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SMOOTH APPROXIMATION OF DATA WITH APPLICATIONS TO INTERPOLATING AND SMOOTHING

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Abstract

In the paper, we are concerned with some computational aspects of smooth approximation of data. This approach to approximation employs a (possibly infinite) linear combinations of smooth functions with coefficients obtained as the solution of a variational problem, where constraints represent the conditions of interpolating or smoothing. Some 1D numerical examples are presented.

1. Introduction

Smooth approximation [2] is an approach to data interpolation that employs the variational formulation of the problem in an inner product space, where constraints represent the interpolation conditions. A possible criterion is to minimize the integral of the squared magnitude of the interpolating function. A more sophisticated criterion is then to minimize, with some weights chosen, the integrals of the squared magnitude of some (or possibly all) derivatives of the interpolating function. We are thus concerned with the exact interpolation of the data at nodes and, at the same time, with the smoothness of the interpolating curve and its derivatives.

Smooth approximation has numerous applications as measurements of the values of a continuous function of one, two, or three independent variables are carried out in many branches of science and technology. We always get a finite number of function values measured at a finite number of points but we are interested also in intermediate values corresponding to other points. Apparently, except for the fixed constraints to be satisfied, the formulation of the problem of smooth approximation can vary and give the resulting interpolant of different smoothness. The cubic spline interpolation is known to be the approximation of this kind.

We confine ourselves to the case of 1D independent variable. We introduce the proper inner product space in Section 2. We formulate the problem and present the existence and uniqueness theorem in Section 3. In the next section, we show results of numerical experiments comparing the classical interpolation formulae and various basis systems for the smooth approximation. We finally sum up the results presented that show some properties of smooth approximation.

A paper containing all proofs has been prepared for a numerical analysis journal.

2. Notation

Let us consider the linear vector space $\widetilde{W}$ of complex functions g continuous together with their derivatives of all orders on the interval (a, b), which may be infinite. Let $\{B_l\}_{l=0}^{\infty}$ be a sequence of nonnegative numbers and let there be the smallest nonnegative integer L such that $B_L > 0$ while $B_l = 0$ for all $l < L$. For $g, h \in \widetilde{W}$ we construct the expression

$$(g, h)_L = \sum_{l=L}^{\infty} B_l \int_a^b g^{(l)}(x)[h^{(l)}(x)]^* \, \mathrm{d}x, \tag{1}$$

where * denotes complex conjugation. Let us further put

$$|g|_L^2 = \sum_{l=L}^{\infty} B_l \int_a^b |g^{(l)}(x)|^2 \, \mathrm{d}x. \tag{2}$$

If $B_0 > 0$ (i.e. $L = 0$) the expression $|g|_0 = \|g\|$ is the *norm* and $(g, h)_0 = (g, h)$ the *inner product*, and the set of all such functions forms a Hilbert space W corresponding to the sequence $\{B_l\}$.

If $L > 0$ then $|g|_L$ is a *seminorm* on W. We construct the quotient space W/P_{L-1} where the subspace $P_{L-1} \subset W$ is the space of polynomials of degree at most $L - 1$. Then $|g|_L$ is the norm and $(g, h)_L$ the inner product on the quotient space W/P_{L-1} of equivalence classes. The choice of the sequence $\{B_l\}$ defines weights of the individual derivatives in the expression (2) and guarantees the convergence of the series (2) as well.

Let us introduce some more notation to be able to formulate the problem of smooth approximation. Let us choose a system of functions $\{g_k\} \subset W$, $k = 1, 2, \ldots$, that is complete and orthogonal (with respect to the inner product (1)), i.e.,

$$(g_k, g_m)_L = 0 \quad \text{for} \quad k \neq m, \qquad (g_k, g_k)_L = |g_k|_L^2 > 0.$$

3. Problem of smooth interpolation

Let us have N (complex, in general) measured (sampled) function values f_1, f_2, $\ldots$, $f_N \in C$ measured at N mutually distinct nodes $X_1, X_2, \ldots, X_N \in R^n$. We are interested also in the intermediate values corresponding to other points. Assume that these $f_j = f(X_j)$ are measured values of some continuous function f while z is an approximating function to be constructed. We put $n = 1$ in what follows.

If $L > 0$ we construct the set $\{\varphi_p\}$, $p = 1, \ldots, L$, of mutually orthogonal complex functions from W such that

$$(\varphi_p, \varphi_q)_L = 0 \quad \text{for} \quad p, q = 1, \ldots, L. \tag{3}$$

This implies $|\varphi_p|_L = 0$. Moreover, assume

$$(\varphi_p, g_k)_L = 0 \quad \text{for} \quad p = 1, \ldots, L, \quad k = 1, 2, \ldots. \tag{4}$$

The natural choice is $\varphi_p(x) = x^{p-1}$, $p = 1, \dots, L$. The relations (3) and (4) are then satisfied. The set $\{\varphi_p\}$ is empty for $L = 0$.

Put

$$z(x) = \sum_{k=1}^{\infty} A_k g_k(x) + t(x), \quad t(x) = \sum_{p=1}^{L} a_p \varphi_p(x). \tag{5}$$

Problem of smooth interpolation. Let us fix nonnegative integers L and N of the above properties. The problem of smooth interpolation of a continuous function f given by its N values $f_j = f(X_j)$ is to find the coefficients a_p and A_k of the expressions (5) such that

$$z(X_j) = f_j, \quad j = 1, \dots, N, \tag{6}$$

and

$$\text{the quantity } \ |z|_L^2 \ \text{ attains its minimum.} \tag{7}$$

The smooth interpolation problem thus consists of the variational problem (7), i.e. minimizing the functional $|z|_L^2$, with constraints (6).

Note that when minimizing $\|z\|^2$, we minimize not only the $L^2(a,b)$ norm of z but also (with a weight B_1 chosen) the $L^2(a,b)$ norm of z', i.e. of the first derivative of z. This can be of importance in processing of such measured data where also a good approximation of the first derivative is needed.

Put

$$R_L(x,y) = \sum_{k=1}^{\infty} \frac{g_k(x) g_k^*(y)}{|g_k|_L^2}. \tag{8}$$

If $L > 0$, introduce the rectangular $N \times L$ matrix Φ with entries $\Phi_{jp} = \varphi_p(X_j)$, $j = 1, \dots, N$, $p = 1, \dots, L$. Now we can formulate the following theorem.

Theorem 1. *Let $X_i \neq X_j$ for all $i \neq j$. Assume that the series* (8) *converges for all $x, y \in (a,b)$. Moreover, let* $\operatorname{rank} \Phi = L$. *Then the problem* (5), (6), *and* (7) *of smooth interpolation has the unique solution*

$$z(x) = \sum_{j=1}^{N} \lambda_j R_L(x, X_j) + \sum_{p=1}^{L} a_p \varphi_p(x),$$

where the coefficients λ_j, $j = 1, \dots, N$, and a_p, $p = 1, \dots, L$, are the unique solution of the linear algebraic system

$$\begin{aligned} \sum_{j=1}^{N} \lambda_j R_L(X_i, X_j) + \sum_{p=1}^{L} a_p \varphi_p(X_i) &= f_i, \quad i = 1, \dots, N, \\ \sum_{j=1}^{N} \lambda_j \varphi_p^*(X_j) \qquad\qquad\qquad\quad &= 0, \quad p = 1, \dots, L. \end{aligned}$$

Proof. The proof is based on the method of Lagrange multipliers for constrained minimization.

4. Numerical examples

We have used three systems $\{g_k\}$ defined in different spaces W with different sequences $\{B_l\}$, cf. [1], [2]. It is $B_l = (\frac{1}{3})^{2l}/(2l)!$, $l = 0, 1, \ldots$, for I and II.

I dashed line The system of transformed complex exponential functions $\exp(\mathrm{i}kx)$, $L = 0$, and the function $R_0(x, y)$ analytically known.

II dotted line The system of monomials x^k orthonormalized numerically on $(-1, 1)$ by the Gram-Schmidt procedure. The function $R_0(x, y)$ computed in double precision by summation until the module of the increment is less than 10^{-12} but at most 40 terms are considered.

III dashed line The same transformed complex exponential functions like in I. $B_l = 0$ for all l except for $B_2 = 1$, i.e. the L^2 norm of z'' is minimized. $R_2(x, y) = |y - x|^3$, $t(x) = a_0 + a_1 x$. This is the well-known *cubic spline interpolation.*

Moreover, we computed the results of

IV dotted line Polynomial interpolation.

V dash-dot line Rational interpolation.

Solid line shows the true solution, i.e. the function f given. We tried two of them, the smooth even function

$$f(x) = \frac{1}{1 + 16x^2} \tag{9}$$

with its maximum at $x = 0$ and the function

$$f(x) = 3(x + 1)^2 + \ln((\tfrac{1}{10}x)^2 + 10^{-5}) + 1 \tag{10}$$

with "almost a singularity" at $x = 0$. The grid is equidistant. Very "wavy" interpolants obtained are not shown.

Numerical experiments performed to present the properties of smooth interpolation show that it is an efficient method.

We were concerned only with the problem of smooth *exact interpolation of function values* at nodes which is controlled by the constraints (6) and, in addition, by the minimum condition (7). Moreover, the smooth approximation approach can be employed also in the *exact Hermite interpolation* and in the *smoothing of data.* The 2D case is much more interesting and makes many important applications possible.

Acknowledgement

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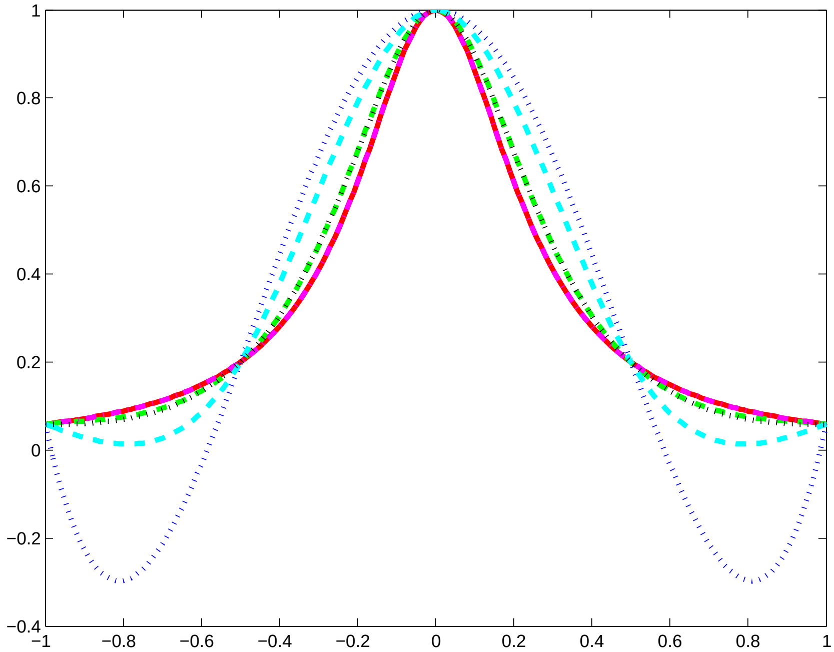

Figure 1: Interpolants of the function (9), $N = 5$. Curves at $x = 0.2$ from top to bottom: IV, III, I identical to II, true identical to V.

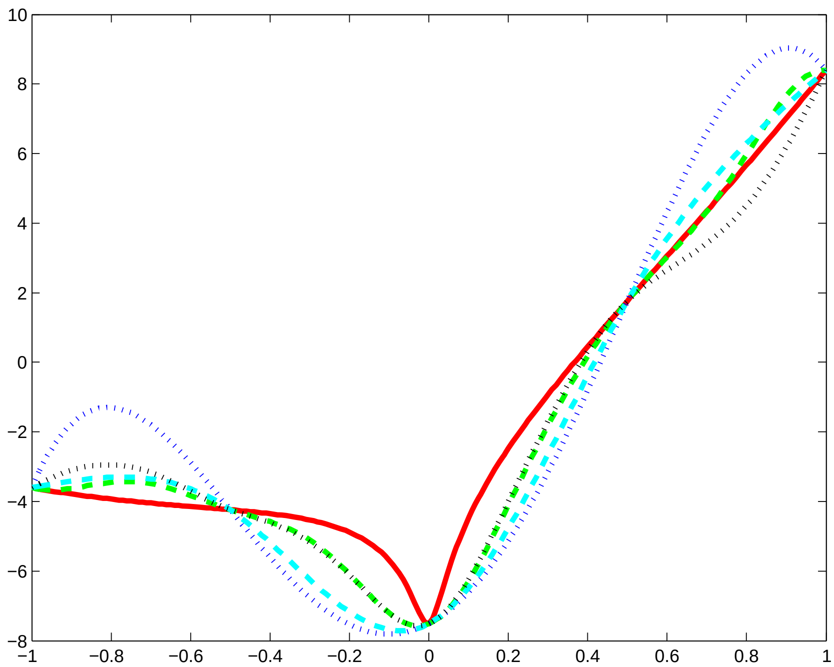

Figure 2: Interpolants of the function (10), $N = 5$. Curves at $x = -0.8$ from top to bottom: IV, II, III, I, true. V not shown.

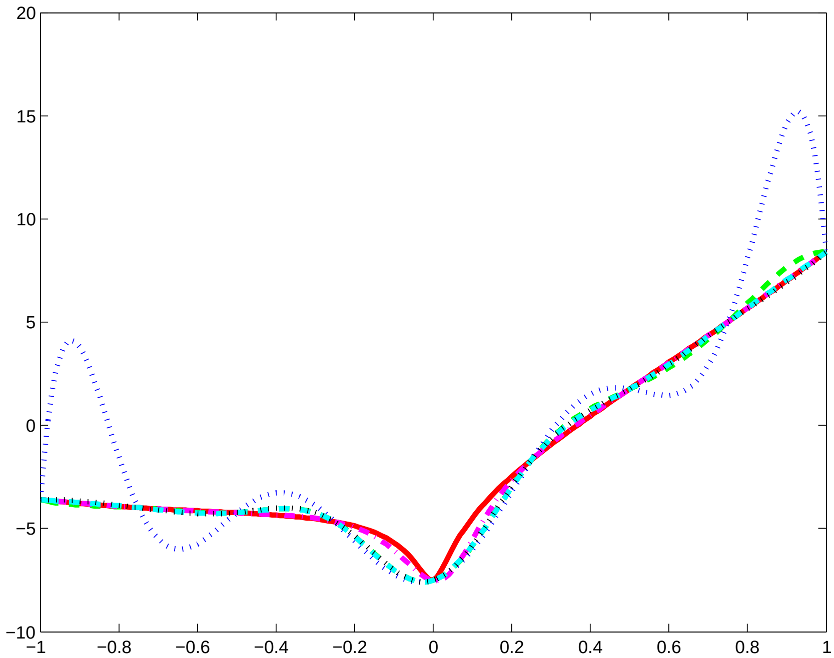

Figure 3: Interpolants of the function (10), $N = 9$. Curves at $x = 0.9$ from top to bottom: IV, I, II identical to III and to true. At $x = -0.1$, the first two from top: true, V. Notice different y scale.

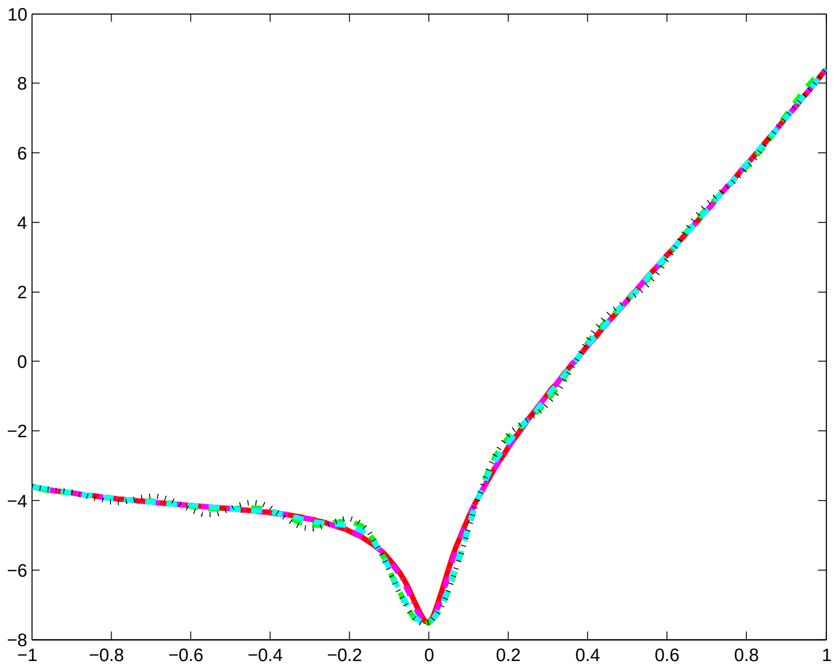

Figure 4: Interpolants of the function (10), $N = 17$. IV not shown, the rest almost identical.