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A COMPUTATIONAL COMPARISON OF METHODS DIMINISHING SPURIOUS OSCILLATIONS IN FINITE ELEMENT SOLUTIONS OF CONVECTION–DIFFUSION EQUATIONS*

Volker John, Petr Knobloch

Abstract

This paper presents a review and a computational comparison of various stabilization techniques developed to diminish spurious oscillations in finite element solutions of scalar stationary convection–diffusion equations. All these methods are defined by enriching the popular SUPG discretization by additional stabilization terms. Although some of the methods can substantially enhance the quality of the discrete solutions in comparison to the SUPG method, any of the methods can fail in very simple situations and hence none of the methods can be regarded as reliable. We also present results obtained using the improved Mizukami–Hughes method which is often superior to techniques based on the SUPG method.

1. Introduction

During the past three decades, much effort has been devoted to the numerical solution of the scalar convection–diffusion equation

$$-\varepsilon\,\Delta u + \boldsymbol{b}\cdot\nabla u = f \quad \text{in } \Omega, \qquad\qquad u = u_b \quad \text{on } \partial\Omega. \tag{1}$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded domain with a polygonal boundary $\partial\Omega$, $\varepsilon > 0$ is the constant diffusivity, $\boldsymbol{b} \in W^{1,\infty}(\Omega)^2$ is a given convective field, $f \in L^2(\Omega)$ is an outer force, and $u_b \in H^{1/2}(\partial\Omega)$ represents the Dirichlet boundary condition. In our numerical tests also less regular boundary conditions are considered.

Problem (1) describes the stationary distribution of a physical quantity u (e.g., temperature or concentration) determined by two basic physical mechanisms, namely the convection and diffusion. The broad interest in solving problem (1) is also caused by the fact that it is a simple model problem for convection–diffusion effects which appear in many more complicated problems arising in applications, e.g., in convection–dominated incompressible fluid flow problems which are described by the Navier–Stokes equations. Despite the apparent simplicity of problem (1), its numerical solution is by no means easy when convection is strongly dominant (i.e., when $\varepsilon \ll |\boldsymbol{b}|$). In this case, the solution of (1) typically possesses interior and boundary layers, which often leads to unwanted spurious (nonphysical) oscillations in the numerical solution.

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In this paper, we concentrate on the solution of (1) using the finite element method. The simplest finite element discretization of (1) is the classical Galerkin formulation which, in simple settings, is equivalent to a central finite difference discretization. Thus, it is not surprising that, in the convection dominated regime, the Galerkin solution is usually globally polluted by spurious oscillations and hence the Galerkin discretization is inappropriate.

To enhance the stability and accuracy of the Galerkin discretization of (1) in the convection dominated case, various stabilization strategies have been developed. The most popular stabilization technique within the framework of finite element discretizations of (1) is the streamline upwind/Petrov–Galerkin (SUPG) discretization proposed by Brooks and Hughes [2], see Section 2. It can be observed that the solutions obtained with the SUPG method possess often spurious oscillations in the vicinity of layers.

To diminish the oscillations of SUPG solutions, a large class of finite element methods has been constructed by adding yet additional stabilization terms to the SUPG discretization of (1). Usually, these terms depend on the element residuals of the discrete solution and therefore the resulting methods are consistent and hence higher–order accurate. We shall discuss such stabilization methods in Section 3. The stabilization terms introduce additional artificial diffusion and often depend on the unknown discrete solution in a nonlinear way. It is believed that, for a proper amount of artificial diffusion, we obtain a discrete solution which represents a good approximation of the solution of (1) and does not contain any spurious oscillations. Therefore, the design of suitable formulas specifying the artificial diffusion introduced by the stabilization terms was a subject of an extensive research during the past two decades.

The main aim of this paper is to present a computational comparison of the above–mentioned stabilization techniques by means of two standard test problems whose solutions possess characteristic features of solutions of (1). In addition, we shall introduce a new simple model problem of the type of (1) for which none of the above–mentioned stabilization methods gives a satisfactory discrete solution. This indicates the necessity to seek other ways of approximating the solution to the convection–diffusion equation (1). We also present results obtained using the improved Mizukami–Hughes method which is often superior to techniques based on the SUPG method and which gives good approximations to the solutions of all three test problems considered in this paper. In the whole paper we confine ourselves to conforming piecewise linear triangular finite elements.

The plan of the paper is as follows. In the next section, we formulate two discretizations of the problem (1): the Galerkin discretization and the SUPG method. In Section 3, we present a review of various additional stabilization terms added to the SUPG discretization to diminish spurious oscillations at layers. Also, we mention the improved Mizukami–Hughes method. Then the results of our numerical tests are reported in Section 4. Finally, the paper is closed by Section 5 containing our conclusions.

Throughout the paper, we use the standard notations $L^p(\Omega)$, $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$, etc. for the usual function spaces. The norm and seminorm in the Sobolev space $H^k(\Omega)$ will be denoted by $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. The inner product in the space $L^2(\Omega)$ or $L^2(\Omega)^2$ will be denoted by $(\cdot,\cdot)$. For a vector $\boldsymbol{a} \in \mathbb{R}^2$, we denote by $|\boldsymbol{a}|$ its Euclidean norm.

2. The Galerkin discretization of (1) and the SUPG method

To define a finite element discretization of (1), we introduce a triangulation $\mathcal{T}_h$ of the domain Ω consisting of a finite number of open triangular elements K possessing the usual compatibility properties. Using this triangulation, we define the finite element space

$$\mathrm{V}_h = \{v \in H_0^1(\Omega)\,;\; v|_K \in P_1(K) \quad \forall\, K \in \mathcal{T}_h\}\,,$$

where $P_1(K)$ is the space of linear functions on K. Further, we introduce a piecewise linear function $\widetilde{u}_{bh} \in H^1(\Omega)$ such that $\widetilde{u}_{bh}|_{\partial\Omega}$ approximates the boundary condition u_b. Then the usual Galerkin finite element discretization of the convection–diffusion equation (1) reads:

Find $u_h \in H^1(\Omega)$ such that $u_h - \widetilde{u}_{bh} \in \mathrm{V}_h$ and

$$a(u_h, v_h) = (f, v_h) \qquad \forall\, v_h \in \mathrm{V}_h\,,$$

where

$$a(u,v) = \varepsilon\,(\nabla u, \nabla v) + (\boldsymbol{b}\cdot\nabla u, v)\,.$$

Since the Galerkin method lacks stability if convection dominates diffusion, Brooks and Hughes [2] proposed to enrich it by a residual–based stabilization term yielding the streamline upwind/Petrov–Galerkin (SUPG) method:

Find $u_h \in H^1(\Omega)$ such that $u_h - \widetilde{u}_{bh} \in \mathrm{V}_h$ and

$$a(u_h, v_h) + (R(u_h), \tau\,\boldsymbol{b}\cdot\nabla v_h) = (f, v_h) \qquad \forall\, v_h \in \mathrm{V}_h\,, \tag{2}$$

where $\tau \in L^\infty(\Omega)$ is a nonnegative stabilization parameter and

$$R(u_h) = \boldsymbol{b}\cdot\nabla u_h - f$$

is the residual (note that $\Delta u_h = 0$ on any element of the triangulation).

A delicate problem is the choice of the stabilization parameter τ in (2). Theoretical investigations of the SUPG method (see, e.g., Roos *et al.* [20]) provide certain bounds for τ for which the SUPG method is stable and leads to (quasi–)optimal convergence of the discrete solution u_h. However, it has been reported many times that the choice of τ inside these bounds may dramatically influence the accuracy of the discrete solution. Therefore, over the last two decades, much research has also been devoted to the choice of τ and various strategies for the computation of τ have been proposed, see, e.g., the review in the recent paper by John and

Knobloch [14]. Let us stress that the definition of τ mostly relies on heuristic arguments and the 'best' way of choosing τ for general convection–diffusion problems is not known. Here we define τ on any element $K \in \mathcal{T}_h$ by the formula

$$\tau|_K \equiv \tau_K = \frac{h_K}{2\,|\boldsymbol{b}|}\,\xi(\mathrm{Pe}_K) \qquad \text{with} \qquad \mathrm{Pe}_K = \frac{|\boldsymbol{b}|\,h_K}{2\,\varepsilon}\,, \tag{3}$$

where h_K is the diameter of K in the direction of the convection $\boldsymbol{b}$, Pe_K is the local Péclet number and ξ is the so–called upwind function defined by $\xi(\alpha) = \coth\alpha - 1/\alpha$. If $\boldsymbol{b}|_K$ is not constant, then the parameters h_K, Pe_K and τ_K are generally functions of the points $x \in K$. The formula (3) is a generalization of an analogous one–dimensional formula which guarantees that, for the one–dimensional case of (1) with constant data, the SUPG solution with continuous piecewise linear finite elements on a uniform division of an interval Ω is nodally exact, c.f. Christie *et al.* [9].

3. A short review of stabilization methods based on the SUPG method

The SUPG method produces to a great extent accurate and oscillation–free solutions but it does not preclude spurious oscillations (overshooting and undershooting) localized in narrow regions along sharp layers. Although these nonphysical oscillations are usually small in magnitude, they are not permissible in many applications. An example are chemically reacting flows where it is essential to guarantee that the concentrations of all species are nonnegative. The small spurious oscillations may also deteriorate the solution of nonlinear problems, e.g., in two–equations turbulence models.

The oscillations along sharp layers are caused by the fact that the SUPG method is neither monotone nor monotonicity preserving (in contrast with the continuous problem (1)). Therefore, various terms introducing artificial crosswind diffusion in the neighborhood of layers have been proposed to be added to the SUPG formulation in order to obtain a method which is monotone, at least in some model cases, or which at least reduces the local oscillations. This procedure is referred to as discontinuity capturing or shock capturing. A detailed review of such methods was recently published by John and Knobloch [14].

Usually, the additional artificial diffusion is introduced by adding either the term

$$(\widetilde{\varepsilon}^{iso}\,\nabla u_h, \nabla v_h) \tag{4}$$

or the term

$$(\widetilde{\varepsilon}^{cd}\,D\,\nabla u_h, \nabla v_h) \qquad \text{with} \qquad D = \begin{cases} I - \dfrac{\boldsymbol{b}\otimes\boldsymbol{b}}{|\boldsymbol{b}|^2} & \text{if } \boldsymbol{b} \neq \boldsymbol{0}, \\ 0 & \text{if } \boldsymbol{b} = \boldsymbol{0} \end{cases} \tag{5}$$

to the left–hand side of (2). The former term introduces isotropic artificial diffusion whereas the latter one adds the artificial diffusion in the crosswind direction only (note that D is the projection onto the line orthogonal to $\boldsymbol{b}$, I being the identity tensor). A basic problem of all these methods is to find the proper amount of artificial diffusion which leads to sufficiently small nonphysical oscillations (requiring that artificial diffusion is not 'too small') and to a sufficiently high accuracy (requiring that artificial diffusion is not 'too large'). The derivation of formulas for $\widetilde{\varepsilon}^{iso}$ and $\widetilde{\varepsilon}^{cd}$ is typically based either on a convergence analysis or on investigations of the discrete maximum principle or (very often) on heuristic arguments. Usually, the parameter $\widetilde{\varepsilon}^{iso}$ or $\widetilde{\varepsilon}^{cd}$ depends on the unknown discrete solution u_h and hence the resulting method is nonlinear.

Many formulas for $\widetilde{\varepsilon}^{iso}$ rely on replacing the convection $\boldsymbol{b}$ in the SUPG weighting function by another upwind direction. This approach is used, e.g., in the methods of Hughes *et al.* [13], Tezduyar and Park [21], Galeão and do Carmo [12], do Carmo and Galeão [8] and in the modifications of these methods mentioned below. Let us mention at least the idea of Galeão and do Carmo [12]. They introduced an approximate streamline direction $\boldsymbol{b}_h$ for which the discrete solution u_h elementwise satisfies the equation (1) with $\boldsymbol{b}$ replaced by $\boldsymbol{b}_h$. Minimizing the difference between $\boldsymbol{b}$ and $\boldsymbol{b}_h$, they found that $\boldsymbol{b}_h = \boldsymbol{b} - \boldsymbol{z}_h$ with

$$\boldsymbol{z}_h = \frac{R(u_h)\,\nabla u_h}{|\nabla u_h|^2}\,.$$

(Here and in the following it is understood that, if $\nabla u_h = \mathbf{0}$ in the denominator, the respective expression is replaced by zero.) Finally, they replaced the function $\tau\,\boldsymbol{b}$ in (2) by $\tau\,\boldsymbol{b} + \sigma\,\boldsymbol{z}_h$ with a nonnegative parameter σ. That leads to the discretization (2) with the additional term (4) on the left–hand side, where

$$\widetilde{\varepsilon}^{iso} = \sigma\,\frac{|R(u_h)|^2}{|\nabla u_h|^2}\,. \tag{6}$$

Based on ideas of Hughes *et al.* [13], Galeão and do Carmo [12] defined the parameter σ by

$$\sigma = \max\{0, \tau(\boldsymbol{z}_h) - \tau(\boldsymbol{b})\}\,. \tag{7}$$

The notation of the type $\tau(\boldsymbol{b}^\star)$ denotes a value computed using the formula (3) with $\boldsymbol{b}$ replaced by some function $\boldsymbol{b}^\star$. Note that $\boldsymbol{b}^\star$ influences the value of $\tau_K(\boldsymbol{b}^\star)$ also through the definition of h_K.

Do Carmo and Galeão [8] proposed to simplify (7) to

$$\sigma = \tau(\boldsymbol{b})\,\max\left\{0, \frac{|\boldsymbol{b}|}{|\boldsymbol{z}_h|} - 1\right\}\,, \tag{8}$$

which assures that the term (4) is added only if the above–introduced vector $\boldsymbol{b}_h$ satisfies the natural requirement $\boldsymbol{b}\cdot\boldsymbol{b}_h > 0$. It may also be advantageous to set

$$\sigma = \tau(\boldsymbol{b})\,\max\left\{0, \frac{|\boldsymbol{b}|}{|\boldsymbol{z}_h|} - \zeta_h\right\} \quad \text{with} \quad \zeta_h = \max\left\{1, \frac{\boldsymbol{b}\cdot\nabla u_h}{R(u_h)}\right\}\,, \tag{9}$$

which was proposed by Almeida and Silva [1]. Further variants of this approach were developed by do Carmo and Galeão [8] and do Carmo and Alvarez [6] who proposed techniques which should suppress the addition of the artificial diffusion in regions where the solution of (1) is smooth. A finer tuning of the stabilization parameters was introduced by do Carmo and Alvarez [7]. Let us also mention that, motivated by assumptions of a rather general error analysis, Knopp *et al.* [18] suggested to replace (6), on any element $K \in \mathcal{T}_h$, by

$$\widetilde{\varepsilon}^{iso}|_K = \sigma\,|Q_K(u_h)|^2 \qquad \text{with} \qquad Q_K(u_h) = \frac{\|R(u_h)\|_{0,K}}{S_K + \|u_h\|_{1,K}}\,, \tag{10}$$

where S_K are appropriate positive constants. The stabilization term (4) was also used by Johnson [15] who proposed to set

$$\widetilde{\varepsilon}^{iso}|_K = \max\{0, \alpha\,[\mathrm{diam}(K)]^\nu\,|R(u_h)| - \varepsilon\} \qquad \forall\, K \in \mathcal{T}_h \tag{11}$$

with some constants α and $\nu \in (3/2, 2)$. He suggested to take $\nu \sim 2$.

The crosswind artificial diffusion term (5) was first considered by Johnson *et al.* [16]. A straightforward generalization of their approach leads to

$$\widetilde{\varepsilon}^{cd}|_K = \max\{0, |\boldsymbol{b}|\,h_K^{3/2} - \varepsilon\} \qquad \forall\, K \in \mathcal{T}_h\,. \tag{12}$$

The value $h_K^{3/2}$ was motivated by a careful analysis of the numerical crosswind spread in the discrete problem, i.e., of the maximal distance in which the right–hand side f significantly influences the discrete solution. The resulting method is linear but non–consistent and hence it is restricted to finite elements of first order of accuracy.

Codina [10] proposed to define $\widetilde{\varepsilon}^{cd}$, for any $K \in \mathcal{T}_h$, by

$$\widetilde{\varepsilon}^{cd}|_K = \frac{1}{2}\,\max\left\{0, C - \frac{2\,\varepsilon\,|\nabla u_h|}{\mathrm{diam}(K)\,|\boldsymbol{b}\cdot\nabla u_h|}\right\}\,\mathrm{diam}(K)\,\frac{|R(u_h)|}{|\nabla u_h|}\,, \tag{13}$$

where C is a suitable constant, and he recommended to set $C \approx 0.7$ for (bi)linear finite elements. To improve the properties of the resulting method for $f \neq 0$, John and Knobloch [14] replaced (13) by

$$\widetilde{\varepsilon}^{cd}|_K = \frac{1}{2}\,\max\left\{0, C - \frac{2\,\varepsilon\,|\nabla u_h|}{\mathrm{diam}(K)\,|R(u_h)|}\right\}\,\mathrm{diam}(K)\,\frac{|R(u_h)|}{|\nabla u_h|}\,. \tag{14}$$

If $f = 0$, it is equivalent to the original method (13). Finally, Knopp *et al.* [18] proposed to use (5) with $\widetilde{\varepsilon}^{cd}$ defined, for any $K \in \mathcal{T}_h$, by

$$\widetilde{\varepsilon}^{cd}|_K = \frac{1}{2}\,\max\left\{0, C - \frac{2\,\varepsilon}{Q_K(u_h)\,\mathrm{diam}(K)}\right\}\,\mathrm{diam}(K)\,Q_K(u_h)\,, \tag{15}$$

which leads to a method having properties convenient for theoretical investigations. The definition of $Q_K(u_h)$ is the same as in (10).

It is also possible to add both isotropic and crosswind artificial diffusion terms to the left–hand side of (2) as proposed by Codina and Soto [11]. They set

$$\widetilde{\varepsilon}^{iso} = \max\{0, \varepsilon_{\rm dc} - \tau(\boldsymbol{b})\,|\boldsymbol{b}|^2\}\,, \qquad \widetilde{\varepsilon}^{cd} = \varepsilon_{\rm dc} - \widetilde{\varepsilon}^{iso}\,,$$

where $\varepsilon_{\rm dc}$ is defined by the formula (14). However, for the test examples considered in the present paper, this method gave very similar results as (5) with $\widetilde{\varepsilon}^{cd}$ given by (14) and hence we shall not consider it in the following.

For triangulations consisting of weakly acute triangles, Burman and Ern [3] proposed to use (5) with $\widetilde{\varepsilon}^{cd}$ defined, on any $K \in \mathcal{T}_h$, by

$$\widetilde{\varepsilon}^{cd}|_K = \frac{\tau(\boldsymbol{b})\,|\boldsymbol{b}|^2\,|R(u_h)|}{|\boldsymbol{b}|\,|\nabla u_h| + |R(u_h)|}\,\frac{|\boldsymbol{b}|\,|\nabla u_h| + |R(u_h)| + \tan\alpha_K\,|\boldsymbol{b}|\,|D\,\nabla u_h|}{|R(u_h)| + \tan\alpha_K\,|\boldsymbol{b}|\,|D\,\nabla u_h|} \tag{16}$$

($\widetilde{\varepsilon}^{cd} = 0$ if one of the denominators vanishes). The parameter α_K is equal to $\pi/2 - \beta_K$ where β_K is the largest angle of K. If $\beta_K = \pi/2$, it is recommended to set $\alpha_K = \pi/6$. To improve the convergence of the nonlinear iterations, we replace $|R(u_h)|$ by $|R(u_h)|_{\rm reg}$ with $|x|_{\rm reg} \equiv x\,\tanh(x/2)$. Based on numerical experiments, John and Knobloch [14] simplified (16) to

$$\widetilde{\varepsilon}^{cd}|_K = \frac{\tau(\boldsymbol{b})\,|\boldsymbol{b}|^2\,|R(u_h)|}{|\boldsymbol{b}|\,|\nabla u_h| + |R(u_h)|}\,. \tag{17}$$

In this case, no regularization of the absolute values is applied.

Another stabilization strategy for linear simplicial finite elements was introduced by Burman and Hansbo [5]. The term to be added to the left–hand side of (2) is defined by

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \Psi_K(u_h)\,\mathrm{sign}(\frac{\partial u_h}{\partial \boldsymbol{t}_{\partial K}})\,\frac{\partial v_h}{\partial \boldsymbol{t}_{\partial K}}\,\mathrm{d}\sigma\,, \tag{18}$$

where $\boldsymbol{t}_{\partial K}$ is a tangent vector to the boundary ∂K of K,

$$\Psi_K(u_h) = \mathrm{diam}(K)\,(C_1\,\varepsilon + C_2\,\mathrm{diam}(K))\,\max_{E\subset\partial K}|\,[|\boldsymbol{n}_E\cdot\nabla u_h|]_E\,|\,, \tag{19}$$

$\boldsymbol{n}_E$ are normal vectors to edges E of K, $[|v|]_E$ denotes the jump of a function v across the edge E and C_1, C_2 are appropriate constants. Further, Burman and Ern [4] proposed to use (18) with $\Psi_K(u_h)$ defined by

$$\Psi_K(u_h)|_E = C\,|\boldsymbol{b}|\,[\mathrm{diam}(K)]^2\,|\,[|\nabla u_h|]_E\,| \qquad \forall\ E\subset\partial K \tag{20}$$

or by

$$\Psi_K(u_h) = C\,|R(u_h)|\,, \tag{21}$$

where C is a suitable constant.

Finally, let us mention the improved Mizukami–Hughes method, originally introduced by Mizukami and Hughes [19] and recently improved by Knobloch [17]. It is a method of another type than the methods presented in this section since its derivation does not start from the SUPG discretization. However, like the SUPG method, it is a Petrov–Galerkin method. The weighting functions generally depend on the unknown discrete solution and hence the method is nonlinear. The advantage of the Mizukami–Hughes method is that the discrete solution always satisfies the discrete maximum principle and is usually rather accurate. Drawbacks of the method are that it is defined for conforming linear triangular finite elements only and that it is not clear how to generalize the Mizukami–Hughes method to more complicated convection–diffusion problems than presented in this paper.

4. Numerical results

In this section, we shall present numerical results obtained using the methods from Sections 2 and 3 for the following three test problems:

Example 1. Solution with parabolic and exponential boundary layers. We consider the convection–diffusion equation (1) in $\Omega = (0,1)^2$ with $\varepsilon = 10^{-8}$, $\boldsymbol{b} = (1,0)^T$, $f = 1$ and $u_b = 0$. The solution $u(x,y)$ of this problem, see Figure 1, possesses an exponential boundary layer at $x = 1$ and parabolic boundary layers at $y = 0$ and $y = 1$. In the interior grid points, the solution $u(x,y)$ is very close to x. This example has been used, e.g., in [19].

Example 2. Solution with interior layer and exponential boundary layers. We consider the convection–diffusion equation (1) in $\Omega = (0,1)^2$ with $\varepsilon = 10^{-8}$, $\boldsymbol{b} = (\cos(-\pi/3), \sin(-\pi/3))^T$, $f = 0$ and

$$u_b(x,y) = \begin{cases} 0 & \text{for } x = 1 \text{ or } y \le 0.7, \\ 1 & \text{else.} \end{cases}$$

The solution, see Figure 3, possesses an interior layer in the direction of the convection starting at $(0, 0.7)$. On the boundary $x = 1$ and on the right part of the boundary $y = 0$, exponential layers are developed. This example has been used, e.g., in [13].

Example 3. Solution with two interior layers. We consider the convection–diffusion equation (1) in $\Omega = (0,1)^2$ with $\varepsilon = 10^{-8}$, $\boldsymbol{b} = (1,0)^T$, $u_b = 0$ and

$$f(x,y) = \begin{cases} 16\,(1 - 2\,x) & \text{for } (x,y) \in [0.25, 0.75]^2, \\ 0 & \text{else.} \end{cases}$$

The solution, see Figure 5, possesses two interior layers layer at $(0.25, 0.75) \times \{0.25\}$ and $(0.25, 0.75) \times \{0.75\}$. In $(0.25, 0.75)^2$, the solution $u(x,y)$ is very close to the quadratic function $(4\,x - 1)(3 - 4\,x)$. This example has not been published before.

All the numerical results discussed in this section were computed on uniform triangulations $\mathcal{T}_h$ of Ω of the type depicted in Fig. 7, which consist of $2(N \times N)$ equal right–angled isosceles triangles ($N = 5$ in Fig. 7). We used either $N = 20$ or $N = 64$. Figs. 2, 4 and 6 show the SUPG solutions for Examples 1, 2 and 3, respectively. Although the formula (3) for the stabilization parameter τ can be regarded as optimal in all three cases, we observe significant spurious oscillations in layer regions.

Denoting

$$\Omega_1 = \{(x,y) \in \Omega\,;\ x \leq 0.5,\, y \geq 0.1\}\,, \qquad \Omega_2 = \{(x,y) \in \Omega\,;\ x \geq 0.7\}\,,$$

we introduce the following measures of oscillations in the discrete solutions u_h of Examples 1 and 2:

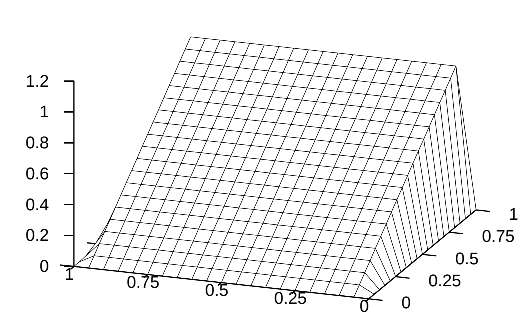

Fig. 1: *Example 1, solution u.*

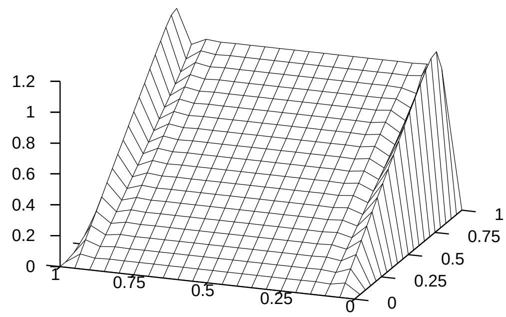

Fig. 2: *Example 1, SUPG,* $N = 20$.

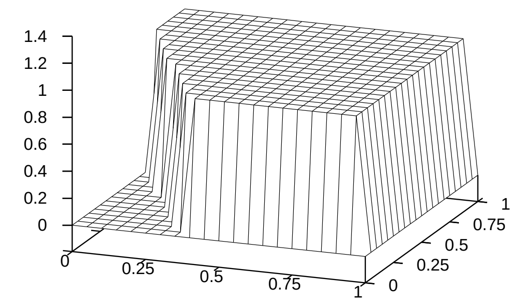

Fig. 3: *Example 2, solution u.*

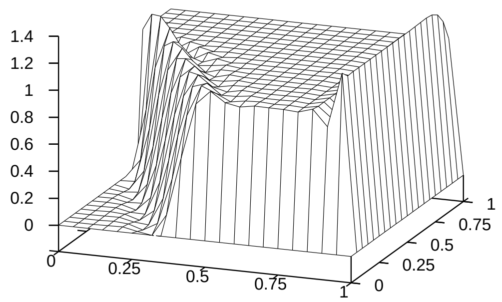

Fig. 4: *Example 2, SUPG,* $N = 20$.

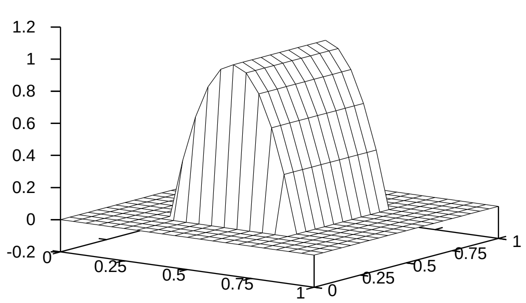

Fig. 5: *Example 3, solution u.*

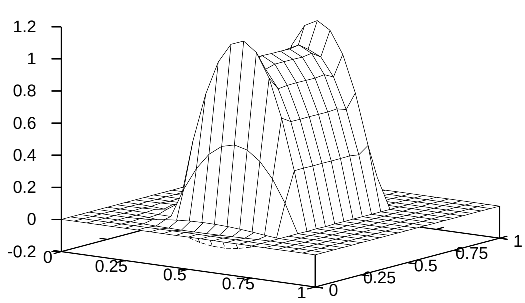

Fig. 6: *Example 3, SUPG,* $N = 20$.

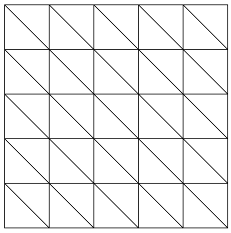

Fig. 7: *Considered type of triangulations.*

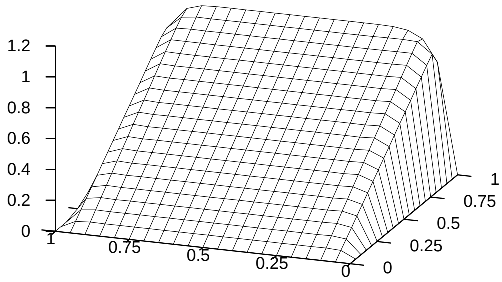

Fig. 8: *Example 1, Cmod,C*$=0.6$,*N*$=20$.

$$osc_{\rm para} := 10 \max_{y\in[0,1]} \{u_h(0.5,y) - u_h(0.5,0.5)\}\,,$$

$$osc_{\rm int} := \left(\sum_{(x,y)\in\Omega_1} (\min\{0,u_h(x,y)\})^2 + (\max\{0,u_h(x,y)-1\})^2\right)^{1/2}\,,$$

$$osc_{\rm exp} := \left(\sum_{(x,y)\in\Omega_2} (\max\{0,u_h(x,y)-1\})^2\right)^{1/2}\,.$$

The measure $osc_{\rm para}$ characterizes the oscillations of u_h in the parabolic boundary layer regions of Example 1 whereas $osc_{\rm int}$ and $osc_{\rm exp}$ measure the oscillations of u_h in the interior and exponential layer regions of Example 2. The summations are performed over the nodes (x,y) of the mesh. Fig. 9 shows the values of these measures for most of the methods discussed in the previous section and we see that there are significant differences between the size of the oscillations. For the six best methods the results are also shown in Fig. 10 which suggests that the best methods are the improved Mizukami–Hughes method and the crosswind artificial diffusion method defined by (5) with (12).

However, a suppression of oscillations does not imply that the respective discrete solution u_h is a good approximation of u since the layers can be smeared considerably. Therefore, we also define the following measures:

$$smear_{\rm para} := \max_{y\in[1/N,1-1/N]} \{u_h(0.5,0.5) - u_h(0.5,y)\}\,, \qquad smear_{\rm int} := x_2 - x_1\,,$$

$$smear_{\rm exp} := \frac{1}{10}\left(\sum_{(x,y)\in\Omega_2} (\min\{0,u_h(x,y)-1\})^2\right)^{1/2}\,.$$

The measure $smear_{\rm para}$ characterizes the smearing of the parabolic boundary layer in Example 1 whereas $smear_{\rm int}$ and $smear_{\rm exp}$ measure the smearing of the interior and exponential layers in Example 2. In the definition of $smear_{\rm int}$, the value x_1 is the x–coordinate of the first point on the cut line $(x,0.25)$ with $u_h(x_1,0.25)\geq 0.1$ and x_2

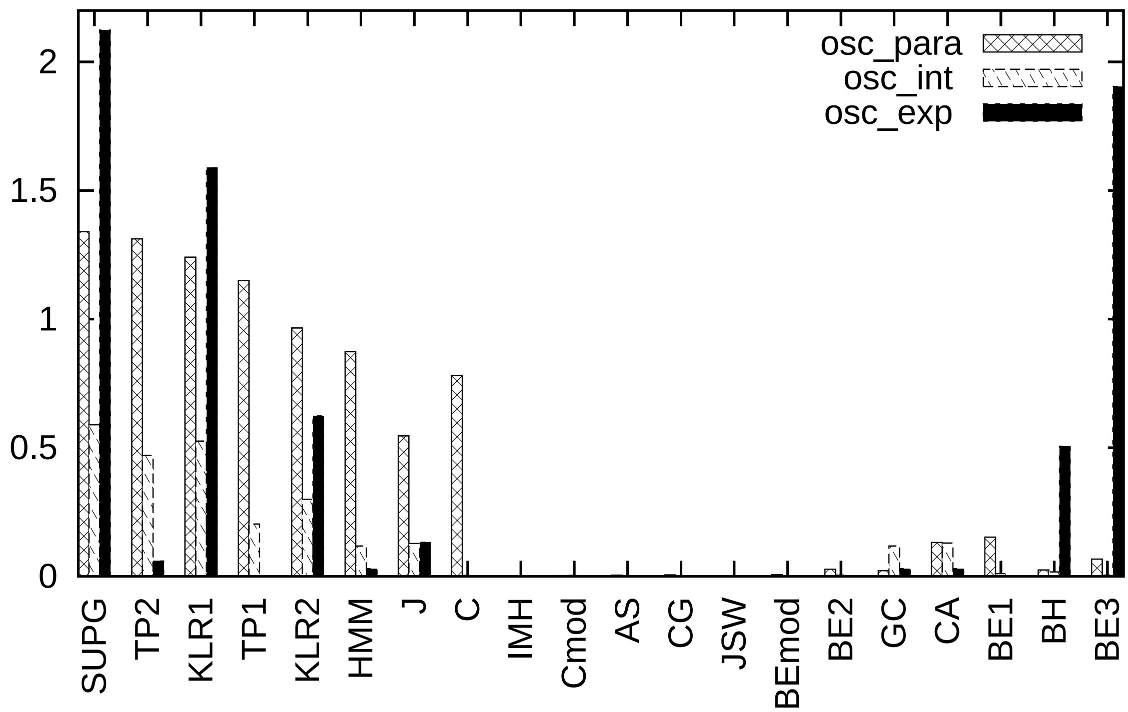

Fig. 9: *Measures of oscillations in discrete solutions of Examples 1 and 2 for* $N = 64$ *and various discretizations. Methods adding isotropic artificial diffusion (4): TP1, TP2 - [21], KLR1 - (10), (7) with* $S_K = 1$, *HMM - [13], J - (11) with* $\alpha = 0.3$ *and* $\nu = 2$, *AS - (6), (9), CG - (6), (8), GC - (6), (7), CA - [6]. Methods adding crosswind artificial diffusion (5): KLR2 - (15) with* $C = 0.6$ *and* $S_K = 1$, *C - (13) with* $C = 0.6$, *Cmod - (14) with* $C = 0.6$, *JSW - (12), BEmod - (17), BE1 - (16). Edge stabilizations (18): BE2 - (21) with* $C = 5 \cdot 10^{-5}$, *BH - (19) with* $C_1 = 0.5$ *and* $C_2 = 0.01$, *BE3 - (20) with* $C = 0.05$. *IMH - improved Mizukami–Hughes method [17].*

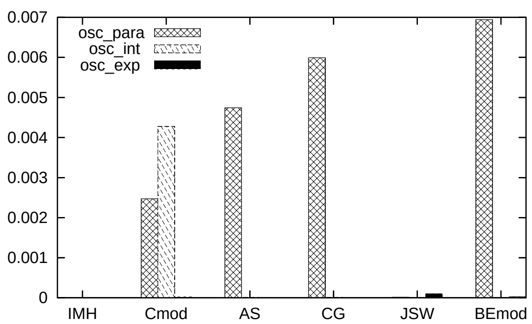

Fig. 10: *Measures of oscillations in discrete solutions of Examples 1 and 2 for IMH, Cmod, AS, CG, JSW and BEmod.*

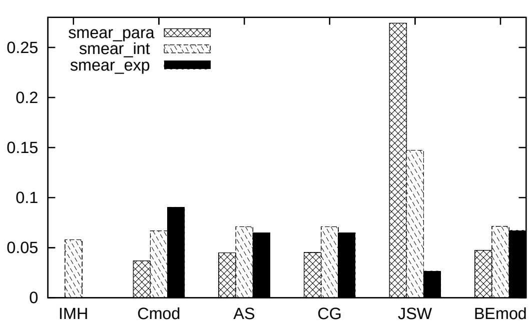

Fig. 11: *Measures of smearing in discrete solutions of Examples 1 and 2 for IMH, Cmod, AS, CG, JSW and BEmod.*

is the x–coordinate of the first point with $u_h(x_2, 0.25) \geq 0.9$. The summation is again performed over the nodes (x, y) of the mesh. The results in Fig. 11 show that the method JSW leads to a considerable smearing of the layers and that the improved Mizukami–Hughes methods does not smear boundary layers for Examples 1 and 2. The remaining four methods (Cmod, AS, CG and BEmod) seem to be comparable.

The above results and many other numerical tests we performed indicate that the best methods are the improved Mizukami–Hughes method [17], the isotropic artificial diffusion methods by do Carmo and Galeão [8] given by (4), (6), (8) and by Almeida and Silva [1] given by (4), (6), (9) and the modified crosswind artificial diffusion methods by Codina [10] given by (5), (14) and by Burman and Ern [3] given by (5), (17). For Example 1, the improved Mizukami–Hughes method gives a nodally exact discrete solution and the remaining four methods give comparable discrete solutions, one of which is depicted in Fig. 8. For Example 2, the discrete solutions obtained using the improved Mizukami–Hughes method and the method by do Carmo and Galeão [8] (denoted by CG) are depicted in Figs. 12 and 13, respectively. For the methods Cmod, AS and BEmod, the discrete solutions are similar as in Fig. 13. Thus, we see that the methods IMH, Cmod, AS, CG and BEmod are able to substantially improve the quality of the discrete solution in comparison to the SUPG method.

Unfortunately, this is not always the case. We observed that often also the methods Cmod, AS, CG and BEmod may produce results with spurious oscillations. This may also happen for Example 2 if a triangulation similar as in Fig. 7 but with different numbers of vertices in x– and y–directions is used. But also for a triangulation of the type from Fig. 7, the methods Cmod, AS, CG and BEmod may give a wrong solution. This is the case for Example 3 as Figs. 14 and 15 show. We see that the oscillations along the interior layers disappeared but the discrete solution is not correct in a region where it should vanish. We observed this phenomenon for all the SUPG based methods discussed in Section 3. Fig. 16 shows an approximation of u obtained using the improved Mizukami–Hughes method which is much better than for the other four methods, however not perfect. Moreover, in contrast with these methods, the IMH solution improves if the mesh is refined.

Let us demonstrate that the phenomenon shown in Figs. 14 and 15 has to be

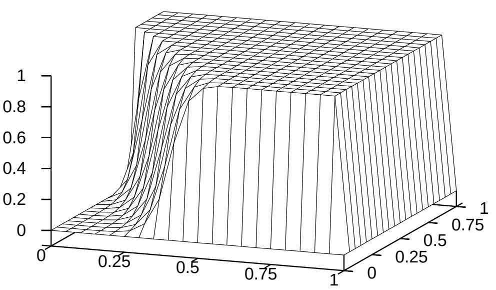

Fig. 12: *Example 2, IMH,* $N = 20$.

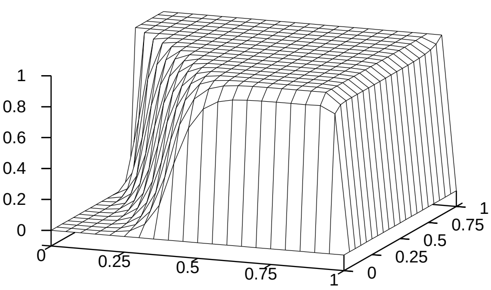

Fig. 13: *Example 2, CG,* $N = 20$.

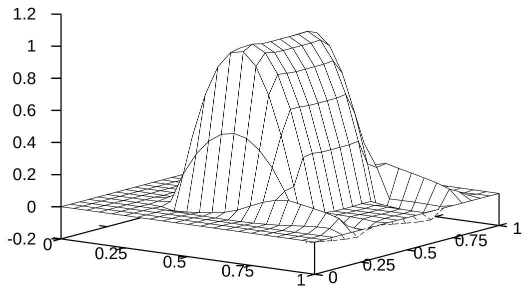

Fig. 14: *Example 3, CG, $N = 20$.*

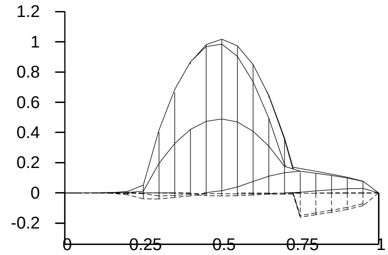

Fig. 15: *As in Fig. 14 (other view).*

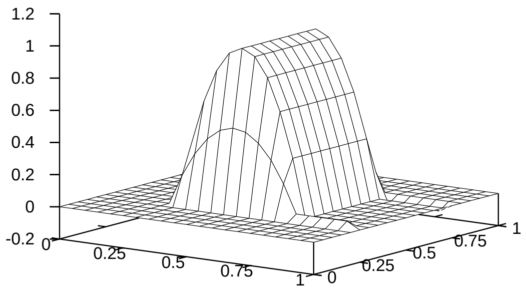

Fig. 16: *Example 3, IMH, $N = 20$.*

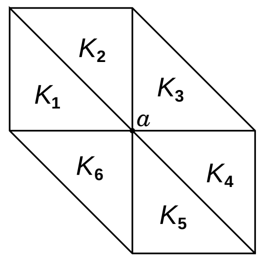

Fig. 17: *Support of a basis function.*

expected if the discrete solution should suppress the spurious oscillations present in the SUPG solution. Thus, let us assume that a solution of the discrete problem obtained by adding the term (4) or (5) to the left–hand side of (2) does not contain spurious oscillations and does not smear the inner layers significantly. Then, on any vertical mesh line intersecting the interval $(0.5, 0.75)$ on the x–axis, we may find a vertex $a \in (0.5, 0.75) \times (0, 0.25)$ surrounded by elements $K_1, \dots, K_6$ as depicted in Fig. 17 such that $\nabla u_h \approx \mathbf{0}$ on $K_4 \cup K_5 \cup K_6$ but $\partial u_h / \partial y$ is positive and nonnegligible on K_2. The elements $K_1, \dots, K_6$ make up the support of the standard piecewise linear basis function equal 1 at a. Using this basis function as v_h in the discrete problem and denoting by $\widetilde{\varepsilon}$ the artificial diffusion in (4) or (5), it is easy to show that

$$\frac{\partial u_h}{\partial x}\Big|_{K_2} \approx \frac{3}{h} \, \frac{\partial u_h}{\partial y}\Big|_{K_2} (2\,\varepsilon + \widetilde{\varepsilon}|_{K_2} + \widetilde{\varepsilon}|_{K_3}) \,,$$

where h denotes the length of a leg of K_2. This means that, on some elements in the region $(0.5, 0.75) \times (0, 0.25)$, the discrete solution has to grow in the x–direction, which is exactly what is observed in Figs. 14 and 15. A deeper explanation of this phenomenon will be a subject of our future research.

5. Conclusions

In this paper we presented a review and a computational comparison of various stabilization techniques based on the SUPG method which have been developed to diminish spurious oscillations in finite element solutions of scalar stationary convection–diffusion equations. We identified the best methods and demonstrated that they are able to substantially enhance the quality of the discrete solutions in comparison to the SUPG method. However, we have also shown that these methods can fail for very simple test problems. An alternative to these methods seems to be the improved Mizukami–Hughes method which gives good results in all the cases considered.

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