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RECENT RESULTS ON QUASILINEAR DIFFERENTIAL EQUATIONS. I

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Abstract. This lecture follows a joint result of the speaker and Daniel Daners. To make the exposition clear and transparent we concentrate here only on the L_∞-estimates for weak solutions for the p-Laplacian with all standard boundary conditions on possibly non-smooth domains. We present $C^{1,\alpha}$-regularity and maximum principle for weak solutions as an application. We also prove existence, continuity and compactness of the resolvent operator.

1. Introduction

In this lecture we give the proof of a priori L_∞-estimates, $C^{1,\alpha}$-regularity and maximum principle for weak solutions to the p-Laplace equation

$$\begin{aligned} -\Delta_p u + c_0|u|^{p-2}u &= f \quad \text{in } \Omega, \\ \mathcal{B}u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

on an open set $\Omega \subset \mathbb{R}^N$. Here $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $p \in (1,\infty)$, and $\mathcal{B}$ a suitable boundary operator associated with the p-Laplacian made more precise later; f is a function only depending on x. In particular, we prove that in certain situations, every weak solution is in $L_\infty(\Omega)$. *These lecture notes are "copy and paste" of selected parts of the joint paper* of the speaker and D. Daners [8], where also some *other cases and estimates than those considered in this lecture are dealt with* (f may depend not only on x but also on u, more general $L_r(\Omega)$-estimates for $r > 1$ are proved there, etc.). For the brevity of the exposition here we concentrate

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mainly on the L_∞-estimates and their main consequences. Problems similar to the above, often with $f = 0$, were considered in [11], [13], [14], [19], [20].

The p-Laplacian is considered here as a prototype of more general quasilinear operator of the second order for which similar results can be proved. In order to keep the main ideas of the lecture as clear as possible, we do not treat here such generalizations.

We would like to emphasize that the choice of *appropriate function space* for weak solutions of (1.1) (as well as for test functions in the definition of the weak solution) will play the crucial role in our proofs.

If $p = 2$ (the linear case), it is well known that the solution u is in $W^2_r(\Omega)$ if $f \in L_r(\Omega)$ and the domain Ω is sufficiently smooth. Also, u satisfies an a priori estimate $\|u\|_{W^2_r} \le c(\|f\|_r + \|u\|_r)$ with $c > 0$, a constant independent of $f \in L_r(\Omega)$, due to [2]. By using embedding theorems for Sobolev spaces, we obtain the estimate

$$\|u\|_{m(r)} \le c(\|f\|_r + \|u\|_r)$$

with $m(r) = Nr(N - 2r)^{-1}$ if $1 < r < N/2$ and $m(r) = \infty$ if $r > N/2$. As shown in [7], such an estimate remains valid for a larger class of operators and non-smooth domains, even if the W^2_r-estimates fail. In the linear case, the exponent $m(r)$ is optimal. Also it is easy to guess from the embedding theorems for Sobolev spaces as mentioned above.

In this lecture we want to generalise these estimates valid for the linear case $p = 2$ to arbitrary $p \in (1, \infty)$. We concentrate on the case $m(r) = \infty$ since the L_∞-estimates are the starting point for the $C^{1,\alpha}$-regularity and the maximum principle.

Actually, there are *no W^2_r-estimates* if $p \ne 2$, that is, if $f \in L_r(\Omega)$, we cannot expect that $u \in W^2_r(\Omega)$ even if Ω is *smooth* (like in the linear case $p = 2$). We demonstrate this point by looking at the Dirichlet problem on an interval and with $p > 2$. It is well known that there exists a principal eigenvalue λ_1 and a principal eigenfunction $\varphi > 0$ to the problem

$$\begin{aligned} -(|\varphi'|^{p-2}\varphi')' &= \lambda_1|\varphi|^{p-2}\varphi \quad \text{in } (0,1), \\ \varphi(0) = \varphi(1) &= 0. \end{aligned} \tag{1.2}$$

Also $\varphi \in C^1([0,1])$, $\varphi'(1/2) = 0$ (see [3]), and it follows from [9, Theorem 10.4] that $|\varphi'|^{p-2}\varphi' \in C^1([0,1])$. Integrating the first equation in (1.2) over $(\frac{1}{2}, x)$ we conclude that

$$\varphi'(x) \sim \left(x - \frac{1}{2}\right)^{1/(p-1)} \quad \text{and} \quad \varphi''(x) \sim \left(x - \frac{1}{2}\right)^{-(p-2)/(p-1)}$$

as $x \to 1/2$. Hence $\varphi \notin W^2_r(0,1)$ for $r > (p-1)/(p-2)$.

The basic method to prove L_∞-estimates originates from the seminal paper [17]. We use suitable cutoff functions of the solution as test functions, interpolation inequalities and then do an iteration based on the validity of a Sobolev-type inequality for functions in our *suitably chosen function space.* This allows to deal with arbitrary domains in case of Dirichlet boundary conditions. It also allows "almost" arbitrary domains in case of Robin-type boundary conditions by using an inequality due to Maz'ya [15], [16] and certain classes of non-smooth domains for Neumann boundary conditions (see Section 4). The proof of the main results is given in Section 3.

2. Assumptions and main results

In this section we state precise assumptions and then discuss main results. We always assume that Ω is an open set. This set is not necessarily bounded or connected. The boundary $\partial\Omega$ is assumed to be the disjoint union of Γ_1, Γ_2 and Γ_3. We study regularity properties of weak solutions of

$$\begin{aligned} -\Delta_p u + c_0|u|^{p-2}u &= f \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \Gamma_1,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_2,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + b_0|u|^{p-2}u &= 0 \quad \text{on } \Gamma_3. \end{aligned} \tag{2.1}$$

Here, $-\Delta_p u := -\operatorname{div}\big(|\nabla u|^{p-2}u\big)$ is the p-Laplacian with $p \in (1,\infty)$. Moreover, $c_0 \in L_\infty(\Omega)$, $b_0 \in L_\infty(\Gamma_3)$, $b_0 \geq 0$ and ν is the outward pointing unit normal to $\partial\Omega$. The boundary conditions are to be understood in a weak sense as explained below. To define *weak solutions* of (2.1), we let

$$a_0(u,v) := \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega c_0(x)|u|^{p-2}uv\,\mathrm{d}x \tag{2.2}$$

for all $u, v \in W_p^1(\Omega)$. Then we define

$$a(u,v) := a_0(u,v) + \int_{\Gamma_3} b_0|u|^{p-2}uv\,\mathrm{d}\mathcal{H}_{N-1} \tag{2.3}$$

whenever the last integral is well defined. Here, $\mathcal{H}_{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure, which coincides with the usual surface measure if Γ_3 is Lipschitz. We next give conditions on the *space of test functions* V_p for the above problem.

Assumption 2.1. We assume that V_p is a Banach space such that

$$\overset{\circ}{W}{}^1_p(\Omega) \hookrightarrow V_p \hookrightarrow W^1_p(\Omega), \tag{2.4}$$

$$\Big\{u \in C^1_c(\overline{\Omega} \setminus \Gamma_1) \cap W^1_p(\Omega) : \int_{\Gamma_3} |u|^p \, \mathrm{d}\mathcal{H}_{N-1} < \infty\Big\} \subset V_p, \tag{2.5}$$

and that the norm

$$\|u\|_{V_p} := \big(\|\nabla u\|_p^p + \|u\|_p^p + \|u \sqrt[p]{b_0}\|^p_{L_p(\Gamma_3)}\big)^{1/p} \tag{2.6}$$

is an equivalent norm on V_p. Finally, we assume that

$$u \min\{|u|^{t-1}, \alpha^{t-1}\} \in V_p$$

for all $\alpha > 0$, $t \geq 1$ and $u \in V_p$.

Remark 2.2. (a) Note that, depending on b_0 and Γ_3, the norm $\|\cdot\|_{V_p}$ may be stronger than the usual W^1_p-norm as shown in Section 4.

(b) Since the L_p-norms are uniformly convex, also the V_p-norm defined by (2.6) is uniformly convex. Hence, by Milman's theorem, V_p is a reflexive space (see [22, Section 5.2]).

Definition 2.3. Let $f \in V_p'$. Then we call $u \in V_p$ a *weak solution* of (2.1) if $a(u, v) = \langle f, v\rangle$ for all $v \in V_p$.

In the above definition $\langle \cdot\,, \cdot\rangle$ denotes duality on V_p. Note that if $f \in L_q(\Omega)$ for some $q \in [1, \infty]$, then

$$\langle f, v\rangle = \int_\Omega f v \, \mathrm{d}x$$

whenever the integral exists. We next introduce the main assumption implying our results on L_∞-estimates.

Assumption 2.4. Suppose that there exist $\delta_0 \geq 0$, $c_{\mathcal{B}} > 0$ and $d > p$ such that

$$\|u\|^p_{dp/(d-p)} \leq c_{\mathcal{B}}\big(a(u, u) + \delta_0 \|u\|_p^p\big) \tag{2.7}$$

for all $u \in V_p$.

By the usual Sobolev embedding theorem, the smallest possible constant is $d = N$ but it may well be that some $d > N$ is optimal. In particular, this is the case for *Neumann boundary conditions* if the *domain is not Lipschitz* (see Section 4). We set

$$\lambda_0 := \|c_0^-\|_\infty + \delta_0. \tag{2.8}$$

If $\lambda_0 = 0$, we call the problem *coercive* since then the functional $a(u,u)$ defined by (2.3) is coercive on V_p. If $\lambda_0 > 0$, we call the problem *non-coercive*.

Note also that the definition of a weak solution only makes sense if $f \in V_p'$. By (2.7), we have $V_p \hookrightarrow L_{dp/(d-p)}(\Omega)$. Also, by Assumption 2.1, $C_c^\infty(\Omega) \subset V_p$ and so V_p is dense in $L_p(\Omega) \cap L_{dp/(d-p)}(\Omega)$. Hence, by duality,

$$L_{(\frac{dp}{d-p})'}(\Omega) \hookrightarrow V_p'.$$

(Here, p' is the conjugate exponent to p given by $1/p + 1/p' = 1$.) Note that

$$L_r(\Omega) \hookrightarrow L_{(\frac{dp}{d-p})'}(\Omega)$$

if $|\Omega| < \infty$ and

$$r \geq \left(\frac{dp}{d-p}\right)' = \frac{dp'}{d+p'}. \tag{2.9}$$

Hence, we consider (2.1) for $f \in L_r(\Omega)$ with r as above.

Next we state *a priori estimates.* We start with an easier case, namely the case of a *coercive problem.*

Theorem 2.5 (Coercive problem)**.** *Suppose that Assumptions 2.1 and 2.2 hold with* $\lambda_0 = 0$*. Moreover, let* $f \in L_r(\Omega) \cap V_p'$ *and let* $u \in V_p$ *be a weak solution of* (2.1)*. Then there exists a constant* $C > 0$ *depending only on p, d and r such that*

$$\|u\|_\infty^{p-1} \leq C c_{\mathcal{B}} |\Omega|^{\frac{p}{d} - \frac{1}{r}} \|f\|_r \tag{2.10}$$

if $r > d/p$ *and* $|\Omega| < \infty$*, and*

$$\|u\|_\infty^{p-1} \leq C c_{\mathcal{B}} \|f\|_r + \|u\|_p^{p-1} \tag{2.11}$$

if $r > d/p$ *(with no restriction on* $|\Omega|$*).*

Next we state estimates valid for *domains of finite measure*, but possibly *non-coercive problems.*

Theorem 2.6 (Finite measure)**.** *Suppose that* $|\Omega| < \infty$*, and that Assumptions 2.1 and 2.2 hold. Moreover, let* $f \in L_r(\Omega)$*,* $r \geq dp'/(d+p')$*, and let* $u \in V_p$ *be a weak solution of* (2.1)*. Then there exists a constant* $C > 0$ *depending only on p, d and r such that*

$$\|u\|_\infty^{p-1} \leq C c_{\mathcal{B}} |\Omega|^{\frac{p}{d} - \frac{1}{r}} G(f,u) \tag{2.12}$$

if $r > d/p$, *where either*

$$G(f,u) = \|f\|_r + \lambda_0 \|u\|_{r(p-1)}^{p-1}$$

or

$$G(f,u) = \begin{cases} \|f\|_r + \lambda_0 |\Omega|^{\frac{1}{r} - \frac{1}{p'}} \|u\|_p^{p-1} & \text{if } r < p', \\ \|f\|_r + c_{\mathcal{B}}^{\mu} \lambda_0^{1+\mu} \|u\|_p^{p-1} & \text{if } r \geq p' \end{cases}$$

with $\mu := \dfrac{d}{p}\Big(\dfrac{1}{p'} - \dfrac{1}{r}\Big)$.

Note that $r(p-1) < 1$ if $r < p'/p$, so $\|\cdot\|_{r(p-1)}$ is not a norm because it does not satisfy the triangle inequality, but we simply understand it to be the integral $\big(\int_\Omega |u|^{r(p-1)}\,\mathrm{d}x\big)^{1/r(p-1)}$.

We finally give an estimate with *no restriction on the measure* of Ω and *possibly for non-coercive problems.*

Theorem 2.7 (Arbitrary measure)**.** *Suppose that Assumptions 2.1 and 2.2 hold. Moreover, let* $f \in L_r(\Omega) \cap V_p'$, $r \geq p'$, *and let* $u \in V_p$ *be a weak solution of* (2.1). *Then there exists a constant* $C > 0$ *depending only on* p, d *and* r *such that*

$$\|u\|_\infty^{p-1} \leq C c_{\mathcal{B}} G(f,u) + \|u\|_p^{p-1} \tag{2.13}$$

if $r > d/p$, *where either*

$$G(f,u) = \|f\|_r + \lambda_0 \|u\|_{r(p-1)}^{p-1}$$

or

$$G(f,u) = \|f\|_r + c_{\mathcal{B}}^{\mu} \lambda_0^{1+\mu} \|u\|_p^{p-1}$$

with μ *as in Theorem 2.6.*

Based on previous L_∞-estimates, we can formulate the following *regularity* result.

Theorem 2.8 (Regularity)**.** *Suppose that Assumptions 2.1 and 2.2 hold,* $f \in L_r(\Omega) \cap V_p'$ *with* $r > d/p$. *Then any weak solution* $u \in V_p$ *of problem* (2.1) *satisfies* $u \in C^{1,\alpha}(\Omega)$ *with some* $\alpha \in (0,1)$.

We also have the *maximum principle* for weak solutions.

Theorem 2.9 (Maximum principle)**.** *Suppose that Assumptions 2.1 and 2.2 hold,* $f \in L_r(\Omega) \cap V_p'$ *with* $r > d/p$, $f \geq 0$ *a.e. in* Ω. *Then any non negative and non trivial weak solution* $u \in V_p$ *is strictly positive in* Ω.

3. Proof of the main results

The proof of the a priori estimates works by iteration. The iteration procedure is based on a basic inequality which we derive first. To make sure all relevant norms involved stay finite, we need to truncate the solution u of (2.1). For $\alpha > 0$ and $t \geq 1$ we set

$$\psi_{\alpha,t}(\xi) := \xi \min\{\alpha^{t-1}, |\xi|^{t-1}\}. \tag{3.1}$$

Further, we define

$$v_{\alpha,q} := \psi_{\alpha,q-p+1} \circ u \quad \text{and} \quad w_{\alpha,q} := \psi_{\alpha,q/p} \circ u$$

if $q \geq p$. It follows from [12, Theorem 7.8]) that $v_{\alpha,q}, w_{\alpha,q} \in W_p^1(\Omega)$ for all $\alpha > 0$ and $q \geq p$ if $u \in W_p^1(\Omega)$. We also need to assume that $v_{\alpha,q}, w_{\alpha,q} \in V_p$ if $u \in V_p$ which is the case in all standard situations as shown in Section 4 (cf. Appendix A).

Proposition 3.1. *Suppose that Assumptions 2.1 and 2.2 hold. Moreover, let u be a weak solution of (2.1) with $f \in L_r(\Omega) \cap V_p'$ for some $r \geq dp'/(d+p')$. If $q \geq p$, then*

$$\|w_{\alpha,q}\|_{dp/(d-p)}^p \leq c_{\mathcal{B}} c(p,q)\big(\|f\|_r \|u\|_{r'(q-p+1)}^{q-p+1} + \lambda_0 \|w_{\alpha,q}\|_p^p\big) \tag{3.2}$$

for all $\alpha > 0$, where

$$c(p,q) := \Big(\frac{q}{p}\Big)^p \frac{1}{q-p+1}. \tag{3.3}$$

Proof. By Assumption 2.1, we have $v_{\alpha,q}, w_{\alpha,q} \in V_p$. It follows from [12, Theorem 7.8] that

$$\nabla(\psi_{\alpha,t} \circ u) = \begin{cases} t|u|^{t-1}\nabla u & \text{if } |u| < \alpha, \\ 0 & \text{if } |u| = \alpha, \\ \alpha^{t-1}\nabla u & \text{if } |u| > \alpha \end{cases} \tag{3.4}$$

for all $\alpha > 0$ and $t \geq 1$. We have

$$\begin{aligned} |\nabla w_{\alpha,q}|^p &= |\nabla(u|u|^{q/p-1})|^p = \Big(\frac{q}{p}\Big)^p |u|^{q-p}|\nabla u|^p \\ &= \Big(\frac{q}{p}\Big)^p \frac{1}{q-p+1} |\nabla u|^{p-2}\nabla u \cdot \nabla(u|u|^{q-p}) \\ &= c(p,q)|\nabla u|^{p-2}\nabla u \cdot \nabla v_{\alpha,q} \end{aligned}$$

whenever $|u| < \alpha$, where $c(p,q)$ is defined by (3.3). Note that $c(p,q) \geq 1$ for all $q \geq p$. If $|u| > \alpha$, then

$$\begin{aligned}|\nabla w_{\alpha,q}|^p &= |\nabla u|^p \alpha^{p(q/p-1)} = |\nabla u|^{p-2} \nabla u \cdot \nabla(\alpha^{q-p} u) \\ &= |\nabla u|^{p-2} \nabla u \cdot \nabla v_{\alpha,q} \leq c(p,q) |\nabla u|^{p-2} \nabla u \cdot \nabla v_{\alpha,q}\end{aligned}$$

for all $q \geq p$ and $\alpha > 0$. If $|u| = \alpha$, then the inequality is trivial. Combining the inequalities we get

$$|\nabla w_{\alpha,q}|^p \leq c(p,q) |\nabla u|^{p-2} \nabla u \cdot \nabla v_{\alpha,q} \tag{3.5}$$

for all $q \geq p$ and $\alpha > 0$. Also, if $|u| \leq \alpha$, then

$$|w_{\alpha,q}|^p = |u|^q = |u|^{p-2} |u|^{q-p+2} = |u|^{p-2} u^2 |u|^{q-p} = |u|^{p-2} u v_{\alpha,q}.$$

If $|u| \geq \alpha$, then

$$|w_{\alpha,q}|^p = \alpha^{p(q/p-1)} |u|^p = |u|^{p-2} u^2 \alpha^{q-p} = |u|^{p-2} u v_{\alpha,q}.$$

Hence,

$$|w_{\alpha,q}|^p = |u|^{p-2} u v_{\alpha,q} \tag{3.6}$$

for all $\alpha > 0$ and $q \geq p$. Since $c(p,q) \geq 1$ and $c_0(x) + \|c_0^-\|_\infty \geq 0$, using (3.5), (3.6) and (2.2) we get

$$\begin{aligned}a_0(w_{\alpha,q}, w_{\alpha,q}) &\leq c(p,q) \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v_{\alpha,q} \, \mathrm{d}x \\ &\quad + \int_\Omega (c_0(x) + \|c_0^-\|_\infty) |w_{\alpha,q}|^p \, \mathrm{d}x \\ &\leq c(p,q) \Big(\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v_{\alpha,q} \, \mathrm{d}x \\ &\qquad + \int_\Omega (c_0(x) + \|c_0^-\|_\infty) |w_{\alpha,q}|^p \, \mathrm{d}x \Big) \\ &= c(p,q) \big(a_0(u, v_{\alpha,q}) + \|c_0^-\|_\infty \|w_{\alpha,q}\|_p^p \big).\end{aligned}$$

Using the definition of a given in (2.3) and the fact that $c(p,q) \geq 1$, we get

$$a(w_{\alpha,q}, w_{\alpha,q}) \leq c(p,q) \big(a(u, v_{\alpha,q}) + \|c_0^-\|_\infty \|w_{\alpha,q}\|_p^p \big).$$

Since $w_{\alpha,q}, v_{\alpha,q} \in V_p$, it finally follows from Assumption 2.2

$$\begin{aligned}\|w_{\alpha,q}\|^p_{dp/(d-p)} &\le c_{\mathcal{B}}\big(a(w_{\alpha,q}, w_{\alpha,q}) + \delta_0\|w_{\alpha,q}\|^p\big)\\ &\le c_{\mathcal{B}}c(p,q)\big(a(u, v_{\alpha,q}) + \lambda_0\|w_{\alpha,q}\|^p_p\big)\end{aligned} \tag{3.7}$$

for all $\alpha > 0$ and $q \ge p$. We next estimate the terms on the right-hand side. First, since u is a weak solution of (2.1) and $f \in L_r(\Omega)$, we get from Hölder's inequality

$$a(u, v_{\alpha,q}) = \langle f, v_{\alpha,q}\rangle \le \|f\|_r\|v_{\alpha,q}\|_{r'}.$$

By definition of $v_{\alpha,q}$, we have $|v_{\alpha,q}| \le |u|^{q-p+1}$, so

$$a(u, v_{\alpha,q}) \le \|f\|_r\|u\|^{q-p+1}_{r'(q-p+1)}.$$

and thus (3.2) follows if we combine everything. □

Corollary 3.2. *Suppose that Assumptions 2.1 and 2.2 hold. Moreover, let u be a weak solution of* (2.1) *with $f \in L_r(\Omega) \cap V_p'$ for some $r \ge dp'/(d+p')$. If $q \ge p$, then*

$$\|w_{\alpha,q}\|^p_{dp/(d-p)} \le c_{\mathcal{B}}c(p,q)\big(\|f\|_r + \lambda_0\|u\|^{p-1}_{r(p-1)}\big)\|u\|^{q-p+1}_{r'(q-p+1)} \tag{3.8}$$

for all $\alpha > 0$, where $c(p,q)$ is defined by (3.3)*.*

Proof. By the definition of $w_{\alpha,q}$ and Hölder's inequality,

$$\|w_{\alpha,q}\|^p_p \le \|u\|^q_q = \int_\Omega |u|^{q-p+1}|u|^{p-1}\,\mathrm{d}x \le \|u\|^{p-1}_{r(p-1)}\|u\|^{q-p+1}_{r'(q-p+1)}.$$

Substituting into (3.2), we obtain the assertion. □

Remark 3.3. If Ω has infinite measure, we cannot expect $u \in L_s(\Omega)$ for $s \in [1,p)$. Hence, in order for the right-hand side of (3.8) to be finite, we need that $r(p-1) \ge p$, that is, $r \ge p'$ if $\lambda_0 \ne 0$. Since $dp'/(d+p') \le p'$ for all $1 < p < d$, we need to assume that $r \ge p'$ if we do not want our estimates to depend on the measure of Ω. In case of finite measure $\|u\|_s \le |\Omega|^{\frac{1}{s}-\frac{1}{p}}\|u\|_p < \infty$ even if we admit $s \in (0,1)$. In the latter case, $\|u\|_s := (\int_\Omega |u|^s\,\mathrm{d}x)^{1/s}$ is not a norm, but for convenience we still use the same notation for the integral. In the coercive case, where $\lambda_0 = 0$, the term involving $\|u\|_{r(p-1)}$ does not appear at all.

We next derive versions of the above inequality replacing $\|u\|_{r(p-1)}$ by $\|u\|_p$. According to the above remark, we need to distinguish two cases, namely the case of Ω having finite measure or not.

Corollary 3.4 (Finite measure). *Suppose that Assumptions 2.1 and 2.2 hold. Moreover, let u be a weak solution of* (2.1) *with $f \in L_r(\Omega)$ for some r, $dp'/(d+p') \le r \le p'$. If $q \ge p$ and $|\Omega| < \infty$, then*

$$\begin{aligned} &\|w_{\alpha,q}\|^p_{dp/(d-p)} \\ &\qquad \le c_{\mathcal{B}} c(p,q)\big(\|f\|_r + \lambda_0 |\Omega|^{1/r-1/p'} \|u\|_p^{p-1}\big)\|u\|^{q-p+1}_{r'(q-p+1)} \end{aligned} \tag{3.9}$$

for all $\alpha > 0$, where $c(p,q)$ is defined by (3.3).

Proof. Since $|\Omega| < \infty$ and $r(p-1) \le p$, we can apply Hölder's inequality with $s := p'/r \ge 1$ to get

$$\|u\|^{p-1}_{r(p-1)} = \Big(\int_\Omega |u|^{r(p-1)}\,\mathrm{d}x\Big)^{1/r} \le |\Omega|^{1/rs'} \|u\|_p^{1/rs} = |\Omega|^{1/r-1/p'} \|u\|_p^{p-1}.$$

Now the desired inequality follows from (3.8). □

We finally establish a version for arbitrary measure. To this end, we need an interpolation inequality. It is similar to the standard one such as found in [12, p. 146] but we admit $\|u\|_s := \big(\int_\Omega |u|^s\,\mathrm{d}s\big)^{1/s}$ also for $s \in (0,1)$. We include the precise statement and proof for completeness.

Lemma 3.5. *Suppose that $0 < s < p < t$ with $p \ge 1$. Set*

$$\mu := \frac{t(p-s)}{s(t-p)}.$$

Then

$$\|u\|_p^p \le \varepsilon^{-\mu}\|u\|_s^p + \varepsilon\|u\|_t^p$$

for all $\varepsilon > 0$ whenever the right-hand side is finite.

Proof. We want to choose $\tau \in (0,1)$ and $\sigma, \rho \ge 1$ with $1/p = 1/\sigma + 1/\rho$ such that $\tau\sigma = s$ and $(1-\tau)\rho = t$ and thus, by Hölder's inequality,

$$\|u\|_p = \||u|^\tau |u|^{1-\tau}\|_p \le \||u|^\tau\|_\sigma \||u|^{1-\tau}\|_\rho = \|u\|_s^\tau \|u\|_t^{1-\tau}.$$

Solving the system of three equations for σ, ρ and τ, we get

$$\sigma = p\frac{t-s}{t-p} \ge 1, \quad \rho = p\frac{t-s}{p-s} \ge 1, \quad \tau = \frac{s(t-p)}{p(t-s)} \in (0,1),$$

so the above works. Next we apply Young's inequality to obtain

$$\|u\|_p \leq \|u\|_s^\tau \|u\|_t^{1-\tau} = \left(\varepsilon^{-(1-\tau)/p\tau}\|u\|_s\right)^\tau \left(\varepsilon^{1/p}\|u\|_t\right)^{1-\tau}$$
$$\leq \tau\varepsilon^{-(1-\tau)/p\tau}\|u\|_s + (1-\tau)\varepsilon^{1/p}\|u\|_t.$$

Finally, note that for $a, b \geq 0$ we have $a \leq (a^p + b^p)^{1/p}$ and $b \leq (a^p + b^p)^{1/p}$, so $\tau a + (1-\tau)b \leq (a^p + b^p)^{1/p}$ and finally

$$(\tau a + (1-\tau)b)^p \leq a^p + b^p.$$

Hence, from the above $\|u\|_p^p \leq \varepsilon^{-(1-\tau)/\tau}\|u\|_s^p + \varepsilon\|u\|_t^p$. Using the value for τ, we get

$$\mu = \frac{1-\tau}{\tau} = \frac{t(p-s)}{s(t-p)}$$

as claimed. □

Corollary 3.6 (Arbitrary measure)**.** *Suppose that Assumptions 2.1 and 2.2 hold. Moreover let u be a weak solution of* (2.1) *with $f \in L_r(\Omega) \cap V_p'$ for some $r \geq p'$. If $q \geq p$, then there exists $c(q) > 0$ depending on r, d and p such that*

$$\|w_{\alpha,q}\|_{dp/(d-p)}^p \leq c_{\mathcal{B}}c(q)\big(\|f\|_r + c_{\mathcal{B}}^\mu \lambda_0^{1+\mu}\|u\|_p^{p-1}\big)\|u\|_{r'(q-p+1)}^{q-p+1} \tag{3.10}$$

for all $\alpha > 0$, where

$$\mu := \frac{d}{p}\Big(\frac{1}{p'} - \frac{1}{r}\Big) \geq 0 \tag{3.11}$$

and the function $c(q) \geq 1$ grows at most polynomially in $q \geq p$.

Proof. We start from (3.2) and use an interpolation argument to absorb part of $\|w_{\alpha,q}\|_p^p$ on the left-hand side. We first note that if $r \geq p'$ and $d > p$, then

$$\frac{pp'r'}{p'+r'} \leq p \leq \frac{dp}{d-p}.$$

Hence, by the interpolation inequality from Lemma 3.5,

$$\|w_{\alpha,q}\|_p^p \leq \varepsilon^{-\mu}\|w_{\alpha,q}\|_{pp'r'/(p'+r')}^p + \varepsilon\|w_{\alpha,q}\|_{dp/(d-p)}^p$$

with $\mu \geq 0$ given by (3.11). Now recall that $|w_{\alpha,q}|^p \leq |u|^q$. Hence, by Hölder's inequality we have

$$\|w_{\alpha,q}\|_{pp'r'/(p'+r')}^p \leq \||u|^{p-1}|u|^{q-p+1}\|_{p'r'/(p'+r')}$$
$$\leq \||u|^{p-1}\|_{p'}\||u|^{q-p+1}\|_{r'} = \|u\|_p^{p-1}\|u\|_{r'(q-p+1)}^{q-p+1}$$

and therefore

$$\|w_{\alpha,q}\|_p^p \le \varepsilon \|w_{\alpha,q}\|_{dp/(d-p)}^p + \varepsilon^{-\mu}\|u\|_p^{p-1}\|u\|_{r'(q-p+1)}^{q-p+1}$$

for all $\varepsilon > 0$ and $q \ge p$. By (3.8),

$$\begin{aligned}\|w_{\alpha,q}\|_{dp/(d-p)}^p \le c_{\mathcal{B}} c(p,q)\big(\|f\|_r + \lambda_0 \varepsilon^{-\mu}\|u\|_p^{p-1}\big)\|u\|_{r'(q-p+1)}^{q-p+1} \\ + \varepsilon c_{\mathcal{B}} c(p,q)\lambda_0 \|w_{\alpha,q}\|_{dp/(d-p)}^p.\end{aligned}$$

Setting $\varepsilon := (2c_{\mathcal{B}} c(p,q)\lambda_{\mathcal{B}})^{-1}$ and moving the last term to the left-hand side, we get

$$\|w_{\alpha,q}\|_{dp/(d-p)}^p \le (2c(p,q))^{1+\mu} c_{\mathcal{B}}\big(\|f\|_r + c_{\mathcal{B}}^{\mu}\lambda_0^{1+\mu}\|u\|_p^{p-1}\big)\|u\|_{r'(q-p+1)}^{q-p+1}$$

for all $q \ge p$. We also used that $c(p,q) \ge 1$ and so $c(p,q) \le c(p,q)^{1+\mu}$. If we set $c(q) := (2c(p,q))^{1+\mu}$, the assertion of the corollary follows. □

All the inequalities derived above have the form

$$\|w_{\alpha,q}\|_{dp/(d-p)}^p \le c(q) c_{\mathcal{B}} G(f,u)\|u\|_{r'(q-p+1)}^{q-p+1} \tag{3.12}$$

for an appropriate function $G(f,u)$, where $c(q)$ grows at most polynomially in $q \ge p$. In particular, we have the following cases:

(1) If $\lambda_0 = 0$ and $r \ge dp'/(d+p')$, then

$$G(f,u) = \|f\|_r \tag{3.13}$$

and $c(q) := c(p,q)$ by Proposition 3.1 (Coercive case).

(2) If $\lambda_0 \ge 0$, and $r \ge dp'/(d+p')$ if $|\Omega| < \infty$ and $r \ge p'$ otherwise, then

$$G(f,u) = \|f\|_r + \lambda_0\|u\|_{r(p-1)}^{p-1} \tag{3.14}$$

and $c(q) := c(p,q)$ by Corollary 3.2.

(3) If $|\Omega| < \infty$, $\lambda_0 \ge 0$ and $dp'/(d+p') \le r \le p'$, then

$$G(f,u) = \|f\|_r + \lambda_0|\Omega|^{1/r-1/p'}\|u\|_p^{p-1} \tag{3.15}$$

and $c(q) := c(p,q)$ by Corollary 3.4.

(4) If $\lambda_0 \geq 0$ and $r \geq p'$, then

$$G(f,u) = \|f\|_r + c_{\mathcal{B}}^{\mu} \lambda_0^{1+\mu} \|u\|_p^{p-1} \tag{3.16}$$

and $c(q) := \big(2c(p,q)\big)^{1+\mu}$ by Corollary 3.6, where μ is defined by (3.11).

We now implement an iteration procedure based on (3.12) which allows us to prove all versions of a priori estimates stated if we take into account the above. Since $|w_{\alpha,q}|^p \nearrow |u|^q$ as $\alpha \to \infty$, it follows from (3.12) and the monotone convergence theorem that

$$\|u\|_{dq/(d-p)}^q \leq c(q) c_{\mathcal{B}} G(f,u) \|u\|_{r'(q-p+1)}^{q-p+1} \tag{3.17}$$

whenever the right-hand side is finite. Assuming that $G(f,u)$ is finite and non-zero, we set

$$v := \frac{u}{(c_{\mathcal{B}} G(f,u))^{1/(p-1)}}. \tag{3.18}$$

Then (3.17) turns into

$$\|v\|_{dq/(d-p)}^q \leq c(q) \|v\|_{r'(q-p+1)}^{q-p+1}. \tag{3.19}$$

The idea then is to iterate the inequality by choosing an initial q_0 and computing q_{n+1} from q_n by solving the equation

$$\frac{dq_n}{d-p} = r'(q_{n+1} - p + 1).$$

It turns out that

$$q_{n+1} = \eta q_n + p - 1 \tag{3.20}$$

with

$$\eta := \frac{d}{r'(d-p)}. \tag{3.21}$$

If we do that, (3.19) turns into

$$\|v\|_{dq_{n+1}/(d-p)}^{q_{n+1}} \leq c(q_{n+1}) \|v\|_{dq_n/(d-p)}^{\eta q_n} \tag{3.22}$$

for all $n \in \mathbb{N}$. The right-hand side is certainly finite for $n = 0$ by (2.7) if we set $q_0 = p$. The above inequality tells us that $u \in L_{dq_1/(d-p)}(\Omega)$. Applying the inequality again, we conclude that $u \in L_{dq_2/(d-p)}(\Omega)$ and iterating n times, we get $u \in L_{dq_n/(d-p)}(\Omega)$ for all $n \in \mathbb{N}$. However, we can utilize it if we know that (q_n) is an increasing sequence. By the recursion formula (3.20) and induction, we have

$$q_n = \eta^n q_0 + (p-1)\sum_{k=0}^{n-1} \eta^k = \eta^n (q_0 - p + 1) + (p-1)\sum_{k=0}^{n} \eta^k,$$

and therefore

$$q_n = \eta^n (q_0 - p + 1) + (p-1)\sum_{k=0}^{n} \eta^k. \tag{3.23}$$

We now prove that (q_n) is increasing if we set $q_0 := p$. Then from (3.23),

$$q_{n+1} - q_n = \eta^{n+1} - \eta^n + (p-1)\eta^{n+1} = \eta^n(\eta p - 1)$$

for all $n \in \mathbb{N}$. This value is positive if and only if $\eta p - 1 > 0$ which is the case if and only if

$$\frac{dp}{d-p} > r',$$

by (3.21). The above is equivalent to the assumption (2.9) on r, so we really have an improvement of regularity at each iteration step. Our a priori estimates are a consequence of the following lemma.

Lemma 3.7. *Suppose that* (3.22) *holds,* v *is defined by* (3.18)*, and* $q_0 \geq p$ *is such that* $u \in L_{dq_0/(d-p)}(\Omega)$*. Then*

$$\|v\|_{dq_n/(d-p)}^{q_n} \leq \Big(\prod_{k=1}^{n} c(q_k)^{\eta^{n-k}}\Big) \|v\|_{dq_0/(d-p)}^{\eta^n q_0} \tag{3.24}$$

for all $n \in \mathbb{N}$.

Proof. We give a proof by induction. For $n = 1$, (3.24) reduces to (3.22) for $n = 0$. If $n > 1$, then by (3.22) and the induction assumption,

$$\begin{aligned}
\|v\|_{dq_{n+1}/(d-p)}^{q_{n+1}} &\leq c(q_{n+1}) \|v\|_{dq_n/(d-p)}^{\eta q_n} \\
&\leq c(q_{n+1}) \Big(\prod_{k=1}^{n} c(q_k)^{\eta^{n-k}}\Big)^{\eta} \|v\|_{dq_0/(d-p)}^{\eta^{n+1} q_0} \\
&= c(q_{n+1}) \Big(\prod_{k=1}^{n} c(q_k)^{\eta^{n-k+1}}\Big) \|v\|_{dq_0/(d-p)}^{\eta^{n+1} q_0} \\
&= \Big(\prod_{k=1}^{n+1} c(q_k)^{\eta^{(n+1)-k}}\Big) \|v\|_{dq_0/(d-p)}^{\eta^{n+1} q_0},
\end{aligned}$$

which is exactly what we need. □

Theorem 3.8. *Suppose that* (3.12) *holds, that* $dp'/(d+p') \leq r$ *and that* $q_0 \geq p$. *If* $G(f,u)$ *is finite and non-zero, then there exists a constant* C *depending only on* d, p, r *and on the function* $c(q)$ *such that*

$$\|u\|_\infty^{q_0+\frac{r(d-p)(p-1)}{rp-d}} \leq C c_{\mathcal{B}} G(f,u)^{\frac{r(d-p)}{rp-d}} \|u\|_{dq_0/(d-p)}^{q_0} \tag{3.25}$$

if $r > d/p$.

Proof. Assume that $r > d/p$ and that $q_0 \geq p$ is such that $u \in L_{dq_0/(d-p)}(\Omega)$. Using that $\eta > 1$ (η is defined by (3.21)), we claim that

$$\eta^n q_0 \leq q_n \leq (2\eta)^n q_0 \tag{3.26}$$

for all $n \in \mathbb{N}$. We give a proof by induction. For $n = 0$ the inequality is obvious. Suppose now that (3.26) holds for some $n \geq 0$. Since $\eta > 1$ and $p - 1 \leq p \leq q_0 \leq \eta^{n+1} q_0 \leq \eta q_n$ it follows that

$$\eta^{n+1} q_0 \leq \eta q_n \leq \eta q_n + p - 1 = q_{n+1} \leq 2\eta q_n \leq (2\eta)^{n+1} q_0$$

as required. Hence, (3.26) holds for all $n \geq 0$. If we take the η^n-th root of (3.24) we get

$$\|v\|_{dq_n/(d-p)}^{q_n \eta^{-n}} \leq \Big(\prod_{k=1}^n c(q_k)^{\eta^{-k}}\Big) \|v\|_{dq_0/(d-p)}^{q_0} \tag{3.27}$$

for all $n \geq 1$. Next we derive a bound for the product in the above inequality by using that $c(q)$ has polynomial growth. By assumption there exist $\beta \geq 1$ and $t \geq 0$ such that

$$c(q) \leq \beta q^t$$

for all $q \geq p$. Using (3.26), we see that

$$c(q_k) \leq \beta (2\eta)^{kt} q_0^t \leq (2\beta q_0 \eta)^{kt}$$

for all $k \geq 1$. Hence, (3.27) implies that

$$\begin{aligned} \|v\|_{dq_n/(d-p)}^{q_n \eta^{-n}} &\leq \Big(\prod_{k=1}^n (2\beta q_0 \eta)^{kt\eta^{-k}}\Big) \|v\|_{dq_0/(d-p)}^{q_0} \\ &\leq (2\beta q_0 \eta)^{t \sum_{k=1}^\infty k\eta^{-k}} \|v\|_{dq_0/(d-p)}^{q_0}, \end{aligned}$$

where the series in the exponent converges since $\eta > 1$. Hence, if we set

$$C := (2\beta q_0 \eta)^{t \sum_{k=1}^{\infty} k\eta^{-k}} < \infty$$

we get

$$\|v\|_{dq_n/(d-p)}^{q_n \eta^{-n}} \le C \|v\|_{dq_0/(d-p)}^{q_0} \tag{3.28}$$

for all $n \in \mathbb{N}$ with $C \ge 1$ independent of $n \in \mathbb{N}$. In order to let $n \to \infty$ we need to compute the limit of $q_n \eta^{-n}$. From (3.23), using that $\eta > 1$, we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{q_n}{\eta^n} &= q_0 - p + 1 + (p-1) \sum_{k=0}^{r} \eta^{-k} \\
&= q_0 - p + 1 + \frac{p-1}{1-\eta^{-1}} \\
&= q_0 - p + 1 + \frac{d(r-1)(p-1)}{rp-d} \\
&= q_0 + \frac{r(d-p)(p-1)}{rp-d}.
\end{aligned}$$

Letting $n \to \infty$ in (3.28) and noting that $q_n \to \infty$, we get

$$\|v\|_{\infty}^{q_0 + \frac{r(d-p)(p-1)}{rp-d}} = \lim_{n\to\infty} \|v\|_{dq_n/(d-p)}^{q_n \eta^{-n}} \le C \|v\|_{dq_0/(d-p)}^{q_0} \tag{3.29}$$

which is equivalent to (3.25) if we take into account (3.18). □

Corollary 3.9. *Suppose the assumptions of the above theorem are satisfied and that $r > d/p$. If $u \in L_\infty(\Omega)$, then there exists a constant $C > 0$ depending only on d, p and r such that*

$$\|u\|_{\infty}^{p-1} \le C c_{\mathcal{B}} |\Omega|^{\frac{p}{d} - \frac{1}{r}} G(f, u) \tag{3.30}$$

if Ω has finite measure and

$$\|u\|_{\infty}^{p-1} \le C c_{\mathcal{B}} G(f, u) + \|u\|_{p}^{p-1} \tag{3.31}$$

or

$$\|u\|_{\infty}^{p-1} \le C c_{\mathcal{B}} G(f, u) + \|u\|_{r(p-1)}^{p-1} \tag{3.32}$$

otherwise.

Proof. Let v be defined by (3.18). Then $v \in L_p(\Omega) \cap L_\infty(\Omega)$ and thus $v \in L_{dq_0/(d-p)}(\Omega)$ for all $q_0 \geq p$. First suppose that Ω has finite measure. Then, by Hölder's inequality,

$$\|v\|^p_{\frac{dp}{d-p}} \leq |\Omega|^{(d-p)/d} \|v\|^p_\infty.$$

Substituting this into (3.29), setting $q_0 = p$, we get

$$\|v\|_\infty^{p+\frac{r(d-p)(p-1)}{rp-d}} \leq C|\Omega|^{(d-p)/d} \|v\|^p_\infty.$$

Rearranging and using the definition of v, we get (3.30) by renaming the constant C. Let $s \geq p$ and choose $q_0 \geq p$ such that $s \leq dq_0/(d-p)$. By interpolation we get

$$\|v\|^{q_0}_{\frac{dq_0}{d-p}} \leq \|v\|_s^{s(1-p/d)} \|v\|_\infty^{q_0 - s(1-p/d)}.$$

If we set

$$\delta := \frac{dr(p-1)}{rp-d},$$

then (3.29) implies

$$\|v\|_\infty^{q_0+\delta(1-p/d)} \leq C\|v\|_s^{s(1-p/d)} \|v\|_\infty^{q_0-s(1-p/d)}.$$

Dividing by $\|v\|_\infty^{q_0-s(1-p/d)}$, we have

$$\|v\|_\infty^{(\delta+s)(1-p/d)} \leq C\|v\|_s^{s(1-p/d)}$$

or equivalently

$$\|v\|_\infty^{(\delta+s)} \leq C^{d/(d-p)} \|v\|^s_s.$$

Using the definition of δ and Young's inequality, we obtain

$$\|v\|_\infty^{p-1} \leq C^{\frac{rp-d}{r(d-p)}\frac{\delta}{s+\delta}} \|v\|_p^{(p-1)\frac{s}{s+\delta}} \leq \frac{\delta}{s+\delta} C^{\frac{rp-d}{r(d-p)}} + \frac{s}{s+\delta}\|v\|_s^{p-1}.$$

Renaming the constant C, we get

$$\|v\|_\infty^{p-1} \leq C + \|v\|_s^{p-1}.$$

Now if we choose $s := p$ we get (3.31), and if $r \geq p'$ and we choose $s := r(p-1)$ we get (3.32)by using the definition of v, completing the proof of the corollary. □

We now derive the main theorems stated in Section 2.

Proof of Theorem 2.5. Suppose that the problem is coercive, that is, $\lambda_0 = 0$. As seen in (3.13) we can set $G(f,u) := \|f\|_r$. Recall that $r > d/p$. If we set $\gamma := r(d-p)/(rp-d)$ and $q_0 := p$ we get from Theorem 3.8 that

$$\|u\|_\infty^{p+\gamma(p-1)} \le C\big(c_{\mathcal{B}}\|f\|_r\big)^{p+\gamma}\|u\|^p_{dp/(d-p)},$$

where the right-hand side is finite because $u \in L_{dp/(d-p)}(\Omega)$ by (2.7). Hence $u \in L_\infty(\Omega)$. Now (2.10) and (2.11) follow from Corollary 3.9, completing the proof of Theorem 2.5. □

Proof of Theorem 2.6. Suppose that $|\Omega| < \infty$ and that $\lambda_0 \ge 0$. If $r < p'$ we let $G(f,u)$ as in (3.15), if $r \ge p'$ we let $G(f,u)$ be as in (3.16). In both cases $G(f,u) < \infty$ since $u \in L_p(\Omega)$ and $f \in L_r(\Omega)$.

Recall that $r > d/p$. Then we see that $u \in L_\infty(\Omega)$ by setting $q_0 = p$ in (3.25). Hence, (2.12) follows from Corollary 3.9.

We get the fact that $u \in L_{r(p-1)}(\Omega)$ since $r(p-1) \le dr(p-1)/(d-p)$. Hence, we can use $G(f,u)$ as defined in (3.14) instead of (3.15), proving the remaining assertion of Theorem 2.6. □

Proof of Theorem 2.7. If $|\Omega|$ is possibly infinite and $\lambda_0 \ge 0$, then the proof is similar to the one of Theorem 2.6, but we can only apply the arguments for $r \ge p'$ (see also Remark 3.3). □

Proof of Theorem 2.8. The $C^{1,\alpha}$-regularity of the weak solution follows from above L_∞-estimates and the regularity result of Tolksdorf [20]. □

Proof of Theorem 2.9. Assume that u is a non negative and non trivial weak solution of (2.1) which is not strictly positive in Ω. Continuity of u (Theorem 2.8) implies that there exists $x_0 \in \Omega$ such that $u(x_0) = 0$ and a cube $K := K(3\rho) \subset \Omega$ of side 3ρ and center x_0 whose sides are parallel to the coordinate axes with the following properties:

(a) $u \not\equiv 0$ in $K(2\rho)$;

(b) there exist $M > 0$ such that $0 \le u < M$ in K (see (i)–(iii) above);

(c) u is a weak supersolution of

$$-\Delta_p u + c_0|u|^{p-2}u = 0 \text{ in } K \tag{3.33}$$

(this is due to the fact that f is non negative in Ω);

(d) $\min_{x\in K(\rho)} u(x) = 0$.

(Notice that $K(\rho)$ and $K(2\rho)$ are cubes of center x_0 and side ρ and 2ρ, respectively.) Now, (a)–(d) contradict Trudinger [21, Thm. 1.2]. Hence, u is strictly positive in Ω. □

4. The choice of function spaces

In this section we discuss some examples where our results apply. We essentially look at the Dirichlet, Neumann and Robin problems separately and identify the spaces V_p and the "dimension" d appearing in the embedding inequality (2.7). We present only some model problems but many kinds of mixed problems are also possible. We give a general criterion for the last part of Assumption 2.1 in Appendix A. We also prove there that it applies to the standard examples discussed below.

4.1. Dirichlet boundary conditions. We assume that $\partial\Omega = \Gamma_1$ and let $V_p := \overset{\circ}{W}{}^1_p(\Omega)$ which is by definition the closure of the set of test functions having compact support in the arbitrary open set $\Omega \subset \mathbb{R}^N$ in $W^1_p(\Omega)$. It is well known that if $N > p$, then there exists a constant c only depending on N and p such that

$$\|u\|_{Np/(N-p)} \leq c\,\|\nabla u\|_p$$

for all $u \in \overset{\circ}{W}{}^1_p(\Omega)$ (see [12, Theorem 7.10]). We can therefore set $d := N$ in (2.7) and $\delta_0 := 0$ in Assumption 2.2. Hence, if $c_0 \geq 0$, then $\lambda_0 = 0$, so the problem is coercive for any open set. If $|\Omega| < \infty$, we can replace N by any $d \geq N$ and find a constant also depending on $|\Omega|$ such that

$$\|u\|_{dp/(d-p)} \leq c\,\|\nabla u\|_p$$

for all $u \in \overset{\circ}{W}{}^1_p(\Omega)$. In particular, given $p > 1$, we can choose $d \geq N$ such that $p < d$ and then apply our results. But of course, the estimates become weaker the larger we choose d.

4.2. Neumann boundary conditions. We assume that Ω is a bounded Lipschitz domain and that $\partial\Omega = \Gamma_2$. We can choose $V_p := W^1_p(\Omega)$. It is well known that if $N > p$, then there exists a constant c depending on N, p and the domain such that the Sobolev inequality

$$\|u\|_{Np/(N-p)} \leq c\,\|u\|_{W^1_p}$$

holds for all $u \in W^1_p(\Omega)$ (see [18, Théorème 3.4]). We can therefore set $d := N$ in (2.7) and $\delta_0 := 1$ in Assumption 2.2. Hence $\lambda_0 = 1 + \|c_0^-\|_\infty > 0$, so the problem is not in general coercive. If $\gamma := \min c_0 > 0$, then

$$\|u\|^p_{W^1_p} \leq \max\{1, \gamma^{-1}\}(\|\nabla u\|^p_p + \gamma\|u\|^p_p) \leq a(u,u)$$

and therefore we can set $\lambda_0 = 0$ if $\min c_0 > 0$, meaning that the problem is coercive in that case. We can replace N by any $d \geq N$ and find a constant also depending on $|\Omega|$ such that

$$\|u\|_{dp/(d-p)} \leq c\,\|u\|_{W_p^1}$$

for all $u \in W_p^1(\Omega)$.

If Ω is not Lipschitz, then (2.7) can fail for any $d \geq N$. An example is a domain with an outward pointing exponential cusp as shown in [1, Theorem 5.32]. On the other hand, there are domains for which (2.7) holds for some optimal $d > N$. Model examples are again domains with outward pointing polynomial cusps (see [1, Theorem 5.35]). If $p > d$, we can simply increase d since for a domain with finite measure, (2.7) holds for any d larger than the minimal possible d and then apply our results.

4.3. Robin boundary conditions. Now we suppose that Ω is a domain, $\partial\Omega = \Gamma_3$ and there exists a constant $\beta > 0$ such that $b_0 \geq \beta$. We then set

$$V_p := W_{p,p}^1(\Omega, \partial\Omega)$$

which is defined to be the completion of the space

$$\{u \in W_p^1(\Omega) \cap C(\overline{\Omega}) : \|u\|_{V_p} < \infty\}$$

with respect to the norm

$$\|u\|_{V_p} = (\|u\|_{W_p^1}^p + \|u|_{\partial\Omega}\|_{L_p(\partial\Omega)}^p)^{1/p},$$

where $L_p(\partial\Omega)$ is given with respect to the $(N-1)$-dimensional Hausdorff measure. These spaces have been introduced by MAZ'YA (see [16, Section 3.6]). We prove in the appendix that Assumption 2.1 is satisfied. Moreover, if $N > 1$, by [16, Corollary 3.6.3] there exists a constant $c > 0$ just depending on N (namely the isoperimetric constant) such that

$$\|u\|_{N/(N-1)} \leq c(\|\nabla u\|_1 + \|u|_{\partial\Omega}\|_{L_1(\partial\Omega)})$$

for all $u \in W_1^1(\Omega) \cap C(\overline{\Omega})$ for which the right hand side is finite. Replacing u by $|u|^p$, we get

$$\|u\|_{Np/(N-1)}^p \leq c\big(p\||u|^{p-1}|\nabla u|\|_1 + \|u|_{\partial\Omega}\|_{L_p(\partial\Omega)}^p\big). \tag{4.1}$$

By Hölder's and Young's inequalities,

$$p\||u|^{p-1}|\nabla u|\|_1 \leq p\|u\|_p^{p-1}\|\nabla u\|_p \leq (p-1)\|u\|_p^p + \|\nabla u\|_p^p,$$

and hence there exists $C > 0$ only depending on N and p such that

$$\|u\|_{Np/(N-1)} \leq C\|u\|_{V_p}$$

for all $u \in V_p$. Now clearly

$$\frac{Np}{N-1} = \frac{(Np)p}{Np-p},$$

so if we set $d := Np$, then (2.7) is satisfied. It was observed in [6] that there are domains for which the embedding $V_p \to L_p(\Omega)$ is not injective (see [4] for an example in the case $p = 2$) since V_p is only defined as an abstract completion of a normed space. Hence, we assume that the embedding is injective and call a domain with that property *admissible*. For such domains we get a priori estimates of the type discussed in this paper if we set $d = Np$. Note that $d > p$ for all $N > 1$.

If the domain has finite measure and $c_0 \geq 0$, then the problem turns out to be coercive. Indeed, using Hölder's and Young's inequality, we have

$$\begin{aligned} cp\||u|^{p-1}|\nabla u|\|_1 &\leq cp|\Omega|^{\frac{1}{Np'}}\|u\|_{Np/(N-1)}^{p-1}\|\nabla u\|_p \\ &\leq c_1\|u\|_{Np/(N-1)}^p + \frac{1}{p}\|\nabla u\|_p^p \end{aligned}$$

for some constant c_1 only depending on N, p and $|\Omega|$. Rearranging (4.1), we get a constant $C > 0$ such that

$$\|u\|_{Np/(N-1)} \leq C(\|\nabla u\|_p^p + \|u|_{\partial\Omega}\|_{L_p(\partial\Omega)}^p)^{1/p}. \tag{4.2}$$

Clearly,

$$\int_{\partial\Omega} b_0|u|^p \, d\sigma \geq \beta \int_{\partial\Omega} |u|^p \, d\sigma = \beta\|u|_{\partial\Omega}\|_{L_p(\partial\Omega)}^p$$

and hence

$$\|u\|_{Np/(N-1)}^p \leq C^p \max\{1, \beta^{-1}\} a(u,u).$$

We can therefore set $d = Np$ and $\delta_0 = 0$ under the above assumptions. This means that the problem is coercive for every admissible bounded domain if $c_0 \geq 0$.

Finally, note that, for instance, for a domain with an outward pointing exponential cusp, $\|\cdot\|_{V_p}$ is stronger than the W_p^1-norm and thus, by the open mapping theorem, the space V_p is a proper subspace of $W_p^1(\Omega)$. That the norm is strictly stronger follows from [1, Theorem 5.32] asserting that for a domain with a sufficiently sharp outward pointing cusp $W_p^1(\Omega) \not\subset L_q(\Omega)$ for all $q > p$, contradicting (4.2) if we assume that $V_p = W_p^1(\Omega)$. The same applies if Ω is an unbounded domain of finite measure (see [1, Theorem 5.30]).

The *notion of admissibility* of a domain Ω is closely related to the properties of its boundary $\partial\Omega$. Roughly speaking, if the boundary $\partial\Omega$ is "wild" in a certain sense, there exists a function $w \in V_p$ such that $w \neq o_{V_p}$ but $w = o_{L_p(\Omega)}$. Here, o_{V_p} and $o_{L_p(\Omega)}$ denote the zero elements in V_p and $L_p(\Omega)$, respectively. Notice that this cannot happen if the trace of a function from V_p is locally defined in a usual sense up to a set of $(N-1)$-dimensional Hausdorff measure zero. It follows from Arendt and Warma [4] that the admissibility of Ω is not essential restriction on the domain Ω (cf. also Biegert [5] and Daners [6, Sec. 3]). An example of a bounded domain which is not admissible is constructed, e.g., in [4, pp. 357 and 358]. One can see that the domains of this kind are rather special. In fact, most of the domains which appear in applications do not possess such complicated structure and, due to our approach, we can go "far beyond" the class of Lipschitz domains.

5. Existence and compactness of the resolvent

In this section we look at existence and compactness of the resolvent to the problem (2.1) under the Assumptions 2.1 and 2.2. We also assume that Ω is *bounded*, that the problem is *coercive*, and that the *embedding*

$$V_p \hookrightarrow\hookrightarrow L_p(\Omega)$$

is *compact*. (We write $\hookrightarrow\hookrightarrow$ for a compact embedding.) We prove that the solution operator $T^{-1}\colon L_r(\Omega) \to V_p \cap L_\infty$ exists for all $r > d/p$ and that it is continuous and compact as an operator into $V_p \cap L_s(\Omega)$ for all $s \in (1, \infty)$. Compact means that it maps bounded sets onto relatively compact sets. We start by constructing the solution operator. By our assumptions, there exists $c > 0$ such that

$$|a(u,v)| \le c\,\|u\|_{V_p}^{p-1}\|v\|_{V_p} \tag{5.1}$$

for all $u, v \in V_p$. In particular, this shows that for every fixed $u \in V_p$ the functional $v \mapsto a(u,v)$ is an element of the dual space V_p'. Hence, for each $u \in V_p$ there exists $T(u) \in V_p'$ such that

$$\langle T(u), v\rangle = a(u,v)$$

for all $u, v \in V_p$ and it therefore defines an operator $T\colon V_p \to V_p'$. This map is continuous essentially because the superposition operator associated with the function $g(\xi) := |\xi|^{p-2}\xi$ is continuous from $L_p(\Omega)$ to $L_{p'}(\Omega)$ and also from $L_p(\Gamma_3)$ to $L_{p'}(\Gamma_3)$ (see [9, page 188]). The monotonicity of g implies that T is a monotone operator and the assumption on the coercivity of the problem guarantees that T is a coercive operator as well. We show that T is bounded, that is, it maps bounded sets of V_p onto bounded sets in V_p'. By definition of the dual norm and (5.1),

$$\|T(u)\|_{V_p'} = \sup_{\|v\|_{V_p}=1} |\langle T(u), v\rangle| = \sup_{\|v\|_{V_p}=1} |a(u,v)| \le c \sup_{\|v\|_{V_p}=1} \|u\|_{V_p}^{p-1} \|v\|_{V_p}$$

and thus T is bounded. It follows from the Browder theorem (see [10, Theorem 5.3.22]) that $T(V_p) = V_p'$, that is, T maps V_p onto V_p'. We next prove existence, continuity and boundedness of the operator T^{-1} also between L_s-spaces.

Theorem 5.1. *Suppose that Assumptions 2.1 and 2.2 hold and that $c_0, b_0 \ge 0$. Then $T^{-1}\colon V_p' \to V_p$ exists, is bounded and continuous. Moreover, if $r > d/p$, then $T^{-1}\colon L_r(\Omega) \to L_\infty(\Omega)$ is bounded and $T^{-1}\colon L_r(\Omega) \to L_s(\Omega)$ is bounded and continuous for all $s \in [1, \infty)$.*

Proof. Since the map $\xi \to |\xi|^{p-2}\xi$ is strictly monotone, it follows that

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) > 0 \tag{5.2}$$

for all $\xi \ne \eta$. Hence, by the coercivity of the problem and the definition of T and $a(u,v)$, it follows that

$$\langle T(u) - T(v), u - v\rangle > 0$$

for all $u, v \in V_p$ with $u \ne v$. Note that the only possibility for the above expression to be zero is if $\nabla u = \nabla v = 0$ almost everywhere and $c_0 = b_0 = 0$. But then the problem is not coercive, contrary to what we assumed. Hence, T is injective and T^{-1} exists. The coercivity implies also the existence of a constant $C > 0$ such that

$$C\|u\|_{V_p}^p \le a(u,u) = \langle T(u), u\rangle$$

for all $u \in V_p$. Hence, if $u \in V_p$ is the weak solution of (2.1) with $f \in V_p'$, then

$$C\|u\|_{V_p}^p \le \langle T(u), u\rangle = \langle f, u\rangle \le \|f\|_{V_p'} \|u\|_{V_p}$$

and since $u = T^{-1}f$,

$$\|T^{-1}(f)\|_{V_p}^{p-1} \le C^{-1}\|f\|_{V_p'}$$

for all $f \in V_p'$. Thus T^{-1} is bounded on V_p'.

We next show that T^{-1} is continuous on V_p'. From Hölder's inequality and (5.2) we get

$$\begin{aligned}
&\langle T(u) - T(v), u - v\rangle \\
&\qquad \ge \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot(\nabla u - \nabla v)\,\mathrm{d}x \\
&\qquad = \|\nabla u\|_p^p + \|\nabla v\|_p^p \\
&\qquad\quad - \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,\mathrm{d}x - \int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u\,\mathrm{d}x \\
&\qquad \ge \|\nabla u\|_p^p + \|\nabla v\|_p^p - \|\nabla u\|_p^{p-1}\|v\|_p - \|\nabla v\|_p^{p-1}\|u\|_p \\
&\qquad = \big(\|\nabla u\|_p^{p-1} - \|\nabla v\|_p^{p-1}\big)\big(\|\nabla u\|_p - \|\nabla v\|_p\big)
\end{aligned} \tag{5.3}$$

for all $u, v \in V_p$. Similarly, for all $u, v \in V_p$,

$$\begin{aligned}
\langle T(u) - T(v), u - v\rangle \ge (&\|c_0^{1/p}u\|_p^{p-1} - \|c_0^{1/p}v\|_p^{p-1}) \\
&\times (\|c_0^{1/p}u\|_p - \|c_0^{1/p}v\|_p)
\end{aligned} \tag{5.4}$$

and if $\Gamma_3 \ne \emptyset$,

$$\begin{aligned}
\langle T(u) - T(v), u - v\rangle \ge (&\|b_0^{1/p}u\|_{L_p(\Gamma_3)}^{p-1} - \|c_b^{1/p}v\|_{L_p(\Gamma_3)}^{p-1}) \\
&\times (\|b_0^{1/p}u\|_{L_p(\Gamma_3)} - \|b_0^{1/p}v\|_{L_p(\Gamma_3)}).
\end{aligned} \tag{5.5}$$

Assume to the contrary that T^{-1} is *not* continuous. Then there exist $f_n \in V_p'$ with $f_n \to f$ in V_p' and $\delta > 0$ such that

$$\|T^{-1}(f_n) - T^{-1}(f)\|_{V_p} \ge \delta \tag{5.6}$$

for all $n \in \mathbb{N}$. Set $u_n := T^{-1}(f_n)$ and $u := T^{-1}(f)$. As (f_n) is a bounded sequence and T^{-1} is bounded, the sequence (u_n) is bounded in V_p. By the reflexivity of V_p (see Remark 2.2), it has a weakly convergent subsequence and so renumbering it, we can assume that (u_n) converges weakly to some

$\tilde{u} \in V_p$. Since by assumption $T(u_n) - T(\tilde{u}) \to f - T(\tilde{u})$ strongly in V_p' and $u_n - \tilde{u} \rightharpoonup 0$ weakly in V_p, we get

$$\langle T(u_n) - T(\tilde{u}), u_n - \tilde{u} \rangle \to 0$$

as $n \to \infty$. Setting $u = u_n$ and $v = \tilde{u}$ in (5.3)–(5.5), we conclude that

$$\begin{aligned} a(u_n, u_n) &= \|\nabla u_n\|_p^p + \|c_0^{1/p} u_n\|_p^p + \|b_0^{1/p} u_n\|_{L_p(\Gamma_3)}^p \\ &\longrightarrow \|\nabla \tilde{u}\|_p^p + \|c_0^{1/p} \tilde{u}\|_p^p + \|b_0^{1/p} \tilde{u}\|_{L_p(\Gamma_3)}^p = a(u, u). \end{aligned}$$

By the coercivity, $a(u,u)^{1/p}$ is a strictly convex equivalent norm on the uniformly convex Banach space V_p, showing that $u_n \to \tilde{u}$ strongly in V_p. Since T is continuous, $f_n = T(u_n) \to T(\tilde{u}) = f$ and by injectivity $\tilde{u} = u$. This means that

$$\|T^{-1}(f_n) - T^{-1}(f)\|_{V_p} = \|u_n - u\|_{V_p} \to 0$$

contradicting (5.6). Hence, T^{-1} is continuous on V_p' as claimed.

To show that $T^{-1} \colon L_r(\Omega) \to L_\infty(\Omega)$ is bounded let $B \subset L_r(\Omega)$ be bounded. From Theorem 2.5 we know that $\|T^{-1}(f)\|_\infty \le C\|f\|_r$ for some constant C independent of $f \in L_r(\Omega)$. Hence, the image of B under T^{-1} is bounded in $L_\infty(\Omega)$.

We finally show that $T^{-1} \colon L_r(\Omega) \to L_s(\Omega)$ is continuous for all $s \in [1, \infty)$. Since $V_p \hookrightarrow L_{dp/(d-p)}(\Omega)$, it is sufficient to look at $s \ge dp/(d-p)$. Suppose now that $r > d/p$ and let (f_n) be a sequence in $L_r(\Omega)$ with $f_n \to f$ in $L_r(\Omega)$. Then as shown above, the sequence $T^{-1}(f_n)$ is bounded in $L_\infty(\Omega)$. Also $T^{-1} \colon L_r(\Omega) \hookrightarrow V_p' \to V_p \hookrightarrow L_{dp/(d-p)}(\Omega)$ is continuous. Hence, if $s \in [dp/(d-p), \infty)$, then by interpolation, for some $\tau \in (0, 1]$,

$$\begin{aligned} &\|T^{-1}(u_n) - T^{-1}(u)\|_s \\ &\qquad \le \|T^{-1}(u_n) - T^{-1}(u)\|_{dp/(d-p)}^\tau \|T^{-1}(u_n) - T^{-1}(u)\|_\infty^{1-\tau} \\ &\qquad \le C\|T^{-1}(u_n) - T^{-1}(u)\|_{V_p}^\tau \to 0 \end{aligned}$$

as $n \to \infty$, by continuity of T^{-1} as a map into V_p. This concludes the proof of the theorem. □

Theorem 5.2. *Suppose that Assumptions 2.1 and 2.2 hold and that $c_0, b_0 \ge 0$. Let $f \in L_r(\Omega)$, $f \neq 0$, $f \ge 0$ a.e. in Ω, $r > d/p$. Then problem* (2.1) *has unique weak solution $u \in L_\infty(\Omega) \cup C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$ which is strictly positive in Ω.*

Proof. The existence of unique weak solution $u \in L_\infty(\Omega) \cup V_p$ follows from Theorem 5.1. Theorem 2.8 implies that $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$ and Theorem 2.9 forces $u > 0$ everywhere in Ω. □

We next look at the compactness of T^{-1}.

Theorem 5.3. *Suppose that Assumptions 2.1 and 2.2 hold and that* $c_0, b_0 \geq 0$*. Then the solution operator* $T^{-1}\colon L_r(\Omega) \to V_p \cap L_s(\Omega)$ *is compact for all* $r > d/p$ *and* $s \in (1, \infty)$*.*

Proof. By Theorem 5.1, $T^{-1}\colon V_p' \to V_p$ exists and is bounded and continuous. By assumption and interpolation, we have

$$V_p \hookrightarrow\hookrightarrow L_t(\Omega)$$

for $1 < t < dp/(d-p)$ and therefore by duality,

$$L_r(\Omega) \hookrightarrow\hookrightarrow V_p'$$

for all $r > dp'/(d+p')$ (and hence also for $r > d/p$). Hence, by continuity and boundedness of $T^{-1}\colon V_p' \to V_p$, it follows that

$$T^{-1}\colon L_r(\Omega) \to V_p$$

is compact for $r > dp'/(d+p')$ (and hence also for $r > d/p$). We now show that T^{-1} is also compact as a map into L_s for $s \in (1, \infty)$. From Theorem 5.1 we know that T^{-1} is bounded. We need to show that the image of every bounded set $B \subset L_\infty$ is relatively compact in $L_s(\Omega)$ for $1 < s < \infty$. For that it is sufficient to show that every sequence in $T^{-1}(B)$ has a convergent subsequence in $L_s(\Omega)$. Hence, let (u_n) be a sequence in $T^{-1}(B)$. Let $f_n := T(u_n) \in B$ and note that because B is bounded, the sequence (f_n) is bounded. We have already seen that T^{-1} is a compact map into V_p, so there is a subsequence (f_{n_k}) such that $T^{-1}(f_{n_k}) \to u$ in V_p. Renumbering the sequence, we can assume that $u_n = T^{-1}(f_n) \to u$ in V_p and therefore in $L_p(\Omega)$. It remains to show that $u_n \to u$ in $L_s(\Omega)$. Let now $s \in (p, \infty)$. Then, as (u_n) is bounded in $L_\infty(\Omega)$, a standard interpolation inequality (see also the proof of Lemma 3.5) implies that there exists $\tau \in (0,1)$ such that

$$\|u_n - u_m\|_s \leq \|u_n - u_m\|_p^\tau \|u_n - u_m\|_\infty^{1-\tau} \leq 2C\|u_n - u_m\|_p^\tau$$

for all $n, m \in \mathbb{N}$. As (u_n) converges in $L_p(\Omega)$ we conclude that (u_n) is a Cauchy sequence in $L_s(\Omega)$ and therefore converges in $L_s(\Omega)$. Hence, $T^{-1}\colon L_r(\Omega) \to L_s(\Omega)$ is compact for $s \in [p, \infty)$. For $s \in (1, p)$ simply observe that $L_p(\Omega) \hookrightarrow L_s(\Omega)$ because Ω has finite measure. □

Remark 5.4. In general, we do not expect $T^{-1}\colon L_r(\Omega) \to L_\infty(\Omega)$ to be compact. The reason is that $V_p \hookrightarrow L_{dp/(d-p)}(\Omega)$ is not compact if $d \geq N$ is optimal. At least in the linear case there is a converse of the a priori estimates which would imply compactness of the embedding (see [7, Section 6]).

Appendix A. Plausibility of Assumption 2.1

The purpose of this appendix is to establish a general criterion to show that $\psi_{\alpha,t} \circ u \in V_p$ for all $u \in V_p$ (the final requirement in Assumption 2.1) for a large class of problems, in particular for the boundary conditions considered in Section 4. The criterion applies in a similar manner to more general mixed problems.

Proposition A.1. *Suppose that V_p is a Banach space such that* (2.4), (2.5) *hold, and that* (2.6) *is an equivalent norm on V_p. If $\psi_{\alpha,t} \circ u \in V_p$ for all u in a dense subset of V_p, then $\psi_{\alpha,t} \circ u \in V_p$ for all $u \in V_p$, that is, Assumption 2.1 is satisfied.*

Proof. By the definition of $\psi_{\alpha,t}$ we clearly have

$$|\psi_{\alpha,t} \circ u| \leq t\alpha^{t-1}|u|.$$

Also, by (3.4),

$$|\nabla(\psi_{\alpha,t} \circ u)| \leq t\alpha^{t-1}|\nabla u|.$$

Now assume that $W \subset V_p$ is a dense set such that $\psi_{\alpha,t} \circ u \in V_p$ for all $u \in W$. Hence, by the definition of the norm (2.6) and the above,

$$\|\psi_{\alpha,t} \circ u\|_{V_p} \leq t\alpha^{t-1}\|u\|_{V_p} \tag{A.1}$$

for all $u \in W$. Given $u \in V_p$, there exist $u_n \in W$ such that $u_n \to u$ in V_p. Clearly, $\psi_{\alpha,t}$ is Lipschitz with

$$|\psi_{\alpha,t}(\xi) - \psi_{\alpha,t}(\eta)| \leq t\alpha^{t-1}|\xi - \eta|$$

and thus $\|\psi_{\alpha,t} \circ u_n - \psi_{\alpha,t} \circ u\|_p \leq t\alpha^{t-1}\|u_n - u\|_p$ for all $n \in \mathbb{N}$. Hence, $\psi_{\alpha,t} \circ u_n \to \psi_{\alpha,t} \circ u$ in $L_p(\Omega)$. By (A.1), the sequence $(\psi_{\alpha,t} \circ u_n)$ is bounded in V_p and therefore $\psi_{\alpha,t} \circ u_n \rightharpoonup \psi_{\alpha,t} \circ u$ weakly in V_p. In particular, this implies that $\psi_{\alpha,t} \circ u \in V_p$. □

We now apply the above to the standard examples. First, if $V_p = W_p^1(\Omega)$, then $\psi_{\alpha,t} \circ u \in W_p^1(\Omega)$ for all $u \in W_p^1(\Omega)$. This covers the examples of the Neumann problem and also the Robin problem on a Lipschitz domain.

Next we look at the Dirichlet problem. We know that $W := C_c(\Omega) \cap W_p^1(\Omega)$ is dense in $\overset{\circ}{W}{}_p^1(\Omega)$. From the above and the continuity of $\psi_{\alpha,t}$ we clearly have $\psi_{\alpha,t} \circ u \in \overset{\circ}{W}{}_p^1(\Omega)$ for all $u \in W$ and therefore for all $u \in \overset{\circ}{W}{}_p^1(\Omega)$ by the above proposition.

We finally consider the space $V_p := W^1_{p,p}(\Omega, \partial\Omega)$ which, as defined in the Section 4, is the completion of the space

$$W := \{u \in W^1_p(\Omega) \cap C(\overline{\Omega}) : \|u\|_{V_p} < \infty\}$$

with respect to the norm

$$\|u\|_{V_p} = \left(\|u\|^p_{W^1_p} + \|u|_{\partial\Omega}\|^p_{L_p(\partial\Omega)}\right)^{1/p}.$$

Since $|\psi_{\alpha,t} \circ u| \leq t\alpha^{t-1}|u|$ on $\overline{\Omega}$, it follows that $\psi_{\alpha,t} \circ u \in W^1_{p,p}(\Omega, \partial\Omega)$ for all $u \in W$, and thus, by definition of $W^1_{p,p}(\Omega, \partial\Omega)$ and by the above proposition, for all $u \in W^1_{p,p}(\Omega, \partial\Omega)$.

References

[1] R. A. Adams: *Sobolev spaces.* Pure and Applied Mathematics, **65**. New York-London: Academic Press, 1975. Zbl 0314.46030, MR0450957.

[2] S. Agmon, A. Douglis and L. Nirenberg: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I.* Commun. Pure Appl. Math. **12** (1959), 623–727. Zbl 0093.10401, MR0125307.

[3] A. Anane: *Simplicité et isolation de la première valeur propre du p-laplacien avec poids.* C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 16, 725–728. Zbl 0633.35061, MR0920052.

[4] W. Arendt and M. Warma: *The Laplacian with Robin boundary conditions on arbitrary domains.* Potential Anal. **19** (2003), no. 4, 341–363. Zbl 1028.31004, MR1988110.

[5] M. Biegert: *The relative capacity.* (Preprint, arXiv: 0806.1417).

[6] D. Daners: *Robin boundary value problems on arbitrary domains.* Trans. Amer. Math. Soc. **352** (2000), no. 9, 4207–4236. Zbl 0947.35072, MR1650081.

[7] D. Daners: *A priori estimates for solutions to elliptic equations on non-smooth domains.* Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), no. 4, 793–813. Zbl 1083.35024, MR1926917.

[8] D. Daners and P. Drábek: *A priori estimates for a class of quasi-linear elliptic equations.* Trans. Amer. Math. Soc. **361** (2009), no. 12, 6475–6500. Zbl 1181.35098, MR2538601.

[9] P. Drábek: *Solvability and bifurcations of nonlinear equations.* Pitman Research Notes in Mathematics Series, **264**. Harlow, Essex: Longman Scientific & Technical, 1992. Zbl 0753.34002, MR1175397.

[10] P. Drábek and J. Milota: *Methods of nonlinear analysis.* Birkhäuser Advanced Texts, Basler Lehrbücher. Basel: Birkhäuser, 2007. Zbl 1176.35002, MR2323436.

[11] J. P. García Azorero and I. Peral Alonso: *Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues.* Comm. Partial Differential Equations **12** (1987), no. 12, 1389–1430. Zbl 0637.35069, MR0912211.

[12] D. Gilbarg and N. S. Trudinger: *Elliptic partial differential equations of second order.* Second ed. Grundlehren der Mathematischen Wissenschaften, **224**. Berlin: Springer-Verlag, 1983. Zbl 0562.35001, MR0737190.

[13] G. M. Lieberman: *Boundary regularity for solutions of degenerate elliptic equations.* Nonlinear Anal., Theory Methods Appl. **12** (1988), no. 11, 1203–1219. Zbl 0675.35042, MR0969499.

[14] P. Lindqvist: *On the equation* $\mathrm{div}(|\nabla u|^{p-2}\nabla u)+\lambda|u|^{p-2}u=0$. Proc. Amer. Math. Soc. **109** (1990), no. 1, 157–164. Zbl 0714.35029, MR1007505.

[15] V. G. Maz'ya: *Classes of domains and imbedding theorems for function spaces.* Soviet Math. Dokl. **1** (1960), 882–885. Transl. from Dokl. Akad. Nauk SSSR 133, 527–530 (1960). Zbl 0114.31001, MR0126152.

[16] V. G. Maz'ya: *Sobolev spaces.* Springer Series in Soviet Mathematics. Berlin: Springer, 1985. Transl. from the Russian by T. O. Shaposhnikova. Zbl 0692.46023, MR0817985.

[17] J. Moser: *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations.* Commun. Pure Appl. Math. **13** (1960), 457–468. Zbl 0111.09301, MR0170091.

[18] J. Nečas: *Les méthodes directes en théorie des équations elliptiques.* Academia, Prague, 1967. MR0227584.

[19] M. Ôtani and T. Teshima: *On the first eigenvalue of some quasilinear elliptic equations.* Proc. Japan Acad. Ser. A Math. Sci. **64** (1988), no. 1, 8–10. Zbl 0662.35080, MR0953752.

[20] P. Tolksdorf: *Regularity for a more general class of quasilinear elliptic equations.* J. Differ. Equations **51** (1984), no. 1, 126–150. Zbl 0488.35017, MR0727034.

[21] N. Trudinger: *On Harnack type inequalities and their application to quasilinear elliptic equations.* Commun. Pure Appl. Math. **20** (1967), 721–747. Zbl 0153.42703, MR0226198.

[22] K. Yosida: *Functional analysis.* 6th ed., Grundlehren der Mathematischen Wissenschaften, **123** Berlin: Springer-Verlag, 1980. Zbl 0435.46002, MR0617913.