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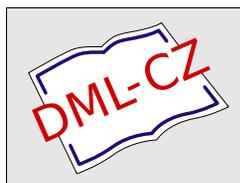
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SOME REMARKS ON MIXED APPROXIMATION PROBLEM

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Dedicated to my father Milan Práger on his 85th birthday

Abstract: Several years ago, we discussed the problem of approximation
polynomials with Milan Práger. This paper is a natural continuation of the
work we collaborated on. An important part of numerical analysis is the
problem of finding an approximation of a given function. This problem can be
solved in many ways. The aim of this paper is to show how interpolation can
be combined with the Chebyshev approximation.

Keywords: interpolation, approximation, Chebyshev approximation, Remez
algorithm

MSC: 65D05, 41A05, 41A50

1. Introduction

Numerical analysis often requires approximating a given real-valued function f,
continuous on a closed interval $[a, b]$, by another function g that is more suitable for
computing and that only slightly differs from the given function. The function g is
in most cases a polynomial. In [2] and [5], there are described three basic ways of
approximation of the given function f.

1. *Interpolation approximation* is such a replacement of the given function f by
   a new function g which satisfies the following condition:

$$f(x_j) = g(x_j) \tag{1}$$

   at the given points $a \le x_0 < x_1 < \cdots < x_n \le b$. Sometimes we also additionally
   require the coincidence of the derivatives $f^{(i)}(x_j) = g^{(i)}(x_j)$ for $i = 1, 2, \ldots, r$.

2. *The Chebyshev approximation* consists in the minimization of the maximum
   norm. The desired polynomial g satisfies

$$E_n(f) = ||f - g|| = \max_{x \in [a,b]} |f(x) - g(x)| \le \max_{x \in [a,b]} |f(x) - h(x)|, \tag{2}$$

   where h is an arbitrary polynomial of degree at most n.

3. *The least squares method* finds a function g which fits with the given function f in such a way that the sum of squares of the differences $f(x_j) - g(x_j)$, sometimes multiplied by a suitable weight function w,

$$\sum_{j=0}^{n} w(x_j)[f(x_j) - g(x_j)]^2$$

is minimal. In the usual case, $g(x) = \sum_{k=0}^{m} c_k g_k(x)$ and the sum to be minimized is

$$\sum_{j=0}^{n} w(x_j) \left[f(x_j) - \sum_{k=0}^{m} c_k g_k(x_j) \right]^2.$$

2. Mixed approximation polynomial

Now we try to construct a mixed polynomial which is a combination of interpolation (1) and the Chebyshev approximation (2). The *mixed approximation polynomial* is a polynomial s of degree at most n which approximates the function f in the Chebyshev sense, i.e.

$$\|f - s\| = \max_{x \in [a,b]} |f(x) - s(x)| \tag{3}$$

is minimal and at the endpoints of the interval it fulfils, in addition, the interpolation conditions

$$s(a) = f(a) \qquad \text{and} \qquad s(b) = f(b). \tag{4}$$

Such a polynomial s has similar properties as the Chebyshev approximation.

If $f \in C[a,b]$ and s is a polynomial of degree at most n such that $s(a) = f(a)$ and $s(b) = f(b)$ then the following holds.

a) Suppose there exist a constant c and n points $x_1 < x_2 < \cdots < x_n$ in the interval (a,b) such that

$$\text{sign}[(-1)^i(f(x_i) - s(x_i))] = c \qquad \text{for} \quad i = 1, \ldots, n.$$

Then

$$E_n(f) \geq \min_{i=1,\ldots,n} |f(x_i) - s(x_i)|.$$

b) The polynomial s is the best approximation of the function f in the sense of Chebyshev if and only if there exist at least n points $x_1 < x_2 < \cdots < x_n$ in the interval (a,b) with the property

$$f(x_i) - s(x_i) = \alpha(-1)^i \|f - s\| \qquad \text{for} \quad i = 1, \ldots, n,$$

where $\alpha = 1$ or $\alpha = -1$ for all i simultaneously. The set of the points $\{x_i\}_{i=1}^{n}$ is called the Chebyshev alternant.

c) The polynomial s is unique.

The proof is given in [3], but it is only a slight modification of corresponding proofs in the standard case.

3. Construction of mixed approximation polynomial

Now we show a construction of the polynomial s which is the best approximation of a given function $f \in C[a,b]$ in the maximum norm and furthermore satisfies $s(a) = f(a)$, $s(b) = f(b)$. We use the Remez algorithm which sequentially improves the polynomial using the alternant property, see [1], [5].

The initial approximation of the alternant $x_1^{(0)}, \ldots, x_n^{(0)}$ can be arbitrary, but the points must be mutually different. From the kth approximation of the alternant $x_1^{(k)}, \ldots, x_n^{(k)}$ we will construct next approximation $x_1^{(k+1)}, \ldots, x_n^{(k+1)}$. Having the kth approximation, we can construct a polynomial

$$s^{(k)}(x) = \sum_{j=0}^{n} c_j^{(k)} x^j$$

of the degree at most n such that it holds

$$f(x_i^{(k)}) - s^{(k)}(x_i^{(k)}) = (-1)^i E^{(k)} \qquad \text{for} \quad i = 1, \ldots, n,$$

where $E^{(k)}$ is some constant which we have to determine. The conditions $s(a) = f(a)$ and $s(b) = f(b)$ are fulfilled at the endpoints of the interval $[a,b]$. We have a system of $(n+2)$ linear equations for the coefficients $c_0^{(k)}, \ldots, c_n^{(k)}$ and the constant $E^{(k)}$.

$$
\begin{aligned}
\sum_{j=0}^{n} c_j^{(k)} \left(x_i^{(k)} \right)^j + (-1)^i E^{(k)} &= f\left(x_i^{(k)} \right), \\
\sum_{j=0}^{n} c_j^k a &= f(a), \\
\sum_{j=0}^{n} c_j^k b &= f(b).
\end{aligned}
\tag{5}
$$

The determinant of this system (5) is nonzero, so there exists a unique solution, see [4].

Now we denote

$$R^{(k)}(x) = f(x) - s^{(k)}(x) \tag{6}$$

and choose arbitrarily the number q such that $q \in (0,1)$.

Let the points $x_1^{(k)}, \ldots, x_n^{(k)}$ be given. Then we are looking for a new set of points $x_1^{(k+1)}, \ldots, x_n^{(k+1)}$ so that $x_i^{(k+1)} \in [x_{i-1}^{(k)}, x_{i+1}^{(k)}]$ for $i = 1, \ldots, n$, and the following conditions are fulfilled:

$$
\max_{1 \le i \le n} \left| R^{(k)}\left(x_i^{(k+1)} \right) \right| \ge |E^{(k)}| + q(\|R^{(k)}\| - |E^{(k)}|), \tag{7}
$$

$$
R^{(k)}\left(x_i^{(k+1)} \right) R^{(k)}\left(x_{i+1}^{(k+1)} \right) \le 0, \qquad i = 1, \ldots, n-1, \tag{8}
$$

$$
\left| R^{(k)}\left(x_i^{(k+1)} \right) \right| \ge |E^k|, \qquad i = 1, \ldots, n. \tag{9}
$$

This choice is not unique. We show that we can construct the $(k + 1)$st approximation of the alternant such that the required properties are satisfied. We choose a point $y^{(k+1)}$ such that it holds

$$\left| R^{(k)} \left(y^{(k+1)} \right) \right| \geq |E^{(k)}| + q(\|R^{(k)}\| - |E^{(k)}|).$$

Since the right-hand side is at most equal to $\|R^{(k)}\|$, then the choice which satisfies the condition (7) is possible.

When we have the kth approximation of the alternant, we can define the $(k+1)$st approximation in the following way. One point is the point $y^{(k+1)}$ and the other points are suitable points from the previous kth approximation.

If $E^k = 0$ the $(k + 1)$st approximation will be a set containing the point $y^{(k+1)}$ and arbitrary $n - 1$ points of the kth approximation, i.e. an arbitrary point of the kth approximation can be replaced by the point $y^{(k+1)}$. By ordering of the points of the $(k + 1)$st approximation, these points will be put in the corresponding intervals. The conditions (8) and (9) are also fulfilled, because $E^k = 0$ and then $R^{(k)} \left(x_i^{(k)} \right) = 0$ for every i, too.

If $E^k \neq 0$, there exists $R^{(k)} \left(x_i^{(k)} \right) \neq 0$ for every i. Now we describe three cases which can occur:

1. $y^{(k+1)} \in [a, x_1^{(k)}]$,

2. $y^{(k+1)} \in [x_n^{(k)}, b]$,

3. $y^{(k+1)} \in [x_1^{(k)}, x_n^{(k)}]$.

In case 1, we put $x_1^{(k+1)} = y^{(k+1)}$ and for $i = 2, \cdots, n$ we define

$$x_i^{(k+1)} = x_i^{(k)} \quad \text{if} \quad R^{(k)} \left(x_1^{(k)} \right) R^{(k)} \left(x_1^{(k+1)} \right) > 0 \qquad \text{or}$$

$$x_i^{(k+1)} = x_{i-1}^{(k)} \quad \text{if} \quad R^{(k)} \left(x_1^{(k)} \right) R^{(k)} \left(x_1^{(k+1)} \right) < 0.$$

We drop the point $x_1^{(k)}$ or $x_n^{(k)}$ from the previous approximation.

In case 2, we put $x_n^{(k+1)} = y^{(k+1)}$ and for $i = 1, \cdots, n - 1$ we define

$$x_i^{(k+1)} = x_i^{(k)} \quad \text{if} \quad R^{(k)} \left(x_n^{(k)} \right) R^{(k)} \left(x_n^{(k+1)} \right) > 0 \qquad \text{or}$$

$$x_i^{(k+1)} = x_{i+1}^{(k)} \quad \text{if} \quad R^{(k)} \left(x_n^{(k)} \right) R^{(k)} \left(x_n^{(k+1)} \right) < 0.$$

We drop the point $x_n^{(k)}$ or $x_1^{(k)}$ from the previous approximation.

In case 3, we denote by i_0 the subscript such that $y^{(k+1)} \in [x_{i_0}^{(k)}, x_{i_0+1}^{(k)}]$. Then we put $x_{i_0}^{(k+1)} = y^{(k+1)}$ and $x_i^{(k+1)} = x_i^{(k)}$ for $i \neq i_0$ if $R^{(k)} \left(y^{(k+1)} \right) R^{(k)} \left(x_{i_0}^{(k)} \right) > 0$. We drop the point $x_{i_0}^{(k)}$ from the previous approximation. Or we put $x_{i_0+1}^{(k+1)} = y^{(k+1)}$ and $x_i^{(k+1)} = x_i^{(k)}$ for $i \neq i_0 + 1$ if $R^{(k)} \left(y^{(k+1)} \right) R^{(k)} \left(x_{i_0+1}^{(k)} \right) > 0$. We drop the point $x_{i_0+1}^{(k)}$ from the previous approximation.

In this choice, the points of the $(k + 1)$st approximation are in the corresponding intervals and all conditions are fulfilled. A convergence of this process for a sufficiently smooth function is proved in [4].

4. Examples

We complete the previous theory with several simple numerical experiments. We present results for the function $f_1(x) = e^x$ on the interval $[0, 1]$. The speed of the convergence in two special cases are summarized in the following Tables 1 and 2. The iterations of the Remez algorithm at the points of the alternant in each table are given. The point which is changed in the corresponding iteration is printed in boldface digits. The points of the uniform partition of the interval are chosen for the first iteration. The accuracy of computing the points of the alternant is 0.01. We solved our task for $n = 3$, $n = 4$, $n = 5$, $n = 6$, but we present only the tables for $n = 3$ and $n = 6$.

Under each table, the mixed approximation polynomial corresponding to the last row is written. Its coefficients are shown to three decimal places.

Table 3 shows the dependence of the approximation error on the degree of the polynomial used. Approximation error is indicated by means of the absolute value of the extremes of the difference $f(x) - s(x)$ between the given function and the

No. of it.	x_1	x_2	x_3
1	0.25	0.50	0.75
2	0.25	0.50	**0.89**
3	**0.15**	0.50	0.89
4	**0.13**	0.50	0.89
5	0.13	**0.52**	0.89

$$s(x) = 0.280x^3 + 0.424x^2 + 1.014x + 1$$

Table 1: Convergence of the alternant, $n = 3$.

No. of it.	x_1	x_2	x_3	x_4	x_5	x_6
1	0.14	0.29	0.43	0.57	0.71	0.86
2	0.14	0.29	0.43	0.57	0.71	**0.96**
3	**0.06**	0.29	0.43	0.57	0.71	0.96
4	0.06	0.29	0.43	0.57	**0.83**	0.96
5	0.06	**0.20**	0.43	0.57	0.83	0.96
6	0.06	0.20	0.43	**0.63**	0.83	0.96
7	**0.05**	0.20	0.43	0.63	0.83	0.96
8	0.05	0.20	**0.40**	0.63	0.83	0.96

$$s(x) = 0.002x^6 + 0.007x^5 + 0.043x^4 + 0.166x^3 + 0.500x^2 + x + 1$$

Table 2: Convergence of the alternant, $n = 6$.

Pol. deg.	max	min	difference
3	0.7489 E−3	0.7471 E−3	0.0018 E−3
4	0.3519 E−4	0.3472 E−4	0.0047 E−4
5	0.1419 E−5	0.1370 E−5	0.0049 E−5
6	0.4972 E−7	0.4717 E−7	0.0255 E−7

Table 3: Approximation convergence in dependence of the polynomial degree.

mixed approximation polynomial. In the table, the maximum and minimum of the absolute value of the extremes and their difference are given. The absolute values of the maximum and the minimum for the theoretical approximation should be identical and the difference should be 0. We did not get the polynomial of the best approximation. Further iterations did not bring any new information in the frame of the chosen accuracy.

The examples were calculated by the mathematical software MATLAB. The examples demonstrate a very fast convergence of the Remez algorithm and very fast increase of the accuracy with increase of the degree of the polynomial.

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