$). The desired result of the present note is to save an aesthetic. Universal metric spaces are given and they should help in a continuation of an investigation of characters of uniformiti s.

Construction: Let a, b be infinite cardinals. $N = \{0,1,2,\dots\}$. Put $H_0 = a \times \{0\}$, $K_0 = \min(a^+, b)$

$$H_n = \left(\sum_{\beta \in K_{n-1}} (\text{card } H_{n-1})^\beta\right) \times \{n\}, \quad K_n = \min((\text{card } H_n)^+, b)$$

Put $A = \sup_n \text{card } H_n$.

Put $\overline{X} = A^N$.

We shall employ the following notation:

$\mathcal{P}(M) = \{L \subset M \mid L \neq \emptyset\}$

$\mathcal{P}_d(M) = \{L \in \mathcal{P}(M) \mid \text{card } L < d\}$.

Clearly, card $\mathcal{P}_b(a) \leq A$. Choos a mapping φ fro A onto $\mathcal{P}_b(H_0)$. We define a relation $R_0 \subset X \times H_0$ by:

(x,h) R_0 iff $h \in \varphi(x_0)$, $x = (x_0, \dots, x_2, \dots$

Choose a one-to-one mapping p_1 from H_1 onto $\mathcal{P}_b(H_o)$. We define a relation $P_o \subset H_1 \times H_o$ by: $(h_1, h_o) \in P_o$ iff $h_o \in p_1(h_1)$

For $M \in \mathcal{P}_b(H_o)$ choose a mapping v_M from A onto $\mathcal{P}(M)$. Let $M \supset Q$ be elements of $\mathcal{P}_b(H_o)$. We put $[M,Q] = \{h \in H_1 \mid Q \subset p_1(h) \subset M\}$, card $\mathcal{P}([M.Q]) \leq A$. We choose a mapping $t_{M,Q}$ from A onto $\mathcal{P}([M,Q])$. Now we are prepared to define a relation $R_1 \subset \bar{X} \times H_1$:

$(x,h) \in R_1$ iff $h \in t_{M(x),Q(x)}(x_2)$ where $M(x) = \varphi(x_o)$ and $Q(x$

$= v_{\varphi(x_o)}(x_1)$.

We put $\mathcal{Z}_1 = \{J \subset H_1 \mid \text{card } J < b \text{ and } \bigcap_{j \in J} R_1^{-1}(j) \neq \emptyset\} =$

$= \bigcup\{\mathcal{P}_b[M,Q] \mid Q \subset M \in \mathcal{P}_b(H_o)\}$ card $\mathcal{Z}_1 = \sum_{\beta < K_o}$ (card

Suppose R_{k-1} and $\mathcal{Z}_{k-1}$ are defined. We shall define R_k a $\mathcal{Z}_k$. We choose a one-to-one mapping p_k from H_k onto $\mathcal{Z}_{k-}$ We define a relation $p_k \subset H_{k-1} \times H_k$ by: $(h_k, h_{k-1}) \in P_k$ iff $h_{k-1} \in p_k(h_k)$. For $M \in \mathcal{Z}_{k-1}$ we choose v^k_M from A onto $\mathcal{P}($

For $Q \subset M \in \mathcal{Z}_{k-1}$ we put $[M,Q] = \{h \in H_k \mid M \supset p_k(h) \supset Q\}$ and we choose a mapping $t^k_{M,Q}$ from A onto $\mathcal{P}_b([M,Q])$. We defi a relation $R_k \subset \bar{X} \times H_k$ by:

$(x,h) \in R_k$ iff $h \in t^k_{M(x),Q(x)}(x_{2k})$ where $M(x) = R_{k-1}(x)$, $Q(x)$

$= v^k_{M(x)}(x_{2k-1})$

We define $\mathcal{Z}_k = \{J \subset H_k \mid \bigcap_{j \in J} R_k^{-1}(j) \neq \emptyset\}$. card $\mathcal{Z}_k =$

$= \sum_{\beta < K_{k-1}}$ card H_k^β

Now we define a pseudometric uniformity $U(a,b)$ on X. For $h \in H_n$, we put $\tilde{h} = \{x \in X \mid (x,h) \in R_n\}$. We define a cover $\mathcal{U}$ of $\bar{X}$ by $\mathcal{U}_n = \{\tilde{h}\}_{h \in H_n}$.

Claim: $\mathcal{U}_n \overset{*}{\prec} \mathcal{U}_{n-1}$, $n = 1,2,3,\dots$

A Hausdorff reflection of $(\bar{X}, U(a,b))$ will be denoted by $M(a,b)$. We put $M(a,b) = (X, U(a,b))$.

Explanation: Being afraid that the very simple idea of Construction has lost in a not very cultured fores of mathematical symbols we add a few human words: the basi idea is to represent a system of subsets of a set X by it

incident graph (the idea well-known to all who are familiar with hypergraphs); relations R' s are yielded by incidence relations of covers $\mathcal{U}_i$; relations P' s describe which members of $\mathcal{U}_{i-1}$ contain some element of $\mathcal{U}_i$; the choice of P_i assures that the uniformity of $M(a,b)$ is involved as much as possible (an necessary); of course, the complexity of $M(a,b)$ is given also by the choice of $\mathcal{V}_M$'s and $t_{M,Q}$ (it is mapping $t^k_{M,Q}$ which gives that $\mathcal{U}_k \overset{*}{<} \mathcal{U}_{k-1}$).
It should be mentioned that K_n's were introduced only for technical needs of the procedure used below.

Definition 1: Let X be a set. Let $\mathcal{P}$ be a cover of X. A point-character pc $\mathcal{P}$ of $\mathcal{P}$ is defined as the least cardinal β such that card $\{P \mid x \in P$ and $P \in \mathcal{P}\} < \beta$ for each $x \in X$.

Definition 2: Let $(X,\mathcal{U})$ be a uniform space. Covering character tc $(X,\mathcal{U})$ (point-character pc $(X,\mathcal{U})$ of $(X,\mathcal{U})$) is defined as follows: tc $(X) \leq \alpha$ (pc $(X) \leq \beta$ resp.) iff there is a base $\mathcal{B}$ of $(X,\mathcal{U})$ such that for each $\mathcal{P} \in \mathcal{B}$ card $\mathcal{P} < \alpha$ (pc $\mathcal{P} \leq \beta$, resp.),
∇c $(X,\mathcal{U}) = \alpha$ if ∇c $(X,\mathcal{U}) \leq \alpha$ and ∇c$(X,\mathcal{U}) \nleq \beta$ for each $\beta < \alpha$ (∇ = t or p).

The following assertion which generalizes [V] seems to be useful.

Proposition 1: Let X be a uniform space. If tc $(X,\mathcal{U}) \leq \alpha^+$, then pc $(X,\mathcal{U}) \leq \alpha$.

Proof: Let $\mathcal{U}$ be an X-uniform cover. Take an X-uniform cover $\mathcal{V}$ which double star-refines $\mathcal{U}$ i.e. $\mathcal{V} \overset{**}{<} \mathcal{U}$ and card $\mathcal{V} \leq \alpha$. Suppose $\mathcal{V}$ is well-ordered by $\prec$ in such a way that $(\mathcal{V},\prec) \simeq \beta \leq \alpha$.

For each $V \in \mathcal{V}$ choose $U_V \in \mathcal{U}$ such that st$\{V, \mathcal{V}\} \subset U_V$. For $V \in \mathcal{V}$, defin $F_V = U_V -$ $- \bigcup \{W \in \mathcal{V} \mid W \prec V\}$. It is easy to check that $\{F_V\}_{V \in \mathcal{V}}$ is an X-uniform cover which refines $\mathcal{U}$ and card $\{F_V \mid x \in F_V\} < \alpha$ for each $x \in X$ (to $x \in X$, put $V_x = \min \{V \in \mathcal{V} \mid x \in V\}$; then $x \in F_V$ for $V \succ V_x$).

For $V \in \mathcal{V}$, put $W_V = \min \{W \in \mathcal{V} \mid W \cap V \neq \emptyset\}$. The

$V \subset F_W$.

Remark: One can derive from Proposition 1 that if th density of a topological space X is less than a^+, then each uniformity inducing the topology of X has point-character less than a.

Definition 3: Let a, b be infinite cardinals. A met rizable uniform space is called (a,b)-universal iff tc (V) and pc $(V) \leq b$, and each uniform space X with tc $(X) \leq a$ and pc $(X) \leq b$ can be embedded in a suitable product of V.

Theorem 1: Let a, b be infinite cardinals. If cf a > $> \omega_0$ then there is no (a,b)-universal space.

Proof: A metrizable uniform space V has a countable se $\mathcal{B}$ of uniform covers. Suppose tc $(V) \leq a$. As cf (a) > there is a cardinal d such that $\sup_{\mathcal{P} \in \mathcal{B}} \operatorname{card} \mathcal{P} < d < a$ which shows that V cannot be (a,b)-universal for any b.

Remark: The quite different situation occurs if we l ve the condition of metrizability of universal spaces.

Theorem 2: Let a, b be infinite cardinals.
1) If a = d and $d^{\beta} = d$ for each $\beta < b$, then there is an (a,b)-universal space V(a,b).
2) If a is a limit cardinal, cf a = ω_0 and

$$(*) \qquad \sum_{\beta < \min(\alpha^+, b)} \alpha^{\beta} < a \text{ for each } \alpha < a$$

then there is an (a,b)-universal space V(a,b).

Proof: 1) In this case, Proposition 1 implies that can suppose that $b \leq d$. Consider a space M(d,b). As $\sum_{\beta < b} d$ = d we gain that card H_n in Construction is equal to d. Hen tc $(M(d,b)) \leq a$. Clearly, pc $(M(d,b)) \leq b$ and we put V(a,b) = = M(d,b).
2) Take an increasing sequence $\{\alpha_n\}_{n \in N}$ of cardinals co verging to a, (*) implies that tc $(M(\alpha_n, b)) \leq a$ for each n We put $V(a,b) = \prod_{n \in N} M(\alpha_n, b)$.

Remark: Under Generalized Continuum Hypothesis, there an (a,b)-universal space for each couple (a,b) such that a an isolated cardinal or cf a = ω_0.

References:

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