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OPTIMIZATION OF PLUNGER CAVITY

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Abstract

In the contribution we present a problem of shape optimization of the cooling cavity of a plunger that is used in the forming process in the glass industry. A rotationally symmetric system of the mould, the glass piece, the plunger and the plunger cavity is considered. The state problem is given as a stationary heat conduction process. The system includes a heat source representing the glass piece that is cooled from inside by water flowing through the plunger cavity and from outside by the environment surrounding the mould. The design variable is the shape of the inner surface of the plunger cavity.

The cost functional is defined as the squared L^2_r norm of the difference between a prescribed constant and the temperature on the outward boundary of the plunger.

1. Introduction

This work deals with the optimal design of the shape of a plunger cavity that controls the cooling of a glass piece during the manufacturing process. The aim of the optimization is to find such a shape of the inner plunger cavity that allows for cooling in such a way that a constant distribution of the temperature is achieved across the surface of the moulding device at the moment of separation of the plunger from the moulded piece.

2. Formulation of the problem

We rotate the system to the horizontal position to be able to describe the optimized plunger cavity surface by a function of one variable.

We define

$$F_2^e(x) = \begin{cases} 0 & \text{for} \quad x \in [0,\, x_2^e] \\ f_2^e(x) & \text{for} \quad x \in [x_2^e,\, 1] \end{cases} \quad , \tag{1}$$

where $x_2^e \in [s_{\min},\, 1]$ ($s_{\min} > 0$ is a fixed constant given by the minimal thickness of the plunger wall), $f_2^e \in C^{(0),1}([x_2^e,\, 1])$, $f_2^e(x_2^e) = 0$ and $0 \le f_2^e(x) \le f_1(x) - s_{\min}$, $|f_2^{e\prime}(x)| < C_D$ for $x \in]x_2^e, 1]$, where f_1 is a fixed function. Further we assume that $a \le f_2^e(x) - s_2$ for $x \in [x_3^e, 1]$, where $a > 0$ represents the radius of a supply tube and

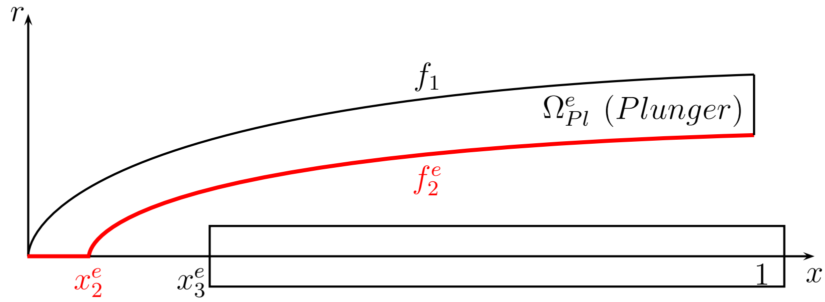

Figure 1: Scheme of the plunger with the optimized part of the boundary.

$s_2 > 0$ is the minimal admissible split width between the inner wall of the plunger cavity and the water supply tube, and $x_3^e \in]x_2, 1]$ is the deepness of the insertion of the tube.

Further we define the set of admissible functions as

$$\begin{aligned} U_{ad}^e = \{ \ F_2^e(x) \in C^{(0),1}([0,1])\,;\ F_2^e(x) = &\begin{cases} 0 & \text{for} \quad x \in [0,\, x_2^e] \\ f_2^e(x) & \text{for} \quad x \in [x_2^e,\, 1] \end{cases} \quad , \\ & x_2^e \in [s_{\min},\, 1],\ s_{\min} > 0,\ f_2^e \in C^{(0),1}([x_2^e,\, 1]),\ f_2^e(x_2^e) = 0, \\ & 0 \le f_2^e(x) \le f_1(x) - s_{\min},\ |f_2^{e\prime}(x)| < C_D \text{ for } x \in]x_2^e, 1], \\ & f_1 \text{ given},\ a \le f_2^e(x) - s_2 \text{ for } x \in [x_3^e, 1],\ a > 0,\ s_2 > 0,\ x_3^e \in]x_2, 1] \} \,, \end{aligned}$$

where the function F_2^e describes the technological constraint for the inner cavity surface.

We assume the region Ω^e_{Pl} that depends on the design function $F_2^e(x)$, and that is defined by the formula

$$\Omega^e_{Pl} = \{(x,\, r) \in R^2\,;\ F_2^e(x) < r < f_1(x), \quad \text{for} \quad x \in [0,\, 1]\} \ .$$

Denote by Θ the set of all admissible regions $\Omega^e_{Pl} \subset R^2$, i.e., regions characterized by $F_2^e \in U^e_{ad}$. Let us define the convergence on the set Θ. Since each Ω^e_{Pl} is uniquely related to F_2^e, we can say that a sequence $\Omega^n_{Pl} \in \Theta$ converges to a region $\Omega^e_{Pl} \in \Theta$ if and only if the sequence of functions ${}^nF_2^e(x)$ converges uniformly in $[0,1]$ to the function $F_2^e(x)$ that defines Ω^e_{Pl}.

Let us consider the union of four planar regions $\Omega = \Omega_{Mo} \cup \Omega_{Gl} \cup \Omega^e_{Pl} \cup \Omega^e_{Ca}$ that represents the planar cross section of the mould, the glass piece, the plunger and the cooling channel of the plunger (see Figure 2).

Furthermore, we denote by Γ_1 the boundary between the plunger Ω^e_{Pl} and the moulded piece Ω_{Gl} and Γ_2^e the boundary between the plunger Ω^e_{Pl} and the plunger cavity Ω^e_{Ca}. We denote by Γ_3 the part of the boundary connecting the mould, the moulded piece and the plunger with the presser, by Γ_4 a part of the axis of symmetry (see Figure 2), by Γ_5 the part of the boundary formed by the tube. Γ_6 is the notation for the part of the boundary between the moulded piece Ω_{Gl} and the mould Ω_{Mo}

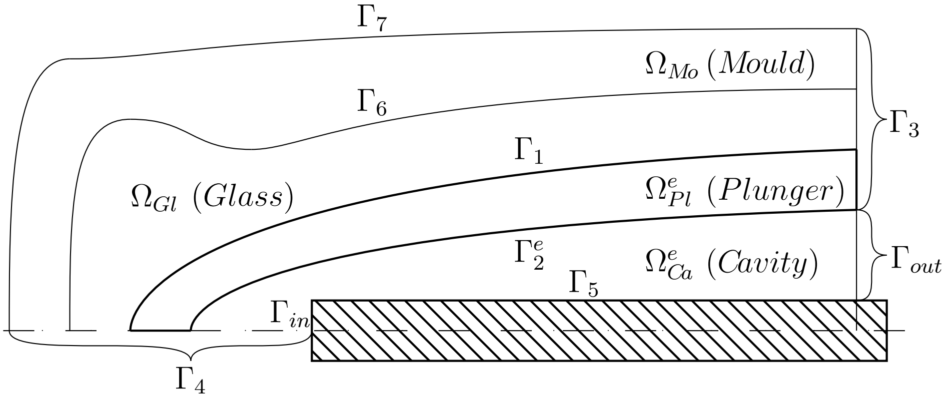

Figure 2: Scheme of the mould, the glass piece, the plunger, the cavity of plunger and the supply tube.

and Γ_7 is the outward boundary of the mould, which is surrounded by an external environment. Γ_{in} denotes the part of the boundary, where the cooling water comes into the cooling channel of the plunger, and Γ_{out} stands for the part of the boundary, where the water exits the channel.
In the three dimensional region G^e_{Ca}, which is created by the rotation of Ω^e_{Ca} around the x axis, we assume an incompressible potential water flow that is rotationally symmetric with respect to the x axis. We split the boundary ∂G^e_{Ca} into the union of four parts as

$$\partial G^e_{Ca} = \Gamma_2^{3D} \cup \Gamma_5^{3D} \cup \Gamma_{in}^{3D} \cup \Gamma_{out}^{3D} \; , \tag{2}$$

where Γ_2^{3D}, Γ_5^{3D}, Γ_{in}^{3D}, and Γ_{out}^{3D} denote the respective parts of the boundary of ∂G^e_{Ca} created by the rotation of Γ^e_2, Γ_5, Γ_{in}, and Γ_{out} around the x axis.

The potential Φ describing the water flow is given as a solution of the Neumann problem

$$\Delta\Phi = 0 \quad \text{in} \qquad G^e_{Ca} \; , \tag{3}$$

$$\frac{\partial \Phi}{\partial n} = g \quad \text{on} \qquad \partial G^e_{Ca} \; , \tag{4}$$

where $g \in L^2(\partial G^e_{Ca})$, representing the normal component of the water flow velocity at the entrance to and the exit from the plunger cavity, is in the form

$$g = \begin{cases} 0 & \text{on} \quad \Gamma_2^{3D} \cup \Gamma_5^{3D} \; , \\ h^{in}_{velo} & \text{on} \quad \Gamma_{in}^{3D} \; , \\ h^{out}_{velo} & \text{on} \quad \Gamma_{out}^{3D} \; , \end{cases} \tag{5}$$

h^{in}_{velo} is the normal velocity at the entrance Γ_{in}^{3D} ($h^{in}_{velo} < 0$) and h^{out}_{velo} is the normal velocity at the exit Γ_{out}^{3D}. Further we assume

$$\int_{\Gamma_{in}^{3D} \cup \Gamma_{out}^{3D}} g \, dS = 0 \; . \tag{6}$$

The variational formulation for the potential function has the form:
We look for the function $\Phi \in H^1(G^e_{Ca})$ such that

$$\int_{G^e_{Ca}} \left(\frac{\partial \Phi}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{\partial \Phi}{\partial x_2} \frac{\partial \varphi}{\partial x_2} + \frac{\partial \Phi}{\partial x_3} \frac{\partial \varphi}{\partial x_3} \right) dV = \int_{\Gamma^{3D}_{in} \cup \Gamma^{3D}_{out}} g\varphi \, dS \quad \forall \varphi \in H^1(G^e_{Ca}) \ . \tag{7}$$

In the cavity G^e_{Ca}, the flowing water velocity field $\boldsymbol{u} = (u_1, u_2, u_3)$ is given as

$$\boldsymbol{u} = \operatorname{grad} \Phi \ . \tag{8}$$

Theorem 1. *(existence and uniqueness of the velocity field)* Under the assumption (6) there exists a unique velocity field of the form (8) satisfying

$$|||\boldsymbol{u}|||_{L^2(G^e_{Ca})} \leq c \left(\|h^{in}_{velo}\|_{L^2(\Gamma^{3D}_{in})} + \|h^{out}_{velo}\|_{L^2(\Gamma^{3D}_{out})} \right) \ , \tag{9}$$

where

$$|||\boldsymbol{u}|||_{L^2(G^e_{Ca})} = \left\| \sqrt{u_1^2 + u_2^2 + u_3^2} \right\|_{L^2(G^e_{Ca})} \ . \tag{10}$$

Proof. See [3]. □

Let us consider the union of four regions $G = G_{Mo} \cup G_{Gl} \cup G^e_{Pl} \cup G^e_{Ca}$ that is created by the rotation of the union $\Omega = \Omega_{Mo} \cup \Omega_{Gl} \cup \Omega^e_{Pl} \cup \Omega^e_{Ca}$ around the x axis. We split ϑ, the searched function representing the distribution of the temperature, into four functions

$$\vartheta = \vartheta_0 + \vartheta_1 + \vartheta_2 + \vartheta_3 \ , \tag{11}$$

where

$$\vartheta_i = \begin{cases} \vartheta|_{G_i} & \text{in} \quad G_i \\ 0 & \text{in} \quad G \setminus G_i \end{cases} \qquad \text{for} \quad i = 0,\, 1,\, 2,\, 3 \ , \tag{12}$$

($G_0 \equiv G^e_{Pl}$, $G_1 \equiv G_{Gl}$, $G_2 \equiv G^e_{Ca}$, $G_3 \equiv G_{Mo}$).
Further we denote by $\vartheta_i|_{\Gamma^{3D}_j}$ the trace of the solution ϑ_i on the boundary Γ^{3D}_j if Γ^{3D}_j is a part of the boundary of G_i for $i = 0, 1, 2, 3,\ j = 1, 2, 3, 4, 5, 6, 7, 8, 9$ ($\Gamma^{3D}_8 = \Gamma^{3D}_{in}\,,\ \Gamma^{3D}_9 = \Gamma^{3D}_{out}$).

By virtue of the rotational symmetry of both the state problem and the function ϑ, the state problem can be formulated variationally in two dimensions. We define the operators

$$\text{Energy}^{velo}_{\Omega}(\vartheta,\, \boldsymbol{w},\, \psi) = c_v \varrho_2 \int_{\Omega^e_{Ca}} \left(\frac{\partial \vartheta_2}{\partial x} w_1 + \frac{\partial \vartheta_2}{\partial r} w_2 \right) \psi r \, d\Omega \ , \tag{13}$$

$$\begin{aligned} \text{Energy}^{cond}_{\Omega}(\vartheta,\, \psi) = {} & k_0 \int_{\Omega^e_{Pl}} \left(\frac{\partial \vartheta_0}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \vartheta_0}{\partial r} \frac{\partial \psi}{\partial r} \right) r \, d\Omega + \\ & + \ k_1 \int_{\Omega_{Gl}} \left(\frac{\partial \vartheta_1}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \vartheta_1}{\partial r} \frac{\partial \psi}{\partial r} \right) r \, d\Omega + \end{aligned} \tag{14}$$

$$
\begin{aligned}
&+\; k_2 \int_{\Omega^e_{Ca}} \left(\frac{\partial \vartheta_2}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \vartheta_2}{\partial r}\frac{\partial \psi}{\partial r} \right) r\, d\Omega \; + \\
&+\; k_3 \int_{\Omega_{Mo}} \left(\frac{\partial \vartheta_3}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \vartheta_3}{\partial r}\frac{\partial \psi}{\partial r} \right) r\, d\Omega \; ,
\end{aligned}
$$

$$
\text{Environment}_\Omega(\vartheta,\, \psi) = \int_{\Gamma_7} \alpha \vartheta_3|_{\Gamma_7} \psi r \, d\Gamma \; , \tag{15}
$$

$$
\text{Source}_\Omega(\psi) = \varrho_1 \int_{\Omega_{Gl}} q \psi r \, d\Omega \; , \tag{16}
$$

$$
\text{Coeff}_\Omega(\psi) = \int_{\Gamma_1} \beta_1 \psi r \, d\Gamma + \int_{\Gamma_6} \beta_6 \psi r \, d\Gamma + \int_{\Gamma_7} \alpha \vartheta_4 \psi r \, d\Gamma \; , \tag{17}
$$

where c_v is the specific heat capacity per unit volume, ϱ_1 is the density of glass, ϱ_2 is the density of water, w_1, w_2 are the water velocity field components expressed in cylindrical coordinates, k_0, k_1, k_2, k_3 are the coefficients of thermal conductivity, α is the coefficient of heat-transfer between the mould and the environment, ϑ_4 is the temperature of the environment, β_1, β_6 are the average power conversion of the unit volume of the glass body (see [4, page 128]) and q is the density of heat sources. Further we denote by

$$
\begin{aligned}
A_\Omega(\vartheta,\, \boldsymbol{w},\, \psi) \;&=\; \text{Energy}^{velo}_\Omega(\vartheta,\, \boldsymbol{w},\, \psi) + \text{Energy}^{cond}_\Omega(\vartheta,\, \psi) \; + \\
&+\; \text{Environment}_\Omega(\vartheta,\, \psi)
\end{aligned} \tag{18}
$$

and

$$
F_\Omega(\psi) = \text{Source}_\Omega(\psi) + \text{Coeff}_\Omega(\psi) \; . \tag{19}
$$

We introduce the weighted Sobolev space $H^1_r(\Omega_i)$ (see [2]) provided with the norm

$$
\|v\|_{1,r,\Omega_i} = \left(\int_{\Omega_i} \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial r} \right)^2 + v^2 \right] r \, d\Omega \right)^{\frac{1}{2}} \qquad i = 0, 1, 2, 3 \; , \tag{20}
$$

$(\Omega_0 \equiv \Omega^e_{Pl},\, \Omega_1 \equiv \Omega_{Gl},\, \Omega_2 \equiv \Omega^e_{Ca},\, \Omega_3 \equiv \Omega_{Mo})$.
Further we introduce

$$
\begin{aligned}
\boldsymbol{H}(\Omega) = \{\; \vartheta;\; \vartheta \text{ defined in (12)},\; \vartheta_i \in H^1_r(\Omega_i) \text{ for any } i = 0, 1, 2, 3, \\
\vartheta_3|_{\Gamma_6} = \vartheta_1|_{\Gamma_6},\, \vartheta_1|_{\Gamma_1} = \vartheta_0|_{\Gamma_1},\, \vartheta_0|_{\Gamma^e_2} = \vartheta_2|_{\Gamma^e_2} \} \; ,
\end{aligned}
$$

where $\vartheta_i|_{\Gamma_j}$ denotes the trace of the function ϑ_i on the boundary Γ_j.
We define the norm in $\boldsymbol{H}(\Omega)$ as

$$
\|\vartheta\|_{\boldsymbol{H}} = \left(\|\vartheta_0\|^2_{1,r,\Omega_0} + \|\vartheta_1\|^2_{1,r,\Omega_1} + \|\vartheta_2\|^2_{1,r,\Omega_2} + \|\vartheta_3\|^2_{1,r,\Omega_3} \right)^{\frac{1}{2}} \; . \tag{21}
$$

Theorem 2. *The set $\boldsymbol{H}(\Omega)$ with the norm (21) is a Hilbert space.*

We denote by $\boldsymbol{H}^*(\Omega)$ the dual space to the space $\boldsymbol{H}(\Omega)$ with the norm

$$\|\psi\|_{\boldsymbol{H}^*} = \sup_{\varphi \neq 0} \frac{A_\Omega(\varphi,\, \boldsymbol{w},\, \psi)}{\|\varphi\|_{\boldsymbol{H}}} \ .$$

We define the sets

$$\Omega_H = \Omega \cup \Gamma_3 \cup \Gamma_{in} \cup \Gamma_{out}$$

and

$${}^e\mathcal{H}^{2D} = \{v \in C^\infty(\Omega_H);\ v|_{\Gamma_3 \cup \Gamma_{in} \cup \Gamma_{out}} = 0\,\}\,.$$

Let $\boldsymbol{H_0}(\Omega)$ be the closure of the set ${}^e\mathcal{H}^{2D}$ in $\boldsymbol{H}(\Omega)$.

We assume the existence of a function $\vartheta^e_\Gamma \in \boldsymbol{H}(\Omega)$ such that

$$\vartheta^e_\Gamma|_{\Gamma_{in}} = 288 \qquad \text{on } \Gamma_{in}, \tag{22}$$

$$\vartheta^e_\Gamma|_{\Gamma_{out}} = h^e_{out} \qquad \text{on } \Gamma_{out}, \tag{23}$$

$$\vartheta^e_\Gamma|_{\Gamma_3} = h_3 \qquad \text{on } \Gamma_3, \tag{24}$$

where $h_3 \in C(\Gamma_3)$ is a given function representing the steady temperature on the boundary Γ_3 (see Figure 2) and $h^e_{out} \in C(\Gamma_{out})$ is a given function representing the temperature distribution on the cavity output Γ_{out}.

We use the variational formulation of the energy equation to formulate

The State Problem:

We look for the function $\vartheta \equiv \vartheta(F^e_2) \in \boldsymbol{H}(\Omega)$ such that

$$A_\Omega(\vartheta,\, \boldsymbol{w^e},\, \psi) \ = \ F_\Omega(\psi) \quad \forall \psi \in \ \boldsymbol{H}_0(\Omega)\ , \tag{25}$$

$$\vartheta - \vartheta^e_\Gamma \in \ \boldsymbol{H}_0(\Omega)\ , \tag{26}$$

where $F^e_2 \in U^e_{ad}$ and $\boldsymbol{w^e}$ is the corresponding flow pattern given as the gradient of the solution to (7).

Remark. The state problem is solved in two steps. First, the potential Φ of the water velocity is found as a solution of the problem (7) in the region G^e_{Ca}. The components of the velocity field $\boldsymbol{u}$ are computed from (8), transformed to cylindrical coordinates and substituted into (13). Then the distribution of the temperature ϑ in the whole system Ω is found as the solution of the state problem (25), (26).

Theorem 3. *(the existence and uniqueness of the solution of the state problem) The state problem (25), (26) has a unique solution $\vartheta(F^e_2)$ for each $F^e_2 \in U^e_{ad}$ and the associated flow pattern $\boldsymbol{w^e}$ obtained as the gradient of the unique solution of (7), moreover, there exists a constant $C > 0$ such that*

$$\|\vartheta(F^e_2)\|_{\boldsymbol{H}} \leq C\|F_\Omega\|_{\boldsymbol{H}^*} \ . \tag{27}$$

Proof. It is sufficient to verify the assumptions of the Lax-Milgram Theorem (see [3]). □

We formulate the **problem of the optimal design for the plunger cavity shape:** We define the **cost functional** as

$$\mathcal{J}^S(F_2^e) = \|\vartheta(F_2^e)|_{\Gamma_1} - T_{\Gamma_1}\|^2_{0,r,\Gamma_1} \ , \tag{28}$$

where $\vartheta(F_2^e)|_{\Gamma_1}$ is the Γ_1-trace of the solution $\vartheta(F_2^e)$ of the state problem (25), (26) in the region Ω^e_{Pl}, where T_{Γ_1} is a given constant representing the known optimal temperature of the plunger surface. We look for the **optimal design** $F_{Opt} \in U^e_{ad}$ such that

$$\mathcal{J}^S(F_{Opt}) \le \mathcal{J}^S(F_2^e) \qquad \forall\ F_2^e \in U^e_{ad} \ . \tag{29}$$

Theorem 4. *The optimal design problem (29) has at least one solution.*

Proof. We refer to Theorem 2.1 [1, page 29], see [3]. □

Remark. A sensitivity analysis can be performed on the basis of temperature evaluation along the boundary Γ_1. Let us introduce a homeomorphism between the outward plunger boundary Γ_1 and the plunger cavity boundary Γ_2^e defined by the gradient lines of the temperature field in the plunger. In the parts of Γ_1 where we need to decrease the temperature, we narrow "the wall" by moving the points of Γ_2^e along the gradient lines to locally achieve more intensive cooling. On the other hand, in places of Γ_1 where we need higher temperature, we increase "the wall thickness" to locally decrease the intensity of cooling. By the term "the wall thickness" we understand the length of the temperature gradient line that connects the related points of Γ_1 and Γ_2^e.

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