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# Recent developments in the theory of function spaces with dominating mixed smoothness

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# RECENT DEVELOPMENTS IN THE THEORY OF FUNCTION SPACES WITH DOMINATING MIXED SMOOTHNESS

Hans-Jürgen Schmeisser

Abstract. The aim of these lectures is to present a survey of some results on spaces of functions with dominating mixed smoothness. These results concern joint work with Winfried Sickel and Miroslav Krbec as well as the work which has been done by Jan Vybíral within his thesis. The first goal is to discuss the Fourier-analytical approach, equivalent characterizations with the help of derivatives and differences, local means, atomic and wavelet decompositions. Secondly, on this basis we study approximation with respect to hyperbolic crosses, embeddings and traces. We follow [42], [43], [44], [59], [63], [64], [70], and [94], [95], [96]. Partial results can be found also in [6], [7], [8], [37] and [48].

## 1. Introduction – Dominating mixed smoothness and related topics

**1.1. Sobolev spaces.** The aim of this section is twofold. On the one hand, we describe how to define dominating mixed smoothness properties of functions by classical means, i.e., by derivatives and differences, and we give precise definitions of the corresponding spaces. On the other hand, we introduce related topics which provide the motivation to study such a type of function spaces and which will be discussed in later sections.

Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^d$ be the Euclidean $d$-space, where $d \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the

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complex plane. As usual, $L_p(\mathbb{R}^d)$, with $0 < p \le \infty$, is the quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f \mid L_p(\mathbb{R}^d)\| = \left(\int_{\mathbb{R}^d} | f(x)|^p \, dx\right)^{1/p}$$

with the obvious modification if $p = \infty$.

Let $\mathbb{N}_0^d$ be the set of all multi-indices

$$\alpha = (\alpha_1, \ldots, \alpha_d) \quad \text{with} \quad \alpha_i \in \mathbb{N}_0 \quad \text{and} \quad |\alpha| = \sum_{i=1}^{d} \alpha_i.$$

Let $S(\mathbb{R}^d)$ be the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^d$. By $S'(\mathbb{R}^d)$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^d$. If $\alpha \in \mathbb{N}_0^d$ and $f \in S'(\mathbb{R}^d)$, we put

$$D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \, f,$$

where the derivatives are understood in the distributional sense.

**Definition 1.1** (Sobolev spaces)**.** Let $1 < p < \infty$ and let $(r_1, \ldots, r_d) \in \mathbb{N}^d$. We put

$$W_p^{r_1,\ldots,r_d}(\mathbb{R}^d) = \left\{ f \in L_p(\mathbb{R}^d) : D^\alpha f \in L_p(\mathbb{R}^d) \text{ for all } \alpha \in \mathbb{N}_0^d, \ \sum_{i=1}^{d} \frac{\alpha_i}{r_i} \le 1 \right\}$$

and

$$\|f \mid W_p^{r_1,\ldots,r_d}(\mathbb{R}^d)\| = \sum_{0 \le \frac{\alpha_1}{r_1} + \cdots + \frac{\alpha_d}{r_d} \le 1} \|D^\alpha f \mid L_p(\mathbb{R}^d)\|.$$

These are the anisotropic Sobolev spaces extensively studied by the Russian school of function spaces (see, for example, [51]). If $r_1 = \cdots = r_d = r \in \mathbb{N}$ then we obtain the standard isotropic Sobolev spaces on $\mathbb{R}^d$, denoted by $W_p^r(\mathbb{R}^d)$ in the following.

Next, let $\alpha^j \in \mathbb{N}^{n_j}$, where $j = 1, \ldots, d$ and $n_j \in \mathbb{N}$. We put

$$\alpha = (\alpha^1, \ldots, \alpha^d) \in \mathbb{N}_0^{n_1} \times \cdots \times \mathbb{N}_0^{n_d}$$

and split

$$\mathbb{R}^{n_1 + \cdots + n_d} = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}.$$

**Definition 1.2** (Sobolev spaces with dominating mixed derivatives)**.** Let $1 < p < \infty$ and let $(r_1, \dots, r_d) \in \mathbb{N}^d$. We put

$$\begin{aligned} S_p^{r_1,\dots,r_d} W(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}) = \{ & f \in L_p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}) : \\ & D^\alpha f \in L_p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}) \text{ for all } \alpha,\ |\alpha^j| \le r_j,\ j = 1, \dots, d\} \end{aligned} \tag{1.1}$$

and

$$\|f \mid S_p^{r_1,\dots,r_d} W(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d})\| = \sum_{\substack{|\alpha^j| \le r^j, \\ j=1,\dots,d}} \|D^\alpha \mid L_p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d})\|.$$

**Remark 1.3.** Note that

$$f_1 \otimes \cdots \otimes f_d \in S_p^{r_1,\dots,r_d} W(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d})$$

if $f_j \in W_p^{r_j}(\mathbb{R}^{n_j})$ for all $j = 1, \dots, d$, where $f \otimes g$ stands for the tensor product of $f$ and $g$. This means that the above Sobolev spaces with dominating mixed derivatives are well adapted to tensor products of functions.

If $n_1 = \cdots = n_d = 1$ then Definition 1.2 suggests that the properties of the space should be determined by the mixed derivative

$$D^{(r_1,\dots,r_d)} f \in L_p(\mathbb{R}^d).$$

For example, in the case $d = 2$ it is well known (cf., e.g., [63]) that

$$\begin{aligned} \|f \mid W_p^{r_1,r_2}(\mathbb{R}^2)\| \sim \|f \mid L_p(\mathbb{R}^2)\| &+ \|D^{(r_1,0)} f \mid L_p(\mathbb{R}^2)\| \\ &+ \|D^{(0,r_2)} f \mid L_p(\mathbb{R}^2)\| \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} \|f \mid S_p^{r_1,r_2} W(\mathbb{R}^2)\| \sim \|f \mid L_p(\mathbb{R}^2)\| &+ \|D^{(r_1,0)} f \mid L_p(\mathbb{R}^2)\| \\ &+ \|D^{(0,r_2)} f \mid L_p(\mathbb{R}^2)\| + \|D^{(r_1,r_2)} f \mid L_p(\mathbb{R}^2)\| \end{aligned} \tag{1.3}$$

if $1 < p < \infty$. The situation is illustrated in Figures 1 and 2 below.

If $n_1 = \cdots = n_d = 1$ and $r_1 = \cdots = r_d = r \in \mathbb{N}$ then we shall write

$$S_p^r W(\mathbb{R}^d) = S_p^{r_1,\dots,r_d} W(\mathbb{R} \times \cdots \times \mathbb{R})$$

for shortness. Obviously, in this case the topological embeddings

$$W_p^{rd}(\mathbb{R}^d) \hookrightarrow S_p^r W(\mathbb{R}^d) \hookrightarrow W_p^r(\mathbb{R}^d)$$

hold true.

Similarly, spaces on the $d$-dimensional torus $\mathbb{T}^d$ and on domains $\Omega \subset \mathbb{R}^d$ can be defined. The spaces $S_p^r W(\mathbb{T}^d)$ have been introduced and used by Babenko in 1960 (see [4]) in the context of multivariate approximation. We shall comment upon that in Subsection 1.3.

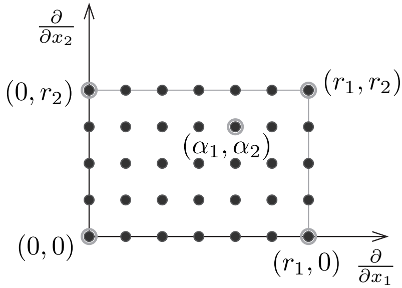


Figure 1: Dominating mixed

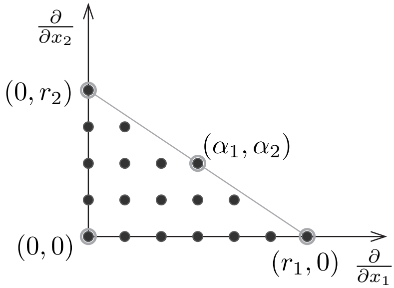


Figure 2: Anisotropic

**1.2. Fractional Sobolev spaces.** We introduce some further notation. If $\varphi \in S(\mathbb{R}^d)$ then

$$(\mathcal{F}\varphi)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-ix\xi}\, \varphi(x)\, dx, \quad \xi \in \mathbb{R}^d, \tag{1.4}$$

denotes the Fourier transform of $\varphi$. As usual, $\mathcal{F}^{-1}$ stands for the inverse Fourier transform, given by the right-hand side of (1.4) with $i$ in place of $-i$. Here $x\xi$ denotes the scalar product in $\mathbb{R}^d$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $S'(\mathbb{R}^d)$ by duality. Occasionally we shall write $\mathcal{F}_d$ and $\mathcal{F}_d^{-1}$ in place of $\mathcal{F}$ and $\mathcal{F}^{-1}$, respectively, to indicate the dependence on the dimension of the underlying Euclidean space. Moreover, if $x \in \mathbb{R}^d$ then we write

$$|x| = (x_1^2 + \cdots + x_d^2)^{1/2} \quad \text{and} \quad \langle x \rangle = (1 + |x|^2)^{1/2}.$$

**Definition 1.4** (Fractional Sobolev spaces)**.** Let $1 < p < \infty$ and let $r \in \mathbb{R}$. We put

$$H_p^r(\mathbb{R}^d) = \left\{ f \in S'(\mathbb{R}^d) : \mathcal{F}^{-1}[\langle \xi \rangle^r \mathcal{F} f] \in L_p(\mathbb{R}^d) \right\}$$

and

$$\| f \mid H_p^r(\mathbb{R}^d) \| = \| \mathcal{F}^{-1}[\langle \xi \rangle^r \mathcal{F} f] \mid L_p(\mathbb{R}^d) \|.$$

The spaces $H_p^r(\mathbb{R}^d)$ are called fractional Sobolev spaces or Bessel-potential spaces. They coincide with $W_p^r(\mathbb{R}^d)$ if $r \in \mathbb{N}$. Here we concentrated on the standard (isotropic) case. This will be sufficient for our later purposes. Next we describe the dominating mixed variant. To this end, we recall the splitting

$$\mathbb{R}^{n_1+\cdots+n_d} = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}.$$

Hence, $\xi \in \mathbb{R}^{n_1+\cdots+n_d}$ will be identified with $(\xi^1, \ldots, \xi^d)$ where $\xi^j \in \mathbb{R}^{n_j}$ for $j = 1, \ldots, d$.

**Definition 1.5** (Fractional Sobolev spaces with dominating mixed smoothness)**.** Let $1 < p < \infty$ and let $(r_1, \ldots, r_d) \in \mathbb{N}^d$. We put

$$\begin{aligned} S_p^{r_1,\ldots,r_d} H(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}) = \{ & f \in S'(\mathbb{R}^{n_1+\cdots+n_d}) : \\ & \mathcal{F}^{-1}[\langle \xi^1 \rangle^{r_1} \ldots \langle \xi^d \rangle^{r_d} \mathcal{F} f] \in L_p(\mathbb{R}^{n_1+\cdots+n_d})\} \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &\|f \mid S_p^{r_1,\ldots,r_d} H(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d})\| \\ &\qquad = \|\mathcal{F}^{-1}[\langle \xi^1 \rangle^{r_1} \ldots \langle \xi^d \rangle^{r_d} \mathcal{F} f] \mid L_p(\mathbb{R}^{n_1+\cdots+n_d})\|. \end{aligned}$$

As above, if $n_1 = \cdots = n_d = 1$ and $r_1 = \cdots = r_d = r \in \mathbb{R}$ then we shall write

$$S_p^r H(\mathbb{R}^d) = S_p^{r_1,\ldots,r_d} H(\mathbb{R} \times \cdots \times \mathbb{R})$$

for shortness. Spaces of this type have been introduced by Lizorkin and Nikol'skii in 1965 (see [47]). We have (equivalent norms):

$$W_p^r(\mathbb{R}^d) = H_p^r(\mathbb{R}^d) \quad \text{if} \quad r \in \mathbb{N},$$

$$\begin{aligned} S_p^{r_1,\ldots,r_d} W(R^{n_1} \times \cdots \times \mathbb{R}^{n_d}) &= S_p^{r_1,\ldots,r_d} H(R^{n_1} \times \cdots \times \mathbb{R}^{n_d}) \\ &\qquad \text{if } (r_1, \ldots, r_d) \in \mathbb{N}^d, \end{aligned} \tag{1.6}$$

$$S_p^r H(R^d) \subset H_p^r(\mathbb{R}^d) \subset S_p^{r/d} H(\mathbb{R}^d) \quad \text{if } r > 0. \tag{1.7}$$

The proofs are based on Fourier multiplier theorems. We refer to [85], [42] and the references given there.

In an analogous way we can define corresponding spaces on the $d$-dimensional torus $\mathbb{T}$. Details can be found in [63, Chap. 3] (the isotropic case).

**1.3. Multivariate approximation.** We want to discuss how we can benefit from dominating mixed smoothness. To this end, we shall consider an example related to the multivariate (high-dimensional) approximation. We deal with the periodic case and follow TEMLYAKOV (cf. [79]).

Let $1 < q < \infty$ and let $F$, $F \hookrightarrow L_q(\mathbb{T}^d)$, be a normed space. If $m \in \mathbb{N}$ then we denote by

$$\varphi_m(F, L_q) := \inf_{(u_j)_{j=1}^m \subset L_\infty(\mathbb{T}^d)} \sup_{\substack{f \in F \\ \|f|F\| \leq 1}} \Big\| f - \sum_{j=1}^{m} (f, u_j) u_j \;\Big|\; L_q(\mathbb{T}^d) \Big\|,$$

where the infimum is taken over all orthnormal systems $(u_j)_{j=1}^m$, the so-called orthowidth (or Fourier width) of order $m$ of $F$ in $L_q$. In other words, we ask which approximation error for an arbitrary $f \in F$ can be achieved in the metric of $L_q$ if we restrict ourselves to the approximation by partial sums of order $m$ with respect to orthonormal systems. Moreover, we are looking for an optimal system.

Let us recall the well-known embedding theorems of Sobolev type for the spaces $H_p^r(\mathbb{T}^d)$ and $S_p^r H(\mathbb{T}^d)$, respectively. We set $a_+ = \max(a, 0)$ for $a \in \mathbb{R}$. If

$$1 < p, q < \infty \quad \text{and} \quad r > d\Big(\frac{1}{p} - \frac{1}{q}\Big)_+ \tag{1.8}$$

then

$$H_p^r(\mathbb{T}^d) \hookrightarrow L_q(\mathbb{T}^d).$$

If

$$1 < p, q < \infty \quad \text{and} \quad r > \Big(\frac{1}{p} - \frac{1}{q}\Big)_+ \tag{1.9}$$

then

$$S_p^r H(\mathbb{T}^d) \hookrightarrow L_q(\mathbb{T}^d).$$

Proofs can be found in [63, Chap. 2 and 3]. The following results are proved in TEMLYAKOV [79]. Assume (1.8). Then

$$\varphi_m(H_p^r(\mathbb{T}^d), L_q(\mathbb{T}^d)) \asymp \Big(\frac{1}{m}\Big)^{\frac{r}{d} + \left(\frac{1}{p} - \frac{1}{q}\right)_+}. \tag{1.10}$$

Suppose (1.9). Then

$$\varphi_m(S_p^r H(\mathbb{T}^d), L_q(\mathbb{T}^d)) \asymp \Big(\frac{\log^{d-1} m}{m}\Big)^{r + \left(\frac{1}{p} - \frac{1}{q}\right)_+}. \tag{1.11}$$

On the other hand, if (1.8) is assumed then it follows from (1.11) that

$$\varphi_m(S_p^{r/d}H(\mathbb{T}^d), L_q(\mathbb{T}^d)) \asymp \left(\frac{\log^{d-1} m}{m}\right)^{\frac{r}{d}+\left(\frac{1}{p}-\frac{1}{q}\right)_+}.$$

**Remark 1.6.** One can ask how many Fourier coefficients have to be computed to reconstruct $f$ with accuracy $O(\varepsilon)$. We consider the case $p = q$ in the above situations.

If $f \in H_p^r(\mathbb{T}^d)$ then (1.10) shows that we need at least $O(\frac{1}{\varepsilon})^{d/r}$ coefficients. In particular, the number grows exponentially with respect to the dimension ("the curse of dimension"). The situation is improved drastically if additional dominating mixed smoothness is assumed. If $f \in S_p^r(\mathbb{T}^d) \subset H_p^r(\mathbb{T}^d)$ then, as a consequence of (1.11), $O(\frac{1}{\varepsilon})^{(1+\delta)/r}$ coefficients (for some $\delta > 0$) are sufficient. This is nearly the one-dimensional situation. The problem of high dimension is shifted to appropriate function spaces.

**Remark 1.7.** The optimal reconstruction by partial sums of orthonormal systems can be realized by means of approximation with respect to hyperbolic crosses. If $m = 2^\ell \ell^{d-1}$ and if (1.9) is assumed, then

$$\Big\| f - \sum_{k\in\mathcal{H}_\ell} \widehat{f}(k)e^{ikx} \;\Big|\; L_q(\mathbb{T}^d)\Big\| \le c\left(\frac{\log^{d-1} m}{m}\right)^{r+\left(\frac{1}{p}-\frac{1}{q}\right)_+} \|f \mid S_p^r H(\mathbb{T}^d)\|$$

for all $f \in S_p^r H(\mathbb{T}^d)$, where $\widehat{f}(k)$ denote the Fourier coefficients with respect to the $d$-dimensional trigonometric system and where

$$\mathcal{H}_\ell = \left\{k \in \mathbb{Z}^d : |k_1| \le 2^{j_1}, \dots, |k_d| \le 2^{j_d},\ j_1 + \cdots + j_d = \ell\right\} \tag{1.12}$$

stands for the dyadic hyperbolic cross of order $\ell$. The interrelations of approximation with respect to hyperbolic crosses and function spaces with dominating mixed smoothness are discussed in Section 3 in the bivariate nonperiodic case.

**Remark 1.8.** The last two remarks apply also to further types of high-dimensional approximation (hyperbolic spline and wavelet approximation, approximation with respect to sparse grids) which arise from the numerical solution of partial differential equations, data analysis and signal processing. As a consequence, there is an increasing interest in function spaces with dominating mixed smoothness in computational mathematics. Here, we only refer to Bungartz, Griebel [15], DeVore, Konyagin, Temlyakov [20], Donoho, Vetterli, DeVore, Daubechies [26], Griebel, Oswald, Schiekofer [34], Kamont [38], Nitsche [52], [53], Sickel [65], [66], Sickel, Ullrich [69], Ullrich [93], and Yserentant [97], [98]. See also the comments in Section 3.

**Remark 1.9.** Formulae (1.9) and (1.11) in particular show that the embedding operator $I : S^r_pH(\mathbb{T}^d) \to L_q(\mathbb{T}^d)$ is compact. The Fourier widths can be used to measure the degree of compactness of the embedding. There exist further quantities related to and measuring the compactness which have been studied in the context of spaces with dominating mixed smoothness and applications to the distribution of eigenvalues of operators and to stochastic processes. The behaviour of entropy numbers will be discussed in Subsection 5.3. We refer also to Belinsky [10], Dinh Dung [23], [24], [25], Temlyakov [75]–[79], Tikhomirov [80], and Vybíral [96].

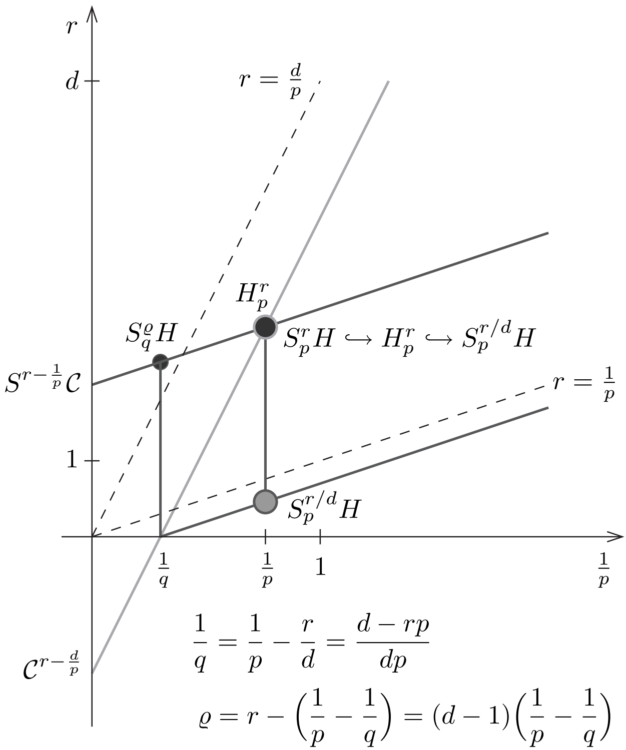


Figure 3: Sobolev embeddings

**1.4. Sobolev embeddings.** First let us recall well-known embeddings for (fractional) Sobolev spaces of functions defined on $\mathbb{R}^d$. Details can be found in [85] (isotropic case) and [63] (dominating mixed case). We have

$$H^r_p(\mathbb{R}^d) \hookrightarrow L_q(\mathbb{R}^d) \quad \text{iff} \quad r \geq d\Big(\frac{1}{p} - \frac{1}{q}\Big),\ 1 < p \leq q < \infty,$$

$$S_p^{r/d}H(\mathbb{R}^d) \hookrightarrow L_q(\mathbb{R}^d) \quad \text{iff} \quad r \geq d\Big(\frac{1}{p} - \frac{1}{q}\Big),\ 1 < p \leq q < \infty,$$

$$S_p^r H(\mathbb{R}^d) \hookrightarrow S_q^\varrho H(\mathbb{R}^d) \hookrightarrow S_q^{r-1/p}\mathcal{C}(\mathbb{R}^d) \quad \text{if} \quad r - \varrho = \frac{1}{p} - \frac{1}{q},\ 1 < p \leq q < \infty.$$

Here, $S_q^{r-1/p}\mathcal{C}(\mathbb{R}^d)$ denotes the Hölder-Zygmund space with dominating mixed smoothness which will be defined in Subsection 1.6. The situation is illustrated in the $(r, 1/p)$-diagram below (see Figure 3).

These embeddings can be extended to the more general spaces introduced in (1.1), (1.5) (cf. also (1.6)). This is proved in [60] (see also [61]). In particular, if

$$r_j \geq n_j\Big(\frac{1}{p} - \frac{1}{q}\Big), \quad j = 1, \dots, d, \qquad 1 < p \leq q < \infty$$

then

$$S_p^{r_1,\dots,r_d}H(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}) \hookrightarrow L_q(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}). \tag{1.13}$$

We have a second look at Sobolev embeddings. As already indicated in formula (1.2) (an)isotropic Sobolev spaces can be completely characterized by the properties of pure derivatives. Indeed, we have the equivalence

$$\|f \mid W_p^m(\mathbb{R}^d)\| \sim \|f \mid L_p(\mathbb{R}^d)\| + \sum_{j=1}^d \Big\| \frac{\partial^m f}{\partial x_j^m} \ \Big|\ L_p(\mathbb{R}^d) \Big\|, \quad 1 < p < \infty,\ m \in \mathbb{N}.$$

In the paper [1] by Adams in the context of embeddings the problem has been discussed whether or not a reduction to mixed derivatives of order $m$ is possible. More precisely, let us consider the set

$$M(m,d) := \big\{\alpha \in \mathbb{N}_0^d : \alpha_j = 0, 1,\ j = 1, \dots, d,\ |\alpha| = m\big\}.$$

If $\mathcal{M} \subset M(m,d)$ and $1 < p < \infty$ then we put

$$W_p^{\mathcal{M}}(\mathbb{R}^d) = \Big\{ \sum_{\alpha \in \mathcal{M}} \sum_{\beta \leq \alpha} \|D^\beta f \mid L_p(\mathbb{R}^d)\| < \infty \Big\}. \tag{1.14}$$

It means that we admit at most one derivative with respect to each direction. Of course, the space is much larger than the Sobolev space $W_p^m(\mathbb{R}^d)$. Nevertheless, one has (cf. e.g. [1]) the embedding

$$W_p^{\mathcal{M}}(\mathbb{R}^d) \hookrightarrow L_q(\mathbb{R}^d) \quad \text{if} \quad m \geq d\Big(\frac{1}{p} - \frac{1}{q}\Big),\ 1 < p \leq q < \infty \quad \text{and} \quad \mathcal{M} = M(m,d). \tag{1.15}$$

Let us mention that this result is a special case of embeddings for general Sobolev spaces as they can be found, for example, in the book [11] by Besov, Il'in and Nikol'skii. We discussed reduced Sobolev embeddings of the type (1.15), in particular, also in the limiting cases $m = d/p$ and $m = d/p + 1$ in [42], [43], and [44] in the context of spaces with dominating mixed smoothness. For example, in the above situation of (1.14) we even have

$$W_p^{\mathcal{M}}(\mathbb{R}^d) \hookrightarrow S_p^{m/d}H(\mathbb{R}^d) \hookrightarrow L_q(\mathbb{R}^d) \quad \text{if} \quad m \geq d\Big(\frac{1}{p} - \frac{1}{q}\Big),\ 1 < p \leq q < \infty \quad \text{and} \quad \mathcal{M} = M(m,d).$$

See Figure 4 for an illustration.

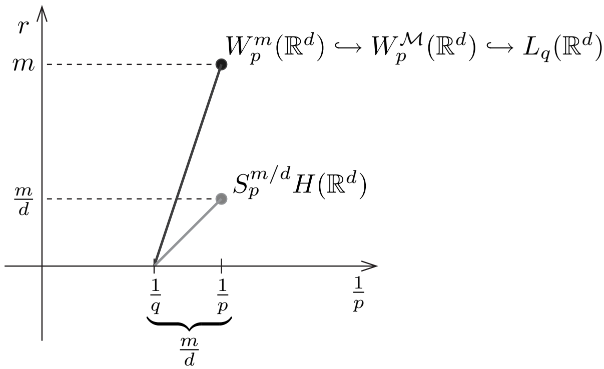


Figure 4

Moreover, further reductions to proper subsets $\mathcal{M} \subset M(m,d)$ are allowed. We give an example in the case $d = 4$.

**Example 1.10.** Let

$$\mathcal{M} = \{(1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1)\}.$$

Then

$$\|f \mid W_p^{\mathcal{M}}(\mathbb{R}^4)\| \sim \|f \mid S_p^{1,1}W(\mathbb{R}^2 \times \mathbb{R}^2)\|$$

and (1.13) implies

$$W_p^2(\mathbb{R}^4) \hookrightarrow W_p^{\mathcal{M}}(\mathbb{R}^4) = S_p^{1,1}H(\mathbb{R}^2 \times \mathbb{R}^2) \hookrightarrow L_q(\mathbb{R}^4),$$

where $\frac{1}{2} = \frac{1}{p} - \frac{1}{q}$, $1 < p < q < \infty$.

**1.5. Traces.** A new aspect in the theory of function spaces with dominating mixed smoothness concerns a trace problem. To be more precise, let us consider an example. Given a function $f$ belonging to some function space on $\mathbb{R}^2$ we are interested in the trace

$$\operatorname{tr} f := f(x,x)$$

on the diagonal. It is well-known ([85]) that in the isotropic case

$$\operatorname{tr}(H^r_p(\mathbb{R}^2)) = B^{r-1/p}_{p\,p}(\mathbb{R}) \quad \text{if} \quad r > \frac{1}{p}.$$

Here $B^{r-\frac{1}{p}}_{p\,p}(\mathbb{R})$ stands for the Besov space on $\mathbb{R}$ (see the next subsection). Moreover the trace does not exist if $r \le \frac{1}{p}$. The same holds true for the trace on any one-dimensional affine subspace of $\mathbb{R}^2$ because of translation and rotation invariance. New effects arise if we consider spaces with dominating mixed smoothness. This was first observed by Triebel in [86] and later studied in detail by Rodriguez (cf. [59]), Vybíral and Sickel (cf. [70] and [95]). We describe a partial result. A more detailed discussion will be given in Subsection 5.1. If

$$\frac{1}{2p} < r < \frac{1}{p}$$

then

$$\operatorname{tr}(S^r_pH(\mathbb{R}^2)) = B^{2r-1/p}_{p\,p}(\mathbb{R}) = \operatorname{tr}(H^{2r}_p(\mathbb{R}^2)).$$

The following interpretation seems to be natural. The additional dominating mixed smoothness leads to the existence of the trace, whereas the trace of $H^r_p(\mathbb{R}^2)$ does not exist. Moreover, the trace space of $S^r_pH(\mathbb{R}^2)$ is the same as for the smaller space $H^{2r}_p(\mathbb{R}^2)$ (cf. the embeddings (1.7)) and we may expect that for a certain range of parameters dominating mixed smoothness will replace higher isotropic smoothness. Recall also Subsection 1.3 for a similar effect.

**1.6. Nikol'skii-Besov spaces.** Dominating mixed smoothness properties can also be described in a classical way using mixed differences or mixed moduli of continuity. Starting point was the paper [50] by Nikol'skii. For simplicity and for transparency we restrict ourselves to the two-dimensional case.

First we define differences

$$\begin{aligned}
\Delta_{t,1} f &= \Delta^1_{t,1} f = f(x_1 + t, x_2) - f(x_1, x_2), \quad t \in \mathbb{R}, \\
\Delta_{t,2} f &= \Delta^1_{t,2} f = f(x_1, x_2 + t) - f(x_1, x_2), \\
\Delta^m_{t,i} f &= \Delta^1_{t,i}(\Delta^{m-1}_{t,i} f), \quad m = 2, \ldots,\ i = 1, 2, \\
\Delta^m_h f &= \Delta^{m_1}_{h_1,1}(\Delta^{m_2}_{h_2,2} f), \quad m = (m_1, m_2) \in \mathbb{N}^2,\ h \in \mathbb{R}^2.
\end{aligned}$$

**Definition 1.11** (Nikol'skii spaces with dominating mixed smoothness)**.** Let $1 \le p \le \infty$, $r_i > 0$ $(i = 1, 2)$ and let $m_i \in \mathbb{N}$, $m_i > r_i$ $(i = 1, 2)$. We put

$$\begin{aligned}
\|f \mid S^{r_1,r_2}_{p\,\infty} B(\mathbb{R}^2)\|_{\Delta,m} &= \|f \mid L_p(\mathbb{R}^2)\| \\
&\quad + \sum_{i=1}^{2} \sup_{|t| \le 1} |t|^{-r_i} \|\Delta^{m_i}_{t,i} f \mid L_p(\mathbb{R}^2)\| \\
&\quad + \sup_{\substack{|h_1| \le 1 \\ |h_2| \le 1}} |h_1|^{-r_1} h_2|^{-r_1} \|\Delta^{m_1}_{h_1,1}(\Delta^{m_2}_{h_2,2}) f \mid L_p(\mathbb{R}^2)\|
\end{aligned} \tag{1.16}$$

and

$$S^{r_1,r_2}_{p\,\infty} B(\mathbb{R}^2) = \{f \in L_p(\mathbb{R}^2) : \|f \mid S^{r_1,r_2}_{p\,\infty} B(\mathbb{R}^2)\|_{\Delta,m} < \infty\}$$

Formula (1.16) can be rewritten as

$$\begin{aligned}
\|f \mid S^{r_1,r_2}_{p\,\infty} B(\mathbb{R}^2)\|_{\Delta,m} &= \|f \mid B^{r_1,r_2}_{p\,\infty}(\mathbb{R}^2)\|_{\Delta,m} \\
&\quad + \sup_{\substack{|h_1| \le 1 \\ |h_2| \le 1}} |h_1|^{-r_1} h_2|^{-r_1} \|\Delta^{m_1}_{h_1,1}(\Delta^{m_2}_{h_2,2}) f \mid L_p(\mathbb{R}^2)\|.
\end{aligned}$$

Here $\|f \mid B^{r_1,r_2}_{p\,\infty}(\mathbb{R}^2)\|_{\Delta,m}$ stands for a norm in the anisotropic Nikol'skii space $B^{r_1,r_2}_{p\,\infty}(\mathbb{R}^2)$. It means that we have added a mixed Hölder condition to describe a new type of dominating mixed smoothness. This is of course in analogy to Sobolev spaces (see also (1.2) and (1.3). If $p = \infty$ this means that we additionally require the existence of a constant $c$ such that

$$|\Delta^{m_1}_{h_1,1}(\Delta^{m_2}_{h_2,2}) f(x)| \le c|h_1|^{r_1} |h_2|^{r_2}$$

for all $x \in \mathbb{R}^2$ and all $h \in [-1, 1]^2$. The spaces $S^{r_1,r_2}\mathcal{C}(\mathbb{R}^2) := S^r_{p\,\infty} B(\mathbb{R}^2)$ are called Hölder-Zygmund spaces with dominating mixed smoothness. Note that the definition does not depend on the size of $m$ (equivalent norms). If $r_1 = r_2 = r > 0$ then we write $S^r_{p\,\infty} B(\mathbb{R}^2)$ in place of $S^{r_1,r_2}_{p\,\infty} B(\mathbb{R}^2)$. Analogously $S^r\mathcal{C}(\mathbb{R}^2)$ is defined.

The above scale of spaces has been extended in the sense of Besov by Amanov [2] in 1965 (see also his book [3]).

**Definition 1.12** (Besov spaces with dominating mixed smoothness)**.** Let $1 \le p \le \infty$, $1 \le q < \infty$, $r_i > 0$ $(i = 1, 2)$ and let $m_i \in \mathbb{N}$, $m_i > r_i$ $(i = 1, 2)$. We put

$$\begin{aligned} \|f \mid S^{r_1,r_2}_{p\,q}B(\mathbb{R}^2)\|_{\Delta,m} &= \|f \mid L_p(\mathbb{R}^2)\| \\ &+ \sum_{i=1}^{2} \Big( \int_{-1}^{1} \| \, |t|^{-r_i} \Delta^{m_i}_{t,i} f \mid L_p\|^q \, \frac{dt}{|t|} \Big)^{1/q} \\ &+ \Big( \int_{-1}^{1} \int_{-1}^{1} \| \, |h_1|^{-r_1} |h_2|^{-r_2} \Delta^{m_1}_{h_1,1} (\Delta^{m_2}_{h_2,2}) f \mid L_p\|^q \, \frac{dh_1}{|h_1|} \, \frac{dh_2}{|h_2|} \Big)^{1/q} . \end{aligned} \tag{1.17}$$

As above we shall write $S^r_{p\,q}B(\mathbb{R}^2)$ if $r_1 = r_2 = r > 0$. Clearly, by definition the embedding

$$S^{r_1,r_2}_{p\,q}B(\mathbb{R}^2) \hookrightarrow B^{r_1,r_2}_{p\,q}(\mathbb{R}^2)$$

holds. All spaces can be extended to functions defined on $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}$ in the spirit of Definition 1.2. This is more or less a matter of clever notation. If we put $m_i = r_i = 1$ in (1.16) and (1.17), respectively, then the resulting spaces are called spaces of bounded mixed variation. These spaces are widely used in high-dimensional approximation and computational mathematics (cf., e.g., the paper [74] by Strömberg).

## 2. Littlewood-Paley analysis – Fourier analytical approach

**2.1. Littlewood-Paley theorems.** The aim of this section is to give a unified Fourier analytical approach to the various spaces with dominating mixed smoothness introduced in Section 1. Moreover, all these spaces can be extended in a natural way to parameters $p$ and $q$ which are less than 1. This describes the state of art of the eighties. Our general reference is Chapter 2 of the book [63] (for isotropic spaces see [85]).

Starting point is the Littlewood-Paley analysis of functions and distributions. First we recall Littlewood-Paley decompositions of $L_p$-spaces for $1 < p < \infty$. We use standard notation introduced in Section 1.

Let $\chi_0 : \mathbb{R}^m \mapsto \mathbb{R}$ $(m \in \mathbb{N})$ be the characteristic function of $[-1, 1]^m$. We put

$$\chi(\xi) = \chi_0(\xi) - \chi_0(2\xi) \qquad \text{and} \qquad \chi_j(\xi) = \chi(2^{-j}\xi), \quad j \in \mathbb{N},\ \xi \in \mathbb{R}^m. \tag{2.1}$$

Then $(\chi_j)_{j\in\mathbb{N}_0}$ represents a dyadic resolution of unity of $\mathbb{R}^m$ satisfying

$$\sum_{j=0}^{\infty} \chi_j(\xi) = 1, \quad \xi \in \mathbb{R}^m,$$
$$\operatorname{supp} \chi_j = \{\xi : 2^{j-1} \le \max_i |\xi_i| \le 2^j\}, \quad j \in \mathbb{N}.$$

Let $(\widetilde{\chi}_k)_{k\in\mathbb{N}_0}$ be the corresponding dyadic resolution of unity of $\mathbb{R}^n$ $(n \in \mathbb{N})$. Taking the tensor products $\chi_j \otimes \widetilde{\chi}_k$ $(j, k \in \mathbb{N}_0)$ we get a dyadic resolution of unity of $\mathbb{R}^m \times \mathbb{R}^n$, i.e. we have

$$\begin{gathered}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \chi_j(\xi)\widetilde{\chi}_k(\eta) = 1, \quad (\xi,\eta) \in \mathbb{R}^m \times \mathbb{R}^n, \\ \operatorname{supp} \chi_j \otimes \widetilde{\chi}_k = \operatorname{supp} \chi_j \times \operatorname{supp} \widetilde{\chi}_k.\end{gathered} \tag{2.2}$$

See Figure 5 below.

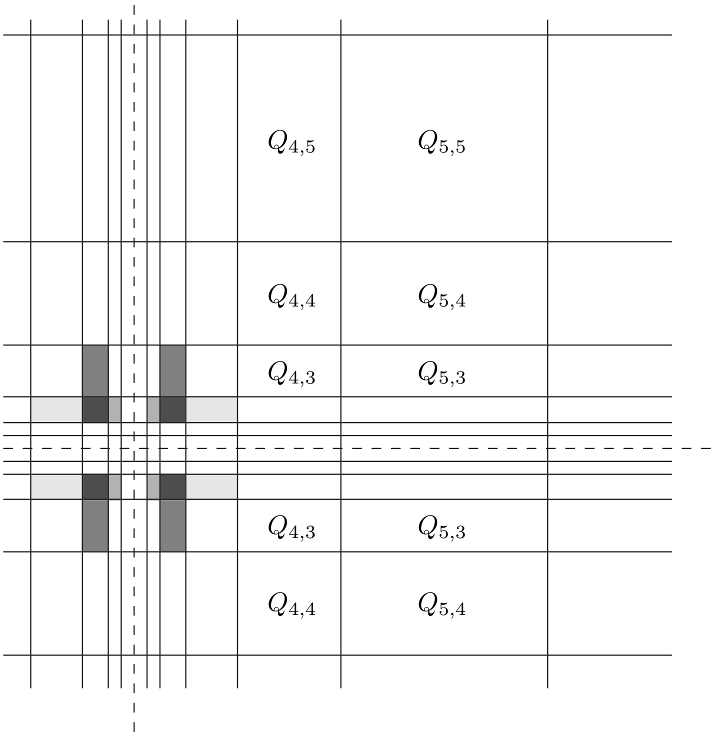


Figure 5: Dyadic partition of $\mathbb{R}^m \times \mathbb{R}^n$ , $Q_{j,\,k} = \operatorname{supp} \chi_j \otimes \widetilde{\chi}_k$

We put

$$f_j^{\chi} = \mathcal{F}_m^{-1}[\chi_j \mathcal{F}_m f], \quad j \in \mathbb{N}_0,$$

$$f_{j,k}^{\chi\widetilde{\chi}} = \mathcal{F}_{m+n}^{-1}[\chi_j \otimes \widetilde{\chi}_k \mathcal{F}_{m+n} f], \quad (j,k) \in \mathbb{N}_0^2. \tag{2.3}$$

These are well-defined smooth functions if $f \in S'(\mathbb{R}^m)$ and $f \in S'(\mathbb{R}^{m+n})$, respectively. For functions belonging to $L_p$ $(1 < p < \infty)$ we use the standard limiting argument.

**Proposition 2.1** (Littlewood-Paley theorem)**.** *Let* $1 < p < \infty$*. Then*

$$\|f \mid L_p(\mathbb{R}^m)\| \sim \Big\| \Big( \sum_{j=0}^{\infty} |f_j^{\chi}(\cdot)|^2 \Big)^{1/2} \Bigm| L_p(\mathbb{R}^m) \Big\|$$

*and*

$$\|f \mid L_p(\mathbb{R}^{m+n})\| \sim \Big\| \Big( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |f_{j,k}^{\chi,\widetilde{\chi}}(\cdot,\cdot)|^2 \Big)^{1/2} \Bigm| L_p(\mathbb{R}^{m+n}) \Big\|. \tag{2.4}$$

In the same way as above we construct smooth dyadic resolutions of unity starting with a real-valued function $\varphi \in S(\mathbb{R}^m)$ which has the properties

$$\varphi_0(\xi) = 1 \quad \text{if} \quad |\xi| \le 1 \qquad \text{and} \qquad \varphi_0(\xi) = 0 \quad \text{if} \quad |\xi| \ge 3/2.$$

We put

$$\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi) \quad \text{and} \quad \varphi_j(\xi) = \varphi(2^{-j}\xi), \quad j \in \mathbb{N}. \tag{2.5}$$

Then

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1, \quad \xi \in \mathbb{R}^m, \tag{2.6}$$

and

$$\operatorname{supp} \varphi_j \subset \big\{ \xi \in \mathbb{R}^m : \ 2^{j-1} \le |\xi| \le \tfrac{3}{2} 2^j \big\}, \quad j \in \mathbb{N} \tag{2.7}$$

(see Figure 6).

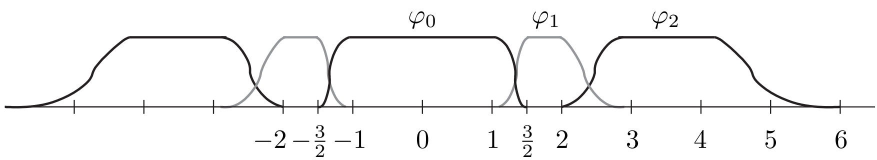


Figure 6: Smooth dyadic resolution of unity

We shall write $\Phi(\mathbb{R}^m)$ for the set of all such resolutions of unity on $\mathbb{R}^m$. If $(\varphi_j)_j \in \Phi(\mathbb{R}^m)$ and $(\widetilde{\varphi}_k)_k \in \Phi(\mathbb{R}^n)$ then

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varphi_j(\xi)\widetilde{\varphi}_k(\eta) = 1, \quad (\xi,\eta) \in \mathbb{R}^m \times \mathbb{R}^n.$$

Again we have a product type resolution of unity of $\mathbb{R}^m \times \mathbb{R}^n$ with some overlapping in contrast to (2.2). We define

$$f_j^{\varphi} := \mathcal{F}_m^{-1}[\varphi_j \mathcal{F}_m f], \quad f \in \mathcal{S}'(\mathbb{R}^m), \tag{2.8}$$

$$f_{j,k}^{\varphi,\widetilde{\varphi}} := \mathcal{F}_{m+n}^{-1}[\varphi_j \otimes \widetilde{\varphi}\mathcal{F}_{m+n} f], \quad f \in \mathcal{S}'(\mathbb{R}^{m+n}). \tag{2.9}$$

By the Paley-Wiener-Schwartz theorem these functions are well-defined entire analytic functions of exponential type for all $f \in S'(R^m)$ and $f \in S'(\mathbb{R}^{m+n})$, respectively. Hence they make sense pointwise.

**Proposition 2.2** (Littlewood-Paley theorem)**.** *Let* $1 < p < \infty$*. Then*

$$\|f \mid L_p(\mathbb{R}^m)\| \sim \Big\|\Big(\sum_{j=0}^{\infty} |f_j^{\varphi}(\cdot)|^2\Big)^{1/2} \Bigm| L_p(\mathbb{R}^m)\Big\|, \tag{2.10}$$

$$\|f \mid L_p(\mathbb{R}^{m+n})\| \sim \Big\|\Big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |f_{j,k}^{\varphi,\widetilde{\varphi}}(\cdot,\cdot)|^2\Big)^{1/2} \Bigm| L_p(\mathbb{R}^{m+n})\Big\|. \tag{2.11}$$

For simplicity we restricted ourselves to the splitting $\mathbb{R}^m \times \mathbb{R}^n$. However, the above statements are also valid for the general splitting $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}$ $(d \in \mathbb{N},\ d > 2)$, too.

**Remark 2.3.** As a consequence the norms on the right-hand sides of (2.10) and (2.11) are equivalent to each other for different $\varphi$ and $\widetilde{\varphi}$, respectively. An alternative proof of this statement can be given using Fourier multipliers for vector-valued $L_p(\ell_2)$-spaces of entire analytic functions. Most important is that we used smooth dyadic resolutions. Moreover, the equivalence can be extended to values $p$, $0 < p \le 1$. This is due to Peetre ([55]) and leads to characterizations of non-homogeneous Hardy spaces. We refer also to [85] and [63]. We comment on this also in Subsection 2.3.

**2.2. Dominating mixed derivatives and differences.** Recall that we have introduced (fractional) Sobolev spaces and Nikol'skii-Besov spaces $S_p^{r_1,r_2}H(\mathbb{R}^m \times \mathbb{R}^n)$ and $S_{p\,q}^{r_1,r_2}B(\mathbb{R}^2)$ with dominating mixed smoothness in Subsections 1.2 and 1.6, respectively. Next we shall give a characterization of these spaces by means of dyadic decompositions in the sense of the previous subsection. This leads us to the so-called Littlewood-Paley analysis of functions (distributions) belonging to such spaces.

**Theorem 2.4** (Littlewood-Paley analysis – Sobolev spaces)**.** *Let* $1 < p < \infty$, $(r_1, r_2) \in \mathbb{R}^2$, *and let* $f_{j,k}^{\chi,\widetilde{\chi}}$ *and* $f_{j,k}^{\varphi,\widetilde{\varphi}}$ *be as in* (2.3) *and* (2.9)*. Then,*

$$\|f \mid S_p^{r_1,r_2}H(\mathbb{R}^m \times \mathbb{R}^n)\| \sim \Big\| \Big( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |2^{jr_1+kr_2} f_{j,k}^{\chi,\widetilde{\chi}}(\cdot,\cdot)|^2 \Big)^{1/2} \Big| L_p(\mathbb{R}^{m+n}) \Big\|$$
$$\sim \Big\| \Big( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |2^{jr_1+kr_2} f_{j,k}^{\varphi,\widetilde{\varphi}}(\cdot,\cdot)|^2 \Big)^{1/2} \Big| L_p(\mathbb{R}^{m+n}) \Big\|.$$

Characterizations of this type for $p \neq 2$ and (an)isotropic Sobolev spaces are due to papers by Lizorkin (characteristic functions) and Triebel (smooth functions) in 1972 (see [45], [46] and [81]). Moreover they introduced new spaces replacing the $\ell_2$-norm on the right-hand side by the $\ell_q$-norm with $1 < q < \infty$. We refer to [83] for details. In the above context of spaces with dominating mixed smoothness one may look at [63] and the references given there. The proof is based on Propositions 2.1 and 2.2 and on vector-valued Fourier multipliers.

**Theorem 2.5** (Littlewood-Paley analysis – Besov spaces)**.** *Let* $f_{j,k}^{\chi,\widetilde{\chi}}$ *and* $f_{j,k}^{\varphi,\widetilde{\varphi}}$ *be as in* (2.3) *and* (2.9) *and let* $0 < q \leq \infty$, $0 < r_1 < m_1$, $0 < r_2 < m_2$.

(i) *If* $1 < p < \infty$, *then*

$$\|f \mid S_{p\,q}^{r_1,r_2}B(\mathbb{R}^2)\|_{\Delta,m} \sim \Big( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \|2^{jr_1+kr_2} f_{j,k}^{\chi,\widetilde{\chi}} \mid L_p(\mathbb{R}^2)\|^q \Big)^{1/q}. \qquad (2.12)$$

(ii) *If* $1 \leq p \leq \infty$, *then*

$$\|f \mid S_{p\,q}^{r_1,r_2}B(\mathbb{R}^2)\|_{\Delta,m} \sim \Big( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \|2^{jr_1+kr_2} f_{j,k}^{\varphi,\widetilde{\varphi}} \mid L_p(\mathbb{R}^2)\|^q \Big)^{1/q}$$

(*with* $\sup_{j,k} \ldots$ *in place of* $\big(\sum_{j,k}(\ldots)^q\big)^{1/q}$ *if* $q = \infty$).

The proof can be found in [63]. The idea to use dyadic decompositions by characteristic functions to characterize Nikol'skii-Besov spaces is due to Lizorkin (1965). Equivalence (2.12) is sometimes called the Lizorkin representation. Nikol'skii (1966) observed that the cases $p = 1$ and $p = \infty$ can be characterized similarly using dyadic decompositions generated by smooth functions (he used differences of de La Vallée-Poussin means). We refer to Nikol'skii's book [51]. Besov spaces with dominating mixed smoothness, in particular embedding theorems, have been studied based on the approximation by entire analytic functions by Nikol'skii (1963, $q = \infty$, see [50]) and Amanov (1965, $1 \le q < \infty$, see [2]). We refer also to Amanov's book [3] in this respect. The decomposition technique allowed also to extend the spaces to negative (non-positive) smoothness as in the case of fractional Sobolev spaces. Peetre (1973) discovered how to extend the theory of Besov spaces to parameters $0 < p < 1$ in a natural way. The main tools are (scalar) Fourier multipliers and inequalities of Plancherel-Pólya-Nikol'skii type for $L_p$-spaces of entire analytic functions (see [54] and his book [56]).

**2.3. Besov and Lizorkin-Triebel spaces.** In the previous subsections we indicated how to measure smoothness via decomposition techniques and Fourier analysis and how to include concrete classical (both Sobolev and Besov) spaces. Now we are prepared to present a unified approach. Although we tacitly assume that the reader is familiar with the theory of isotropic spaces, for convenience and for later purposes we shall describe both the isotropic and the dominating mixed case. In the latter one we follow [63], Chapter 2 and restrict ourselves to the splitting $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ for simplicity. The isotropic theory, which always was the forerunner, is treated in [85].

**Definition 2.6** (Besov and Lizorkin-Triebel spaces)**.** Let $\{\varphi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$ be a smooth dyadic resolution of unity according to (2.5)–(2.7) and let $f_j^{\varphi}$ be as in (2.8).

(i) Let $0 < p \le \infty$, $0 < q \le \infty$, $r \in \mathbb{R}$,

$$\|f \mid B^r_{p\,q}(\mathbb{R}^n)\|_{\varphi} = \Big(\sum_{j=0}^{\infty} 2^{jrq} \|f_j^{\varphi} \mid L_p(\mathbb{R}^n)\|^q\Big)^{1/q} \tag{2.13}$$

if $0 < q < \infty$ and

$$\|f \mid B^r_{p\,\infty}(\mathbb{R}^n)\|_{\varphi} = \sup_{j=0,1,\dots} 2^{jr} \|f_j^{\varphi} \mid L_p(\mathbb{R}^n)\|. \tag{2.14}$$

Then

$$B^r_{p\,q}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f \mid B^r_{p\,q}(\mathbb{R}^n)\|_{\varphi} < \infty\}. \tag{2.15}$$

(ii) Let $0 < p < \infty$, $0 < q \le \infty$, $r \in \mathbb{R}$ and

$$\|f \mid F^r_{p\,q}(\mathbb{R}^n)\|_{\varphi} = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jrq} |f_j^{\varphi}(\cdot)|^q\Big)^{1/q} \ \Big|\ L_p(\mathbb{R}^n) \Big\| \tag{2.16}$$

(with the modification as in (i) if $q=\infty$). Then

$$F^r_{p\,q}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f \mid F^r_{p\,q}(\mathbb{R}^n)\|_\varphi < \infty\}. \tag{2.17}$$

We refer to [85], [87], [89], [90] for a systematic treatment of these spaces. Due to the papers [45], [46] and [81] the spaces $F^r_{p\,q}(\mathbb{R}^n)$ are called Lizorkin-Triebel spaces. The extension to $p<1$ is due to Peetre 1975 (see [55] and his book [56] as well as Triebel [82], [84]).

Next we give the definitions of the spaces with dominating mixed smoothness.

**Definition 2.7** (Besov and Lizorkin-Triebel spaces – dominating mixed smoothness)**.** Let $\{\varphi_j\}_{j=0}^\infty \in \Phi(\mathbb{R}^m)$ and $\{\widetilde{\varphi}_j\}_{j=0}^\infty \in \Phi(\mathbb{R}^n)$ be smooth dyadic resolutions of unity according to (2.5)–(2.7) and let $f^{\varphi,\widetilde{\varphi}}_{j,k}$ be as in (2.9).

(i) Let $0<p\le\infty$, $0<q\le\infty$, $(r_1,r_2)\in\mathbb{R}^2$ and

$$\begin{aligned}\|f \mid S^{r_1,r_2}_{p\,q}B(\mathbb{R}^m\times\mathbb{R}^n)\|_{\varphi,\widetilde{\varphi}}& \\ &= \Big(\sum_{k=0}^\infty\sum_{j=0}^\infty 2^{(jr_1+kr_2)q}\|f^{\varphi,\widetilde{\varphi}}_{j,k} \mid L_p(\mathbb{R}^{m+n})\|^q\Big)^{1/q}\end{aligned} \tag{2.18}$$

(with the modification as in Definition 2.6 if $q=\infty$). Then

$$\begin{aligned}&S^{r_1,r_2}_{p\,q}B(\mathbb{R}^m\times\mathbb{R}^n)\\ &\qquad := \{f\in\mathcal{S}'(\mathbb{R}^{m+n}) : \|f \mid S^{r_1,r_2}_{p\,q}B(\mathbb{R}^m\times\mathbb{R}^n)\|_{\varphi,\widetilde{\varphi}}<\infty\}.\end{aligned} \tag{2.19}$$

(ii) Let $0<p<\infty$, $0<q\le\infty$, $(r_1,r_2)\in\mathbb{R}^2$ and

$$\begin{aligned}\|f \mid S^{r_1,r_2}_{p\,q}F(\mathbb{R}^m\times\mathbb{R}^n)\|_{\varphi,\widetilde{\varphi}}& \\ &= \Big\|\Big(\sum_{k=0}^\infty\sum_{j=0}^\infty 2^{(jr_1+kr_2)q}|f^{\varphi,\widetilde{\varphi}}_{j,k}(\cdot,\cdot)|^q\Big)^{1/q} \Bigm| L_p(\mathbb{R}^{m+n})\Big\|\end{aligned}$$

(with the modification as in Definition 2.6 if $q=\infty$). Then

$$\begin{aligned}&S^{r_1,r_2}_{p\,q}F(\mathbb{R}^m\times\mathbb{R}^n)\\ &\qquad := \{f\in\mathcal{S}'(\mathbb{R}^{m+n}) : \|f \mid S^{r_1,r_2}_{p\,q}F(\mathbb{R}^m\times\mathbb{R}^n)\|_{\varphi,\widetilde{\varphi}}<\infty\}.\end{aligned}$$

All spaces introduced in Definition 2.6 and and Definition 2.7 are quasi-Banach spaces equipped with the corresponding quasi-norms and independent of the underlying resolutions of unity in the sense of equivalent quasi-norms. We refer to [85] and [63] for the details. It is easy to see that

$$\|f\otimes g \mid S^{r_1,r_2}_{p\,q}A(\mathbb{R}^m\times\mathbb{R}^n)\|_{\varphi,\widetilde{\varphi}} = \|f \mid A^{r_1}_{p\,q}(\mathbb{R}^m)\|_\varphi\,\|g \mid A^{r_2}_{p\,q}(\mathbb{R}^n)\|_{\widetilde{\varphi}}, \tag{2.20}$$

where $A$ stands for $B$ or $F$ and $\otimes$ denotes the tensor product of distributions $f \in \mathcal{S}'(\mathbb{R}^m)$ and $g \in \mathcal{S}'(\mathbb{R}^n)$. For brevity we shall write $S^r_{p\,q}A(\mathbb{R}^m \times \mathbb{R}^n)$ if $r_1 = r_2 = r$. The extension of Definition 2.7 to the splitting $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}$, $d > 2$, suggests itself.

There exist several attempts to deal with the basic properties of the above spaces and further (anisotropic, generalized smoothness) spaces from a unified (decomposition) point of view. We refer to Triebel [82], Stöckert, Triebel [72] and Gol'dman [33].

**Remark 2.8.** The investigation of the above spaces for the full range of parameters $p$ and $q$ is based on pointwise estimates and scalar as well as vector-valued inequalities for Peetre and Hardy-Littlewood maximal functions. We roughly sketch an example adapted to the case of spaces with dominating mixed smoothness. Let $\varphi$, $\widetilde{\varphi}$, $\psi$, $\widetilde{\psi}$, $\varphi_0$, $\widetilde{\varphi}_0$, $\psi_0$, $\widetilde{\psi}_0$ be (sufficiently) smooth functions. Let $\varphi_j = \varphi(2^{-j}\cdot)$ and let $\widetilde{\varphi}_k$, $\psi_j$ and $\widetilde{\psi}_k$ be defined analogously. We use the notation $f^{\varphi,\widetilde{\varphi}}_{j,k}$ as in (2.9) and define the Peetre maximal functions as

$$\big(f^{\varphi,\widetilde{\varphi}}_{j,k}\big)^\star_{a,b}(x,y) := \sup_{\substack{u\in\mathbb{R}^m\\ v\in\mathbb{R}^n}} \frac{|(f^{\varphi,\widetilde{\varphi}}_{j,k})(x-u,y-v)|}{(1+|2^j u|)^a(1+|2^k v|)^b}$$

for $(j,k) \in \mathbb{N}_0^2$ and $(a,b) \in \mathbb{R}^2$. Then we have the pointwise estimates

$$\big(f^{\psi,\widetilde{\psi}}_{j,k}\big)^\star_{a,b}(x,y) \le c\big(M_2(M_1|f^{\varphi,\widetilde{\varphi}}_{j,k}|^\mu)(x,y)\big)^{1/\mu}$$

for appropriate $\varphi$, $\widetilde{\varphi}$, $\psi$, $\widetilde{\psi}$, $a > m/\mu$, $b > n/\mu$, $0 < \mu < \infty$. Here $M_1$ and $M_2$ denote the Hardy-Littlewood maximal function with respect to the first and second variable, respectively. Choosing $\max(\frac{m}{a},\frac{n}{b}) < \mu < \min(p,q)$ and applying the vector-valued maximal inequality by Feffermann, Stein [29] as well as its extended version by Bagby [5] we obtain

$$\big\|\big(f^{\psi,\widetilde{\psi}}_{j,k}\big)^\star_{a,b}(x,y) \mid L_p(\ell_q)\big\| \le c\big\|\big(f^{\varphi,\widetilde{\varphi}}_{j,k}\big)^\star_{a,b}(x,y) \mid L_p(\ell_q)\big\|$$

for $0 < p < \infty$, $0 < q \le \infty$, $a > \frac{m}{\min(p,q)}$, $b > \frac{n}{\min(p,q)}$. We refer to Vybíral [94], [96]. For extensions to mixed $L_p$-quasinorms see also [60], [62], [63] and Bazarkhanov [6].

**Remark 2.9.** Fractional Sobolev spaces are special cases of Lizorkin-Triebel spaces. As a consequence of Theorem 2.4, we have

$$S^{r_1,r_2}_{p\,2}F(\mathbb{R}^m \times \mathbb{R}^n) = S^{r_1,r_2}_{p}H(\mathbb{R}^n \times \mathbb{R}^n)$$

for $1 < p < \infty$ and $(r_1, r_2) \in \mathbb{R}^2$. Recall also that in the isotropic case

$$F^r_{p\,2}(\mathbb{R}^n) = H^r_p(\mathbb{R}^n), \quad 1 < p < \infty,\ r \in \mathbb{R}.$$

**Remark 2.10.** We have defined Nikol'skii-Besov spaces with dominating mixed smoothness in Definitions 1.11 and 1.12 by means of iterated mixed differences. Theorem 2.5 shows that these spaces coincide with the spaces introduced in Definition 2.7(i). It is possible to give characterizations of the spaces $S^{r_1,r_2}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n)$ and also of the spaces $S^{r_1,r_2}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n)$ by means of differences for a larger range of parameters $(r_1, r_2)$, $p$, $q$, in particular, for $p < 1$. We refer to [63] and to the recent results by Ullrich [92]. In the case of Besov spaces (for simplicity we choose $r_1 = r_2 = r$, $n = m$) the range of admissible parameters is $r > \sigma_p$, $0 < q \leq \infty$, (see Figure 7 below).

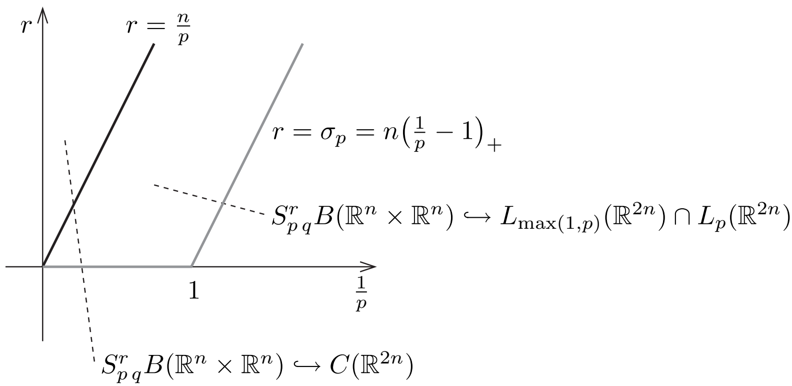


Figure 7

**Remark 2.11.** The following elementary embeddings can be proved easily (see [63]):

$$S^{r_1,r_2}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n) \hookrightarrow S^{r_1,r_2}_{p\,u}B(\mathbb{R}^m \times \mathbb{R}^n),$$
$$S^{r_1,r_2}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n) \hookrightarrow S^{r_1,r_2}_{p\,u}F(\mathbb{R}^m \times \mathbb{R}^n),$$

if $0 < q \leq \infty$,

$$S^{r_1,r_2}_{p\ \min(p,q)}B(\mathbb{R}^m \times \mathbb{R}^n) \hookrightarrow S^{r_1,r_2}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n) \hookrightarrow S^{r_1,r_2}_{p\ \max(p,q)}B(\mathbb{R}^m \times \mathbb{R}^n),$$

and

$$S^{\varrho_1,\varrho_2}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n) \hookrightarrow S^{r_1,r_2}_{p\,v}B(\mathbb{R}^m \times \mathbb{R}^n),$$
$$S^{\varrho_1,\varrho_2}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n) \hookrightarrow S^{r_1,r_2}_{p\,v}F(\mathbb{R}^m \times \mathbb{R}^n),$$

if $r_1 < \varrho_1$, $r_2 < \varrho_2$ and $0 < v \leq \infty$.

## 3. Approximation with respect to hyperbolic crosses

**3.1. Approximation spaces.** We follow [64]. We restrict ourselves to the case $d=2$. With respect to the notation in the previous section this means $n=m=1$, $r_1=r_2=r$ and $\mathcal{F}_2=\mathcal{F}$, $\mathcal{F}_2^{-1}=\mathcal{F}^{-1}$. Let $\chi$ have the meaning of (2.1). As in (2.3) we define

$$f^{\chi}_{j,k}(x,y)=\mathcal{F}^{-1}[\chi_j\otimes\chi_k\mathcal{F}f](x,y),\quad (j,k)\in\mathbb{N}_0^2.$$

The results of this section are mainly based on the following Lizorkin-type characterization of the spaces with dominating mixed smoothness which can be found in [63, Chap. 2] (cf. also Theorem 2.4 and 2.5).

**Proposition 3.1** (Lizorkin representation)**.** *Let $-\infty<r<\infty$ and $1<p<\infty$.*
(i) *If $0\le q\le\infty$, then*

$$S^r_{p\,q}B(\mathbb{R}^2)=\Big\{f\in\mathcal{S}'(\mathbb{R}^2):\\ \|f\mid S^r_{p\,q}B(\mathbb{R}^2)\|_\chi=\Big(\sum_{j=0}^\infty\sum_{k=0}^\infty 2^{(j+k)rq}\,\|f^\chi_{j,k}\mid L_p(\mathbb{R}^2)\|^q\Big)^{1/q}<\infty\Big\}.$$

(ii) *If $1<q<\infty$, then*

$$S^r_{p\,q}F(\mathbb{R}^2)=\Big\{f\in\mathcal{S}'(\mathbb{R}^2):\\ \|f\mid S^r_{p\,q}F\,(\mathbb{R}^2)\|_\chi=\Big\|\Big(\sum_{j=0}^\infty\sum_{k=0}^\infty 2^{(j+k)rq}|f^\chi_{j,k}|^q\Big)^{1/q}\Bigm| L_p(\mathbb{R}^2)\Big\|<\infty\Big\}. \tag{3.1}$$

We are led to the approximation with respect to the hyperbolic cross in a natural way by the observation

$$\begin{aligned} f&=\sum_{j=0}^\infty\sum_{k=0}^\infty f^\chi_{j,k}=\lim_{\ell\to\infty}\sum_{j+k\le\ell}f^\chi_{j,k}\\ &=\lim_{\ell\to\infty}\mathcal{F}^{-1}\Big[\Big(\sum_{j+k\le\ell}\chi_j\otimes\chi_k\Big)\mathcal{F}f\Big]=\lim_{\ell\to\infty}\mathcal{F}^{-1}[\chi_{\mathcal{H}_\ell}\mathcal{F}f],\end{aligned}$$

where

$$\chi_{\mathcal{H}_\ell}(\xi,\eta)=\begin{cases}1 & (\xi,\eta)\in\mathcal{H}_\ell,\\ 0 & \text{otherwise}\end{cases}$$

is the characteristic function of the dyadic hyperbolic cross of order $\ell$ in $\mathbb{R}^2$ defined in (1.12) (see Figure 8).

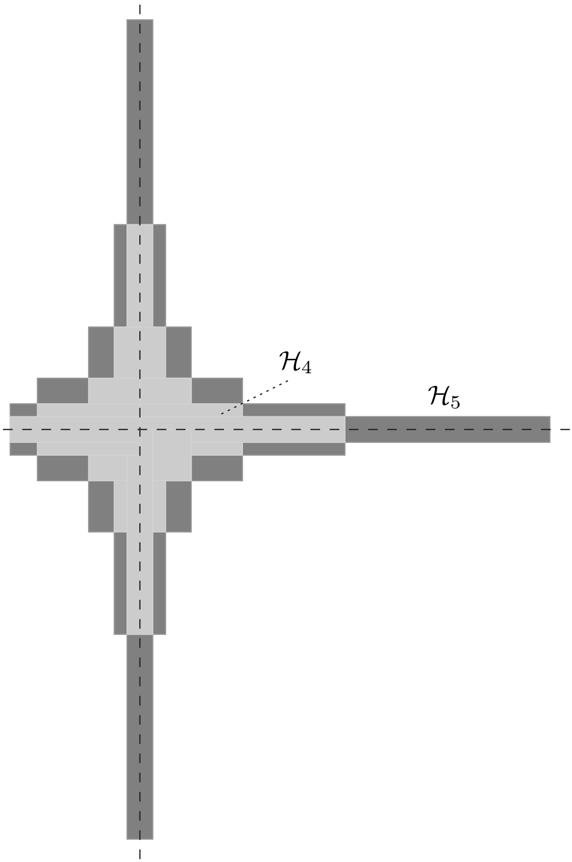


Figure 8: Dyadic hyperbolic cross

**Definition 3.2.** Let $\ell \in \mathbb{N}_0$. The function

$$\mathcal{S}_\ell^{\mathcal{H}} f(x,y) = \sum_{0 \le j+k \le \ell} f_{j,k}^{\chi}(x,y)$$

is called partial sum with respect to the dyadic hyperbolic cross of order $\ell$.

The interrelations with spaces of dominating smoothness are revealed by the following propositions. For the sake of simplicity we shall abbreviate $L_p = L_p(\mathbb{R}^2)$, $S^r_{p\,q}B = S^r_{p\,q}B(\mathbb{R}^2)$ and $S^r_{p\,q}F = S^r_{p\,q}F(\mathbb{R}^2)$ in this section.

**Proposition 3.3.** *Let $1 < p < \infty$, $r > 0$. Then there exists a constant $c_p$ such that*

$$\|f - \mathcal{S}_\ell^{\mathcal{H}} f \mid L_p\| \leq c_p 2^{-\ell r} \|f \mid S^r_{p\,2}F\|_\chi$$

*holds for all $f \in S^r_{p\,2}F$.*

**Proof.** The proof is a consequence of the the Lizorkin-type representation in Proposition 3.1, the Littlewood-Paley characterization in Proposition 2.1 and the estimates

$$\begin{aligned} 2^{\ell r}\|f - \mathcal{S}_\ell^{\mathcal{H}} f \mid L_p\| &\leq \Big\| \underbrace{\sum_{j+k>\ell} 2^{(j+k)r} f^\chi_{j,k}}_{f^\ell} \Bigm| L_p \Big\| \\ &\overset{\text{L.P.}}{\leq} c_p \Big\| \Big( \sum_{\mu=0}^\infty \sum_{\nu=0}^\infty \big|\mathcal{F}^{-1}[\chi_\mu \otimes \chi_\nu \mathcal{F} f^\ell]\big|^2 \Big)^{1/2} \Bigm| L_p \Big\| \\ &\leq c_p \Big\| \Big( \sum_{\mu+\nu>\ell} |2^{(\mu+\nu)r} f^\chi_{\mu,\nu}|^2 \Big)^{1/2} \Bigm| L_p \Big\| \\ &\leq c_p \|f \mid S^r_{p,2}F\|_\chi . \end{aligned}$$

**Proposition 3.4.** *Let $1 < p < \infty$, $r > 0$. Then there exists a constant $C_p$ such that*

$$\|f \mid S^r_{p,\infty}B\|_\chi \leq C_p \big( \sup_{\ell \in \mathbb{N}_0} 2^{\ell r} \|f - \mathcal{S}_\ell^{\mathcal{H}} f \mid L_p\| + \|f \mid L_p\| \big).$$

**Proof.** Let $0 < \ell = j + k$. By similar arguments as in the previous proof we get

$$\begin{aligned} 2^{(j+k)r}\|f^\chi_{j,k} \mid L_p\| &\leq 2^{\ell r} \Big\| \Big( \sum_{j+k\geq\ell} |f^\chi_{j,k}|^2 \Big)^{1/2} \Bigm| L_p \Big\| \\ &\overset{\text{L.-P.}}{\leq} C_p 2^{\ell r} \Big\| \sum_{j+k\geq\ell} f^\chi_{j,k} \Bigm| L_p \Big\| \\ &\leq C_p 2^{\ell r} \|f - \mathcal{S}_\ell^{\mathcal{H}} f \mid L_p\|. \end{aligned}$$

□

It will turn out later on that one cannot achieve better estimates within these two scales of spaces with dominating mixed smoothness. Next we introduce approximation spaces with respect to hyperbolic crosses.

**Definition 3.5.** Let $\ell \in \mathbb{N}_0$, $1 < p < \infty$.

$$E^{\mathcal{H}}(2^\ell, f, L_p) := \inf\{\|f - g \mid L_p\| : g \in L_p \text{ and } \operatorname{supp} \mathcal{F}g \subset \mathcal{H}_\ell\}$$

is called best approximation of $f$ by entire analytic functions with spectrum in the dyadic hyperbolic cross of order $\ell$.

The corresponding approximation spaces are given as follows.

**Definition 3.6** (Approximation spaces)**.** Let $1 < p < \infty$, $1 \le q \le \infty$, $r > 0$.

$$A^r_{p\,q} := \Big\{ f \in L_p : \sum_{\ell=0}^{\infty} 2^{\ell r q} E^{\mathcal{H}}(2^\ell, f, L_p)^q < \infty \Big\},$$

$$\|f \mid A^r_{p\,q}\| = \|f \mid L_p\| + \Big( \sum_{\ell=0}^{\infty} 2^{\ell r q} E^{\mathcal{H}}(2^\ell, f, L_p)^q \Big)^{1/q}$$

($\sup\limits_{\ell} \ldots$ if $q = \infty$).

Our aim is to give equivalent characterizations of these approximation spaces by means of decomposition and by means of real interpolation.

**Theorem 3.7.** *Let* $1 < p < \infty$, $1 \le q \le \infty$ *and* $r > 0$. *Then*

$$\begin{aligned} \|f \mid A^r_{p\,q}\| &\sim \|f \mid L_p\| + \Big( \sum_{\ell=0}^{\infty} 2^{\ell r q} \|f - \mathcal{S}^{\mathcal{H}}_\ell f \mid L_p\|^q \Big)^{1/q} \\ &\sim \|\mathcal{S}^{\mathcal{H}}_0 f \mid L_p\| + \Big( \sum_{\ell=1}^{\infty} 2^{\ell r q} \Big\| \sum_{j+k=\ell} f^{\chi}_{j.k} \Bigm| L_p \Big\|^q \Big)^{1/q}. \end{aligned}$$

**Remark 3.8.** The theorem is due to Lizorkin, Nikol'skii 1989 [48]. Note that

$$\operatorname{supp} \mathcal{F} \Big( \sum_{j+k=\ell} f^{\chi}_{j.k} \Big) = \mathcal{H}_\ell \backslash \mathcal{H}_{\ell-1}.$$

It means that our approximation spaces are characterized as decomposition spaces with respect to dyadic hyperbolic annuli. For the latter see Figure 9. Moreover, best approximation can be realized asymptotically by corresponding partial sums. Combining this with Proposition 3.3 and Proposition 3.4 we obtain the embeddings

$$S^r_p H = S^r_{p\,2} F \subset A^r_{p\,\infty} \subset S^r_{p\,\infty} B.$$

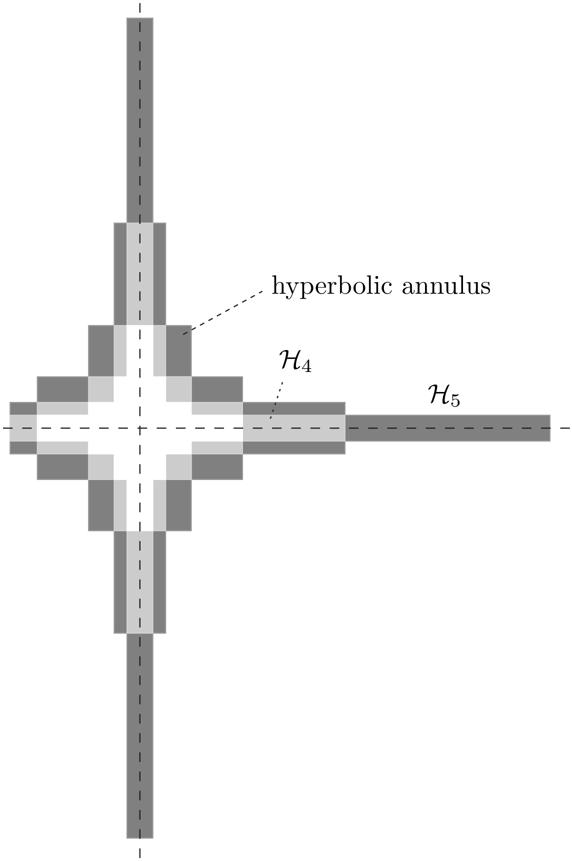


Figure 9: Hyperbolic annulus

In the following $(\cdot,\cdot)_{\Theta,\,q}$ denotes the real interpolation space (for details we refer to TRIEBEL [83]). Recall that $S^r_{p\,2}F = S^r_pH$ if $1 < p < \infty$.

**Theorem 3.9.** *Let* $1 < p < \infty$, $1 \le q \le \infty$ *and* $r_0, r_1 \ge 0$ *such that* $r_0 \ne r_1$. *Let* $0 < \Theta < 1$. *We put* $r = (1-\Theta)r_0 + \Theta r_1$. *Then*

$$\left(S^{r_0}_{p\,2}F, S^{r_1}_{p\,2}F\right)_{\Theta,q} = A^r_{p\,q}.$$

**Remark 3.10.** The proof can be reduced to Jackson and Bernstein-type inequalities which allow to apply the abstract scheme of approximation spaces due to DEVORE and POPOV (see [21]). Let us sketch this reduction. By the lift property of the spaces $A^r_{p\,q}$ and $S^r_{p\,2}F$ it is sufficient to consider the case $r_0 = 0$ and $r_1 = r$. We define $X_0 = \{0\}$ and

$$X_\ell = \{f \in L_p : \operatorname{supp} \mathcal{F} f \subset \mathcal{H}_\ell\}, \quad \ell = 1, 2, \dots .$$

Using the Lizorkin representation (3.1) and the Littlewood-Paley theorem (2.4) we deduce the Jackson-type inequality:

$$\begin{aligned} \|f - S^{\mathcal{H}}_\ell f \mid L_p\| &\le c_1 \Big\| \Big( \sum_{j+k>m} |f^{\chi}_{j,k}|^2 \Big)^{1/2} \Bigm| L_p \Big\| \\ &\le c_2 2^{-\ell r} \|f \mid S^r_{p\,2}F\|. \end{aligned}$$

Similar arguments lead to the Bernstein-type inequality:

$$\|f \mid S^r_{p\,2}F\| \le c 2^{mr} \|f \mid L_p\|$$

which is valid for all $f \in X_m$. It is sufficient to apply the abstract scheme of approximation spaces.

The next theorem shows that the approximation spaces are stable with respect to real interpolation.

**Theorem 3.11.** *Let* $1 < p < \infty$, $1 \le q, q_1 \le \infty$ *and* $0 < r_1$. *Let* $0 < \Theta < 1$.

(i) *We put* $r = \Theta r_1$. *Then*

$$(L_p, A^{r_1}_{p\,q_1})_{\Theta,q} = A^r_{p\,q}.$$

(ii) *Let* $1 \le q_0 \le \infty$ *and* $0 < r_0 < r_1$. *We put* $r = (1-\Theta) r_0 + \Theta r_1$. *Then*

$$(A^{r_0}_{p\,q_0}, A^{r_1}_{p\,q_1})_{\Theta,q} = A^r_{p\,q}.$$

**Remark 3.12.** As a consequence of real interpolation and Theorem 3.9 we find that $S^r_{p\,2}F = A^r_{p\,2}$. In all other cases the spaces $A^r_{p\,q}$ do not coincide with spaces belonging to the two scales $S^r_{p\,q}B$ and $S^r_{p\,q}F$, respectively. This will follow from the results in the next subsection.

**3.2. Comparison theorems.** Our aim is to give a detailed comparison of approximation spaces with respect to hyperbolic crosses and spaces of Besov-Lizorkin-Triebel type with dominating mixed smoothness in the sense of sharp embeddings. We follow [64].

**Theorem 3.13** ($F$-spaces)**.** *Suppose* $1 < p < \infty$, $1 \leq q, u \leq \infty$ *and* $r > 0$.

(i) *Let* $q < p$. *Then*

$$S^r_{p\,1}F \not\subset A^r_{p\,q}.$$

(ii) *Let* $q \geq p$.

$$S^r_{p\,u}F \hookrightarrow A^r_{p\,q}$$

*holds if and only if* $u \leq \min(2, q)$.

(iii) *Let* $q > p$. *Then*

$$A^r_{p\,q} \not\subset S^r_{p\,\infty}F.$$

(iv) *The embedding*

$$A^r_{p\,q} \hookrightarrow S^r_{p\,u}F$$

*holds if and only if* $q \leq p$ *and* $u \geq \max(2, q)$.

The situation is illustrated by Figure 10.

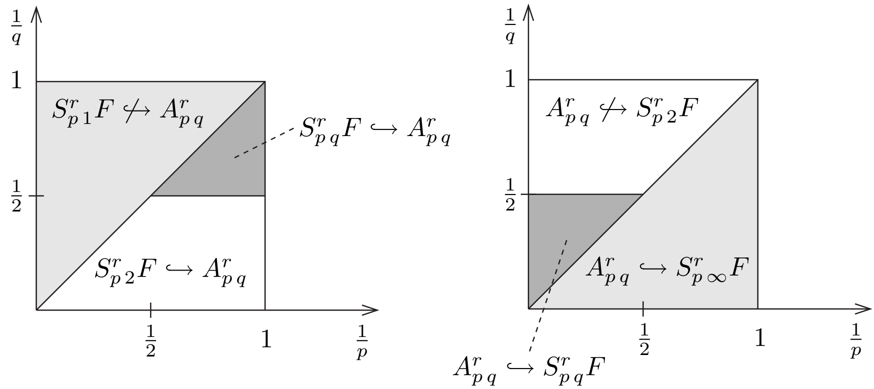


Figure 10: Comparison, $F$-spaces

**Theorem 3.14** ($B$-spaces)**.** *Suppose* $1 < p < \infty$, $1 \leq q, u \leq \infty$ *and* $r > 0$.

(i) *The embedding*

$$S^r_{p\,u}B \hookrightarrow A^r_{p\,q}$$

*holds if and only if* $u \leq \min(2, q, p)$.

(ii) *The embedding*

$$A^r_{p\,q} \hookrightarrow S^r_{p\,u}B$$

*holds if and only if* $u \geq \max(2, q, p)$.

The situation is illustrated by Figure 11.

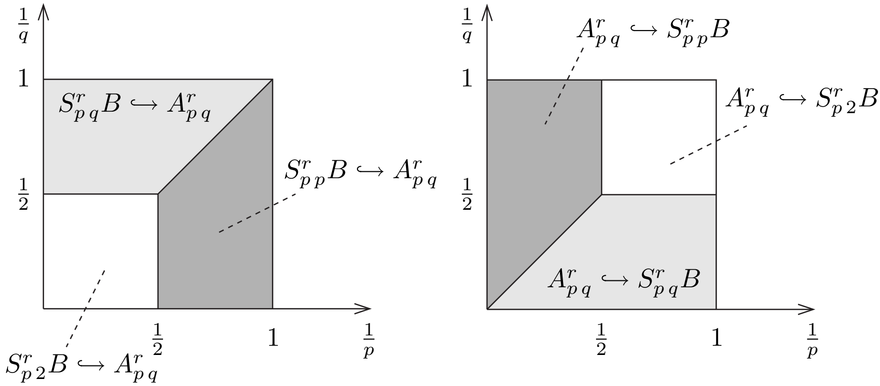


Figure 11: Comparison, $B$-spaces

**Corollary 3.15.** *Suppose* $1 < p < \infty$, $1 \le q, u \le \infty$ *and* $r > 0$.

(i) *The embedding*

$$S^r_{p\,u}B \hookrightarrow A^r_{p\,\infty}$$

*holds if and only if* $u \le \min(2, p)$.

(ii) *The embedding*

$$S^r_{p\,u}F \hookrightarrow A^r_{p\,\infty}$$

*holds if and only if* $u \le 2$.

(iii) *Whenever*

$$S^r_{p\,u}B \hookrightarrow A^r_{p\,\infty},$$

*then*

$$S^r_{p\,u}B \hookrightarrow S^r_{p\,2}F.$$

**Remark 3.16.** Hence, within the scales of Besov and Lizorkin-Triebel classes the optimal embeddings for $A^r_{p\,\infty}(\mathbb{R}^2)$ are:

$$S^r_{p\,2}F \hookrightarrow A^r_{p\,\infty} \hookrightarrow S^r_{p\,\infty}B$$

(cf. also Remark 3.8). The right-hand side in this formula can be found in Lizorkin, Nikol'skii [48]. The "if-parts" have been known in different contexts, cf. Temlyakov [79], Kamont [38] and DeVore, Konyagin and Temlyakov [20].

**Corollary 3.17.** *Suppose* $1 < p < \infty$, $1 \le q, u \le \infty$ *and* $r > 0$.

(i) *The embedding*

$$A^r_{p\,1} \hookrightarrow S^r_{p\,u}B$$

*holds if and only if* $u \ge \max(2, p)$.

(ii) *The embedding*

$$A^r_{p\,1} \hookrightarrow S^r_{p\,u}F$$

*holds if and only if* $u \ge 2$.

(iii) *Whenever*

$$A^r_{p\,1} \hookrightarrow S^r_{p\,u}B,$$

*then*

$$A^r_{p\,1} \hookrightarrow S^r_{p\,2}F \hookrightarrow S^r_{p\,u}B.$$

**Remark 3.18.** The only Besov space which is a subspace of $A^r_{p\,1}$ is given by $S^r_{p\,1}B$, cf. Theorem 3.14. Hence, within the scales of Besov and Lizorkin-Triebel classes the optimal embeddings for $A^r_{p\,1}$ are:

$$S^r_{p\,1}B \hookrightarrow A^r_{p\,1} \hookrightarrow S^r_{p\,2}F.$$

Suppose $1 < p_0, p_1 < \infty$, $1 \le q_0, q_1 \le \infty$, $r_0 > 0$ and $r_1, r_2 \in \mathbb{R}$. The classes $A^{r_0}_{p_0\,q_0}$ and $S^{r_1}_{p_1\,q_1}B$ coincide if and only if $r_0 = r_1$, $p_0 = p_1 = 2$ and $q_0 = q_1 = 2$.

The classes $A^{r_0}_{p_0\,q_0}$ and $S^{r_1}_{p_1\,q_1}F$ coincide if and only if $r_0 = r_1$, $p_0 = p_1 = 2$ and $q_0 = q_1 = 2$.

**Remark 3.19** (Sobolev-type embeddings)**.** We finish this subsection presenting some particular results on diagonal embeddings which are related to Sobolev-type embeddings. We concentrate on embeddings of $A^r_{p\,q}$ into $L_{p_1}(\mathbb{R}^2)$ and $C(\mathbb{R}^2)$, respectively.

Let $1 < p < p_1 < \infty$ and $1 \le q \le \infty$.

(i) The embedding $A^r_{p\,q} \hookrightarrow L_{p_1}$ holds if and only if either $r > \frac{1}{p} - \frac{1}{p_1}$ and $q$ is arbitrary or $r = \frac{1}{p} - \frac{1}{p_1}$ and $q \le p_1$.

(ii) The embedding $A^r_{p\,q} \hookrightarrow C$ holds if and only if $r > \frac{1}{p}$.

In particular, $A^{1/p}_{p\,1}$ contains unbounded functions and is quite different from $S^{1/p}_{p\,1}B$.

**3.3. Aproximation by partial sums.** Next we investigate the norm of the operators

$$I - S^{\mathcal{H}}_m$$

($I$ = id = identity) considered as a mapping from $S^r_{p\,q}F$ into $L_p$ and from $S^r_{p\,q}B$ into $L_p$. Because of Corollary 3.15 this is of interest only if $q > 2$ in the case of the $F$-spaces and $q > \min(p, 2)$ in case of the $B$-spaces. Otherwise the above spaces are embedded into $A^r_{p\,\infty}(\mathbb{R}^2)$ and it follows that

$$\|f - S^{\mathcal{H}}_m f \mid L_p\| \le c(f) 2^{-mr}.$$

We can even show that

$$\|I - S^{\mathcal{H}}_m : S^r_{p\,q}F \to L_p\| \sim 2^{-mr} \quad \text{if} \quad q \le 2,$$
$$\|I - S^{\mathcal{H}}_m : S^r_{p\,q}B \to L_p\| \sim 2^{-mr} \quad \text{if} \quad q \le \min(p, 2).$$

**Theorem 3.20.** *Suppose* $1 < p < \infty$, $2 < q \le \infty$ *and* $r > 0$. *Then*

$$\|I - S^{\mathcal{H}}_m : S^r_{p\,q}F \mapsto L_p\| \sim m^{\frac{1}{2} - \frac{1}{q}} 2^{-rm}.$$

For illustration see Figure 12.

**Theorem 3.21.** *Let* $r > 0$.
(i) *Suppose* $1 < p \le 2$ *and* $p \le q \le \infty$. *Then*

$$\|I - S^{\mathcal{H}}_m : S^r_{p\,q}B \mapsto L_p\| \sim m^{\frac{1}{p} - \frac{1}{q}} 2^{-rm}.$$

(ii) *Suppose* $2 < p < \infty$ *and* $q > 2$. *Then*

$$\|I - S^{\mathcal{H}}_m : S^r_{p\,q}B \mapsto L_p\| \sim m^{\frac{1}{2} - \frac{1}{q}} 2^{-rm}.$$

For illustration see Figure 13.

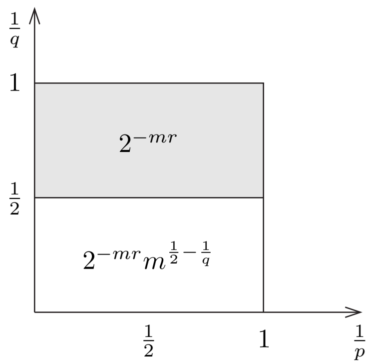


Figure 12: $F$-spaces

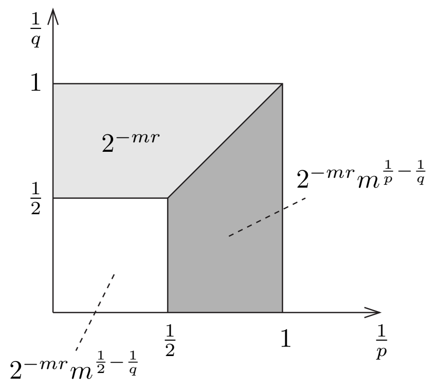


Figure 13: $B$-spaces

**Remark 3.22.** For $q = \infty$ the assertion of Theorem 3.20 was known, cf. Lizorkin and Nikol'skii [48]. In the periodic setting it has been known even for a longer time, cf. Bugrov [14] ($p = 2$), Nikol'skaya [49] and Temlyakov [77].

For related results we refer to DeVore, Petrushev, Temlyakov [22] (periodic setting), Kamont [38] (unit cube, hyperbolic spline approximation) and DeVore, Konyagin, Temlyakov [20] (hyperbolic wavelet approximation).

**Remark 3.23.** There exist further approximation processes related to hyperbolic crosses and spaces with dominating mixed smoothness (cf. also Remark 1.8). Let us mention here the so-called Smolyak algorithm (due to [71]). Application of the Smolyak algorithm to one-dimensional sampling operators leads to sampling on sparse grids (on $\mathbb{R}^d$ or $\mathbb{T}^d$ or the unit cube). This has close connection to the problem of optimal reconstruction of functions of several variables by means of a set of discrete function values. A lot of work has been done in this direction. We refer to the papers by Temlyakov [76], [79], Dinh Dung [23], Sickel, Sprengel [67], [65], [66] and, in particular, to the recent works of Sickel and Ullrich [69], [93]).

## 4. New instruments

**4.1. Local means.** Note that we have used Lizorkin-type representations and Littlewood-Paley decompositions based on characteristic functions as main tools in the previous section. As a consequence our results are restricted to spaces $S^r_{p\,q}B$ and $S^r_{p\,q}F$ with $1 < p < \infty$. Recent developments in the theory of function spaces are characterized by the use of new instruments such as local means, atoms and wavelets, which are successfully applied to tackle key problems as limiting embeddings, traces, extensions, entropy numbers or sampling numbers. Moreover, far-reaching applications to fractals and to spectral theory have been achieved. We refer to the books by Triebel [87], [88], [89], [90] and Haroske [36]. The aim of this section is to develop these new tools for spaces with dominating mixed smoothness. It yields the basis for applications which will be described in the next section.

Recall the definition of spaces $S^{r_1,\,r_2}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n)$ and $S^{r_1,\,r_2}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n)$ in Subsection 2.3 (Definition 2.7). To avoid technicalities we restrict ourselves to the case $m = n$. Rewriting formula (2.9) we get (choosing $\widetilde{\varphi} = \varphi$ and using well-known properties of the Fourier transform)

$$f^{\varphi,\widetilde{\varphi}}_{j,k} = \mathcal{F}^{-1}_{2n}(\varphi_j \otimes \widetilde{\varphi}_k \mathcal{F}_{2n} f) = c(\psi_j \otimes \psi_k) * f,$$

where $c$ is a constant (depending on the definition of $\mathcal{F}_{2n}$), $*$ denotes the convolution in $\mathcal{S}'(\mathbb{R}^{2n})$ and

$$\psi_0 = \mathcal{F}_n^{-1}\varphi_0, \qquad \psi_j(\cdot) = 2^{jn}\psi(2^j\cdot) \quad \text{for} \quad \psi = \mathcal{F}_n^{-1}\varphi,\ j \in \mathbb{N}.$$

Formally we have

$$[(\psi_j \otimes \psi_k) * f](x,y) = 2^{(j+k)n}\int_{\mathbb{R}^{2n}} (\psi \otimes \psi)(2^j u, 2^k v) f(x-u, y-v)\, d(u,v).$$

The functions $\psi_j$ do not have compact support by the Paley-Wiener theorem. As a consequence we would need all values $f(u,v)$ for computation of this quantity for fixed $(x,y)$. This is in contrast to the calculation of higher order differences of $f$ which can be used for an equivalent characterization of our spaces for a certain restricted range of parameters $r_1$, $r_2$, $p$, $q$ as we have pointed out in Subsections 1.6, 2.2 and Remark 2.6. To see this let $m > \max(r_1, r_2)$ be sufficiently large, and $|h_1| = 2^{-j}$, $|h_2| = 2^{-k}$. We can reformulate

$$\begin{aligned}\Delta^m_{h_1,1}(\Delta^m_{h_2,2} f) &= \mathcal{F}_2^{-1}[(e^{i2^{-j}\xi} - 1)^m (e^{i2^{-k}\eta} - 1)^m \mathcal{F}_2 f] \\ &= \mathcal{F}_2^{-1}[\varphi_j \otimes \varphi_k \mathcal{F}_2 f],\end{aligned}$$

where

$$\varphi_j(\xi) = \varphi(2^{-j}\xi), \quad \varphi(\xi) = (e^{i\xi} - 1)^m.$$

Hence, to compute the differences

$$\mathcal{F}_2^{-1}[\varphi_j \otimes \varphi_k \mathcal{F}_2 f](x,y) = \Delta^m_{2^{-j},1}(\Delta^m_{2^{-k},2} f)(x,y)$$

at the point $(x,y)$ we only need values $f(u,v)$ with $|x-u| \le m2^{-j}$, $|y-v| \le m2^{-k}$. Moreover, observe that

$$\begin{gathered}\operatorname{supp}\varphi = \mathbb{R}, \\ D^\alpha\varphi(0) = 0 \quad \text{for} \quad 0 \le \alpha < m, \\ |\varphi(\xi)| = O(|\xi|^m), \quad |\xi| \to 0.\end{gathered}$$

It turns out that we can modify the assumptions with respect to the properties of $\varphi_0$ and $\varphi$ to get equivalent descriptions of the spaces under consideration by local means. We choose $\varphi_0$ and $\varphi$ such that $\psi_0 = \mathcal{F}_n^{-1}\varphi_0$, $\psi = \mathcal{F}_n^{-1}\varphi$ have compact support. Assume $\psi_0$, $\psi \in \mathcal{S}(\mathbb{R}^n)$ and let

$$\operatorname{supp}\psi_0,\ \operatorname{supp}\psi \text{ compact}, \tag{4.1}$$

$$(\mathcal{F}_n\psi_0)(0) \neq 0, \tag{4.2}$$

$$(\mathcal{F}_n\psi_0)(\xi) \neq 0 \quad \text{if} \quad 0 < |\xi| < \varepsilon \tag{4.3}$$

$$(D^\alpha\mathcal{F}_n\psi)(0) = 0 \quad \text{for} \quad |\alpha| \le r,\ r \ge 0, \tag{4.4}$$

be satisfied for some $\varepsilon > 0$. The condition (4.4) is empty if $r < 0$. Observe that (4.4) can be reformulated as

$$\int_{\mathbb{R}^n} u^\alpha \psi(u)\, du = 0 \quad \text{for} \quad |\alpha| \le r.$$

**Example 4.1.** It is not difficult to find functions satisfying the above conditions (4.1)–(4.2). For example, choose $\psi^0 \in \mathcal{S}(\mathbb{R}^n)$ with support in the unit ball such that $\mathcal{F}_n\psi_0(0) \neq 0$ and put $\psi := \Delta^N \psi_0$, $2N > r$.

Given functions $\psi_0$ and $\psi$ satisfying conditions (4.1)–(4.4) we define $\psi_j(u) := 2^{jn}\psi(2^j u)$, $j \in \mathbb{N}$, and

$$\begin{aligned} \Psi_{j,k} f(x,y) &:= (\psi_j \otimes \psi_k) * f(x,y), \quad j,k \in \mathbb{N}_0 \\ &= 2^{(j+k)n} \int_{\mathbb{R}^{2n}} (\psi \otimes \psi)(2^j u, 2^k v) f(x-u, y-v)\, d(u,v),\ j,k \in \mathbb{N} \end{aligned} \tag{4.5}$$

(modification if $j \cdot k = 0$). Hence, to calculate the value of $\Psi_{j,k} f$ at the point $(x,y)$ we need only the restriction of $f$ to the set

$$\{(u,v) : |x-u| < 2^{-j},\ |y-v| < 2^{-k}\}.$$

The next theorem shows that we can replace in the definitions of $S^r_{p\,q}B(\mathbb{R}^n \times \mathbb{R}^n)$ and $S^r_{p\,q}F(\mathbb{R}^n \times \mathbb{R}^n)$ (cf. (2.18) and (2.19) the building blocks $f^{\varphi,\widetilde{\varphi}}_{j,k}$ (cf. (2.9) by the local means defined in (4.5). In contrast to the characterization by differences this works for the full range of parameters $p$, $q$ and $r$.

**Theorem 4.2** (Characterization by local means)**.** *Let* $0 < p \le \infty$, $0 < q \le \infty$, $(r_1, r_2) \in \mathbb{R}^2$ *and let* $\psi_0$, $\psi$ *be as in* (4.1)–(4.4) *with* $r = \max(r_1, r_2)$. *Let* $\Psi_{j,k} f$ *have the meaning of* (4.5).

(i) *Then*

$$\|f \mid S^{r,r_2}_{p\,q}B(\mathbb{R}^n \times \mathbb{R}^n)\| \sim \|2^{(jr_1+kr_2)}\Psi_{j,k} f(x,y) \mid L_p(\mathbb{R}^{2n})|\ell_q(\mathbb{N}_0^2)\|. \tag{4.6}$$

(ii) *If additionally* $p < \infty$ *then*

$$\|f \mid S^{r_1,r_2}_{p\,q}F(\mathbb{R}^n \times \mathbb{R}^n)\| \sim \|2^{(jr_1+kr_2)}\Psi_{j,k} f(x,y) \mid \ell_q(\mathbb{N}_0^2) \mid L_p(\mathbb{R}^{2n})\|. \tag{4.7}$$

**Remark 4.3.** The proof of the theorem can be found in Vybíral [96] (see also [94]). There are also more general results weakening the smoothness assumptions of $\psi_0$ and $\psi$. Moreover, it is shown that the local means $\Psi_{j,k}f$ in (4.6) and (4.7) can be replaced by Peetre's maximal function

$$(\Psi_{j,k}f)^\star_a(x,y) := \sup_{u,v} \frac{|(\psi_j \otimes \psi_k) * f|(x-u,y-v)}{(1+|2^j u|)^a(1+|2^k v|^a)}$$

if $a$ is large enough. Similar results have been obtained by Bazarkhanov [6]. The idea to use local means is due to Triebel [87] in the isotropic case, see also [90] for an updated essentially improved version. The case of Besov spaces with dominating mixed smoothness has been investigated by Rodriguez [59]. The key inequalities in the proof presented in [96] read as (for $a$ large enough)

$$\begin{aligned} &\| \sup_{|u|\le c2^{-j}} \sup_{|v|\le c2^{-k}} (\psi_j \otimes \psi_k) * f(x-u,y-v) \mid L_p(\ell_q)\| \\ &\qquad \le \left\| \sup_{u,v} \frac{|(\psi_j \otimes \psi_k) * f|(x-u,y-v)}{(1+|2^j u|)^a(1+|2^k v|^a)} \;\Big|\; L_p(\ell_q)\right\| \\ &\qquad \sim \|f \mid S^0_{p\,q}F\|. \end{aligned}$$

**4.2. Atoms.** A crucial step in the modern theory of function spaces has been made by the development of atomic and molecular characterizations. In the case of Besov and Lizorkin-Triebel spaces this is due to Frazier, Jawerth [30], [31] and Frazier, Jawerth, Weiss [32]. Sub-atomic (quarkonial) decompositions have been introduced by Triebel (cf. [88]). The use of (sub)atomic decompositions has lead to essential progress in applications to singular integrals and pseudodifferential operators. It paved the way to new approaches in fractal analysis and spectral theory. Moreover, quarks, atoms and molecules are closely connected with wavelet bases and frames in function spaces with far-reaching consequences in applications to computational mathematics, signal and image processing. The adaption to spaces with dominating mixed smoothness has been investigated by Rodriguez [59], Hochmuth [37], Bazarkhanov [6], [7] and Vybíral [94], [96]. Here we follow [96] and concentrate on the splitting $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. Based on the Fourier-analytical approach and the characterization by local means we are able to localize and discretize the spaces $S^{r_1,r_2}_{p\,q}B(\mathbb{R}^n \times \mathbb{R}^n)$ and $S^{r_1,r_2}_{p\,q}F(\mathbb{R}^n \times \mathbb{R}^n)$. We deal with representations by atoms and compactly supported wavelets (next subsection) but we do not touch quarkonial decompositions and wavelet frames (see [96] for the latter topics). Roughly speaking any function (distribution) $f$ belongs to such a space if and only if it can be represented as

$$f = \sum_{(j,k)\in\mathbb{N}_0^2} \sum_{(\mu,\nu)\in\mathbb{Z}^2} b^{\mu,\nu}_{j,k}\, a^{\mu,\nu}_{j,k}$$

(convergence in an appropriate sense, in particular in $\mathcal{S}'(\mathbb{R}^{2n})$), with coefficients $b_{j,k}^{\mu,\nu} \in \mathbb{C}$ and distinguished (smooth) functions $a_{j,k}^{\mu,\nu}$ having compact support in a neighborhood of the lattice point $(2^{-j}\mu, 2^{-k}\nu)$. Moreover, the sequence $\{b_{j,k}^{\mu,\nu}\}$ of coefficients has to belong to a corresponding weighted sequence space which depends on the parameters $r$, $p$ and $q$. We consider two cases:

- the atomic decomposition, where the building blocks $a_{j,k}^{\mu,\nu}$ have to fulfil qualitative properties, the representation is not unique but leads to some flexibility;
- (next subsection) the wavelet decomposition, where the building blocks $a_{j,k}^{\mu,\nu}$ are constructed as tensor products of compactly supported one-dimensional wavelets, the representation is unique, and the coefficients can be calculated explicitly.

In the latter case we shall have an isomorphism between function and sequence spaces. To be more precise we need some notation.

In the following we assume $(x, y) \in \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, $(\mu, \nu) \in \mathbb{Z}^{2n} = \mathbb{Z}^n \times \mathbb{Z}^n$, $(j, k) \in \mathbb{N}_0 \times \mathbb{N}_0$. Let us denote by $Q_j^\mu \subset \mathbb{R}^n$ a cube with centre $2^{-j}\mu$ and sides parallel to the coordinate axes of length $2^{-j+1}$. Let $dQ_j^\nu$ be the cube with the same centre and side-length $d2^{-j+1}$. We put $Q_{j,k}^{\mu,\nu} = Q_j^\mu \times Q_k^\nu$.

**Definition 4.4** (Atoms). Let $K = (K_1, K_2)$, $L = (L_1, L_2) \in \mathbb{N}_0^2$, $d \geq 1$. A function $a_{j,k}^{\mu,\nu} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is called a $(K, L)$-atom if and only if

$$\operatorname{supp} a_{j,k}^{\mu,\nu} \subset dQ_{j,k}^{\mu,\nu},$$

$$|D^{(\alpha,\beta)} a_{j,k}^{\mu,\nu}(x, y)| \leq 2^{|\alpha|j+|\beta|k}, \quad |\alpha| \leq K_1,\ |\beta| \leq K_2,$$

$$\int_{\mathbb{R}^n} x^\alpha a_{j,k}^{\mu,\nu}(x, y)\, dx = 0, \quad |\alpha| < L_1,\ j \in \mathbb{N}, \tag{4.8}$$

$$\int_{\mathbb{R}^n} y^\beta a_{j,k}^{\mu,\nu}(x, y)\, dy = 0, \quad |\beta| < L_2,\ j \in \mathbb{N} \tag{4.9}$$

((4.8) is empty if $j = 0$ and $L_1 = 0$, (4.9) is empty if $k = 0$ and $L_2 = 0$).

**Definition 4.5** (Sequence spaces). Suppose $0 < p \leq \infty$, $0 < q \leq \infty$, $(r_1, r_2) \in \mathbb{R}^2$ and let $\lambda = (\lambda_{j,k}^{\mu,\nu})$ be a sequence of complex numbers. We denote by $\chi_{j,k}^{\mu,\nu}$ the characteristic function of the cube $Q_{j,k}^{\mu,\nu}$ and put

$$g_{j,k}(x, y) := \sum_{\mu,\nu} \lambda_{j,k}^{\mu,\nu} \chi_{j,k}^{\mu,\nu}.$$

(i) A sequence $\lambda$ belongs to the space $s^{r_1,r_2}_{p\,q}b$ if and only if

$$\|\lambda \mid s^{r_1,r_2}_{p\,q}b\| := \|2^{jr_1+kr_2} g_{j,k} \mid L_p(\mathbb{R}^{2n}) \mid \ell_q(\mathbb{N}_0^2)\| < \infty. \tag{4.10}$$

(ii) A sequence $\lambda$ belongs to the space $s^{r_1,r_2}_{p\,q}f$ if and only if

$$\|\lambda \mid s^{r_1,r_2}_{p\,q}f\| := \|2^{jr_1+kr_2} g_{j,k} \mid \ell_q(\mathbb{N}_0^2) \mid L_p(\mathbb{R}^{2n})\| < \infty.$$

Note that (4.10) can be reformulated. We have

$$\|\lambda \mid s^{r_1,r_2}_{p\,q}b\| \sim \Big\|2^{(j-n/p)r_1+(k-n/p)r_2}\Big(\sum_{\mu,\nu} |\lambda^{\mu,\nu}_{j,k}|^p\Big)^{1/p} \;\Big|\; \ell_q(\mathbb{N}_0^2)\Big\|.$$

Next we want to sketch the ideas how to derive atomic characterizations from local means described in the previous subsection. Starting point is a representation formula due to Triebel [88]. It was shown that there exist smooth kernels $\psi_0$, $\psi$, $|\psi(x)| = O(|x|^N)$ near to 0 ($N$ large) such that

$$f = \sum_{j,k} (\psi_j \otimes \psi_k) * f.$$

The next step is a localization at the scale $(2^{-j}, 2^{-k})$. To this end we choose a smooth function $\varrho$ with

$$\operatorname{supp} \varrho \subset \{x : |x| \le d\} \quad \text{and} \quad \sum_{\mu \in \mathbb{Z}^n} \varrho(x-\mu) = 1.$$

Setting

$$\varrho^{\mu,\nu}_{j,k}(x,y) := \varrho(2^j x - \nu)\varrho(2^k y - \nu)$$

we obtain

$$\sum_{\mu,\nu} \varrho^{\mu,\nu}_{j,k}(x,y) = 1, \quad \operatorname{supp} \varrho^{\mu,\nu}_{j,k} \subset dQ^{\mu,\nu}_{j,k}$$

and the decomposition

$$f = \sum_{j,k} \sum_{\mu,\nu} \underbrace{\varrho^{\mu,\nu}_{j,k}(x,y) \cdot (\psi_j \otimes \psi_k) * f(x,y)}_{=: b^{\mu,\nu}_{j,k}(x,y)}.$$

If $|\alpha| \le K_1$, $|\beta| \le K_2$, $(x,y) \in \operatorname{supp} b^{\mu,\nu}_{j,k}$ then there exist constants $c$, $C$ independent of $j$, $k$, $\mu$, $\nu$ such that

$$|D^{(\alpha,\beta)} b^{\mu,\nu}_{j,k}(x,y)| \le 2^{j|\alpha|+k|\beta|} \underbrace{C \sum_{|\alpha|\le K_1,\,|\beta|\le K_2} \sup_{cQ^{\mu,\nu}_{j,k}} |(D^\alpha\psi)_j \otimes (D^\beta\psi)_k * f}_{=:\lambda^{\mu,\nu}_{j,k}}.$$

Consequently, the functions

$$a^{\mu,\nu}_{j,k}(x,y) := (\lambda^{\mu,\nu}_{j,k})^{-1} b^{\mu,\nu}_{j,k}(x,y)$$

are $(K,0)$-atoms. (Moment conditions are not satisfied.) Moreover, using Peetre's maximal function, the estimates

$$\begin{aligned}\|\lambda \mid s^{r_1,r_2}_{p\,q} f\| &\le C_1 \sum_{|\alpha|,|\beta|} \left\| 2^{jr_1+kr_2} \big((D^\alpha\psi)_j \otimes (D^\beta\psi)_k * f\big)^\star_a \mid L_p(\ell_q)\right\| \\ &\le C_2 \|f \mid S^{r_1,r_2}_{p\,q} F\|\end{aligned}$$

can be proved. Altogether this leads to atomic decompositions for large $r_1$, $r_2$ in the sense of the following theorem.

**Theorem 4.6** (Atomic decomposition)**.** *Let* $0 < p \le \infty$, $0 < q \le \infty$, $(r_1,r_2) \in \mathbb{R}^2$ *and let*

$$\sigma_p := n(1/p-1)_+, \quad \sigma_{pq} := n(1/\min(p,q)-1)_+.$$

(i) *Let* $K_i > r_i$, $L_i > \sigma_p - r_i$ $(i=1,2)$. *Then*

$$f \in S^{r_1,r_2}_{p\,q} B(\mathbb{R}^n \times \mathbb{R}^n) \Longleftrightarrow f = \sum_{j,k} \sum_{\mu,\nu} \lambda^{\mu,\nu}_{j,k} a^{\mu,\nu}_{j,k}, \tag{4.11}$$

(*convergence in* $\mathcal{S}'(\mathbb{R}^{2n})$) *where* $a^{\mu,\nu}_{j,k}$ *are* $(K,L)$*-atoms and* $\|\lambda \mid s^{r_1,r_2}_{p\,q} b\| < \infty$. *Moreover,*

$$\|f \mid S^{r_1,r_2}_{p\,q} B\| \sim \inf \|\lambda \mid s^{r_1,r_2}_{p\,q} b\|,$$

*where the infimum is taken over all admissible representations* (4.11).

(ii) *Let* $p < \infty$, $K_i > r_i$, $L_i > \sigma_{pq} - r_i$, $i = 1,2$. *Then*

$$f \in S^{r_1,r_2}_{p\,q} F(\mathbb{R}^n \times \mathbb{R}^n) \Longleftrightarrow f = \sum_{j,k} \sum_{\mu,\nu} \lambda^{\mu,\nu}_{j,k} a^{\mu,\nu}_{j,k}, \tag{4.12}$$

(*convergence in* $\mathcal{S}'(\mathbb{R}^{2n})$) *where* $a^{\mu,\nu}_{j,k}$ *are* $(K,L)$*-atoms and* $\|\lambda \mid s^{r_1,r_2}_{p\,q} f\| < \infty$. *Moreover,*

$$\|f \mid S^{r_1,r_2}_{p\,q} F\| \sim \inf \|\lambda \mid s^{r_1,r_2}_{p\,q} f\|, \tag{4.22}$$

*where the infimum is taken over all admissible representations* (4.12).

**4.3. Wavelet bases.** We do not comment on the history of wavelets. There is a huge list of books and papers which appeared in the last 25 years. Were we are interested in wavelets with compact support (so-called Daubechies wavelets) and corresponding bases and refer to DAUBECHIES [18], [19] for basic concepts. We consider the one-dimensional case. Let $\psi_0$, $\psi$ be real-valued functions defined on $\mathbb{R}$. We put

$$\psi_0^\mu(x) := \psi_0(x-\mu), \quad \mu \in \mathbb{Z},\ x \in \mathbb{R},$$

and

$$\psi_j^\mu(x) := 2^{j/2}\psi(2^j x - \mu), \quad j \in \mathbb{N},\ \mu \in \mathbb{Z},\ x \in \mathbb{R}.$$

It means that the system $(\psi_j^\mu)_{j,\mu}$ is generated from two functions by means of $L_2$-norm preserving dyadic dilation and translation with respect to uniform grid of mesh-size $2^{-j}$. The task is to construct (smooth) functions $\psi_0$ and $\psi$ such that the corresponding system is an orthonormal basis in $L_2(\mathbb{R})$ and an unconditional bases in (quasi) Banach spaces of functions such as Besov-Lizorkin-Triebel spaces. The following proposition is due to DAUBECHIES [19].

**Proposition 4.7** (Daubechies wavelets)**.** *Let $\varkappa \in \mathbb{N}$.*

(i) *There exist functions $\psi_0$, $\psi \in C^\varkappa(\mathbb{R})$ with compact support and satisfying*

$$\int_{-\infty}^{\infty} \psi_0(x)\, dx = 1,$$

$$\int_{-\infty}^{\infty} x^\alpha \psi(x)\, dx = 0 \quad \textit{for} \ \ \alpha = 0, \ldots, \varkappa$$

*such that $(\psi_j^\mu)_{j,\mu}$ is an orthonormal basis in $L_2(\mathbb{R})$.*

(ii) *Let $\psi_0$, $\psi$ be as in* (i)*. Then $(\psi_j^\mu \otimes \psi_k^\nu)_{j,k;\mu,\nu}$ is an orthonormal basis in $L_2(\mathbb{R}^2)$.*

**Remark 4.8.** The relations to atoms and local means considered in the previous subsections can be seen from the following. Let $(\psi_j^\mu)_{j;\mu}$ be as in Proposition 4.7. Then

$$a_{j,k}^{\mu,\nu} := 2^{-(j+k/2)}\psi_j^\mu \otimes \psi_k^\nu$$

are atoms in the sense of Definition 4.4 with vanishing moments up to order $\varkappa$. On the other hand (putting $\widetilde{\psi}(x) = \psi(-x)$) we see that the wavelet coefficients

$$(f, \psi_j^\mu \otimes \psi_k^\nu) = 2^{-(j+k/2)}(\widetilde{\psi}_j \otimes \widetilde{\psi}_k) * f(2^{-j}\mu, 2^{-k}\nu)$$

are building blocks in the sense of (4.5) up to a factor (local means). Using the results of Subsections 4.1 and 4.2 wavelet bases in spaces with dominating mixed smoothness are constructed in [96].

**Theorem 4.9** (Wavelet decomposition)**.** *Let* $(\psi_j^\mu)_{j;\mu}$ *be as in Proposition 4.7, where* $\varkappa$ *is a sufficiently large natural number* (*depending on* $r_1$, $r_2$, $p$, $q$). *Let us further agree that* $A = B$ *or* $A = F$, $a = b$ *or* $a = f$. *Let* $0 < p, q \leq \infty$ ($p < \infty$) *if* $A = F$ *and let* $(r_1, r_2) \in \mathbb{R}^2$.

(i) *We have*

$$f \in S^{r_1,r_2}_{p\,q} A(\mathbb{R}^2) \Longleftrightarrow f = \sum_{j,k} \sum_{\mu,\nu} (f, \psi_j^\mu \otimes \psi_k^\nu) \psi_j^\mu \otimes \psi_k^\nu, \tag{4.13}$$

*where*

$$\lambda(f) = \left(2^{(j+k)/2}(f, \psi_j^\mu \otimes \psi_k^\nu)\right)^{\mu,\nu}_{j,k} \in s^{r_1,r_2}_{p\,q} a,$$

(ii) *The representation* (4.13) *is unique and the mapping* $f \to \lambda(f)$ *is an isomorphism from* $S^{r_1,r_2}_{p\,q} A(\mathbb{R}^2)$ *onto* $s^{r_1,r_2}_{p\,q} a$.

**Remark 4.10.** Similar results on the isomorphism of spaces with dominating mixed smoothness and corresponding sequence spaces have been obtained by Hochmuth [37] and Bazarkhanov [7]. Let us emphasize that we did not take care about the best possible smoothness $\varkappa$ in the above theorem. Recently, Triebel (unpublished notes) has obtained sharp estimates for $\varkappa$ in the isotropic case of spaces $B^r_{p\,q}(\mathbb{R}^n)$ and $F^r_{p\,q}(\mathbb{R}^n)$. Following his method should lead to $\varkappa > \max(r_i, \sigma_p - r_1)$ for $S^{r_1,\,r_2}_{p\,q} B$ and to $\varkappa > \max(r_i, \sigma_{pq} - r_i)$ for $S^{r_1,\,r_2}_{p\,q} F$ as optimal conditions.

## 5. Applications

Atomic and wavelet decompositions have been successfully applied to the study of embeddings and traces. We shall present a survey of recent results concerning

- the investigation of traces on the diagonal $\{(x, y) \in \mathbb{R}^2 : \ x = y\}$ (see [59], [86], [95] and [70] for an analogous problem in $\mathbb{R}^3$),
- critical embeddings establishing sharp estimates for growth envelope functions based on multivariate rearrangements in the spirit of [36] and [89] (see [42], [43], [44]),
- the asymptotic behavior of entropy numbers of compact embeddings in spaces on domains where the problem is reduced to sequence spaces via wavelet decompositions (see [96]).

**5.1. Traces.** Let us consider the trace problem for the spaces $S^{r_1,r_2}_{p\,q}B(\mathbb{R}^2)$ and $S^{r_1,r_2}_{p\,q}F(\mathbb{R}^2)$ with dominating mixed smoothness described in Subsection 1.5. If a function $f$ belongs to such a space we formally define

$$\operatorname{tr} f := f(x,x), \quad x \in \mathbb{R}$$

(the so-called trace on the diagonal). If $r_1$ and $r_2$ are sufficiently large then $\operatorname{tr} f$ can be rigorously defined. This will be the case in the results which will be presented below. A space $B(\mathbb{R})$ is called the trace on the diagonal of a space $A(\mathbb{R}^2)$ ($\operatorname{tr} A(\mathbb{R}^2) = B(\mathbb{R})$ for short notation) if

$$\operatorname{tr} : A(\mathbb{R}^2) \to B(\mathbb{R})$$

is linear and bounded and if there exists an extension operator

$$\operatorname{ext} : B(\mathbb{R}) \to A(\mathbb{R}^2)$$

which is a linear and bounded such that

$$\operatorname{tr} \circ \operatorname{ext} = \operatorname{id} : B(\mathbb{R}) \to B(\mathbb{R}).$$

The problem was first studied by TRIEBEL [86] in the special case $A(\mathbb{R}^2) = S^{r_1,r_2}_{p\,1}B(\mathbb{R}^2)$, $1 \le p \le \infty$. Using atomic decompositions RODRIGUEZ [59] was able to extend his results to the general case of spaces $S^{r_1,r_2}_{p\,q}B(\mathbb{R}^2)$, where some borderline cases remained open. A nearly final solution has been achieved by VYBÍRAL [95] for both $S^{r_1,r_2}_{p\,q}B(\mathbb{R}^2)$ and $S^{r_1,r_2}_{p\,q}F(\mathbb{R}^2)$. Here we shall present these results.

**Theorem 5.1** (Traces, $B$-spaces)**.** *Let $0 < p,q \le \infty$, $r_1, r_2 > 0$, $\sigma_p := (1/p-1)_+$.*

(i) *If $r_2 \ge r_1 > \sigma_p$ and $r_2 > 1/p$ then*

$$\operatorname{tr} S^{r_1,r_2}_{p\,q}B(\mathbb{R}^2) = B^{r_1}_{p\,q}(\mathbb{R}).$$

(ii) *If $r_1 \ge r_2 > \sigma_p$ and $r_1 > 1/p$ then*

$$\operatorname{tr} S^{r_1,r_2}_{p\,q}B(\mathbb{R}^2) = B^{r_2}_{p\,q}(\mathbb{R}).$$

(iii) *If $\sigma_p < r_1 < 1/p$, $\sigma_p < r_2 < 1/p$ and $r_1 + r_2 - 1/p > \sigma_p$ then*

$$\operatorname{tr} S^{r_1,r_2}_{p\,q}B(\mathbb{R}^2) = B^{r_1+r_2-1/p}_{p\,q}(\mathbb{R}).$$

**Remark 5.2.** Whereas in part (i) and (ii) the trace coincides with the trace on the coordinate axes $(x_1, 0)$ and $(0, x_2)$, respectively something new happens in part (iii). In particular, if $r_1 = r_2 = r$ it means $r_1 = r_2 = r$ that the trace of $S^{r}_{p\,q}B(\mathbb{R}^2)$ on the diagonal coincides with the trace of the (smaller) isotropic Besov space $B^{2r}_{p\,q}(\mathbb{R}^2)$. For an illustration of the above three cases see Figure 14 and 15.

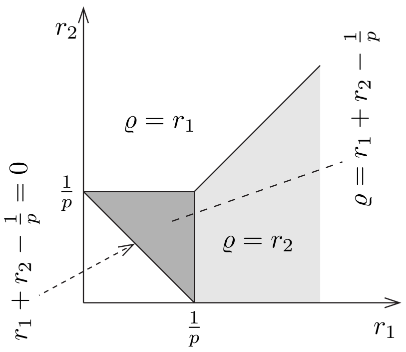


Figure 14: $p \geq 1$

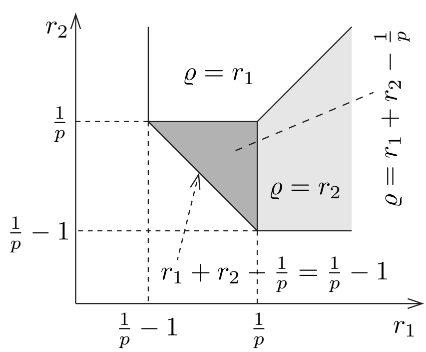


Figure 15: $p < 1$

**Remark 5.3.** We discuss the borderline cases (see [95] for the details). We restrict ourselves to the case $r_1 < r_2$. In the converse case one has to change the roles of $r_1$ and $r_2$ in an obvious way.

If $r_2 = 1/p$, $\sigma_p < r_1 < 1/p$ and $q \leq \min(1, p)$ then we have

$$\operatorname{tr} S^{r_1,r_2}_{p\,q} B(\mathbb{R}^2) = B^{r_1}_{p\,q}(\mathbb{R})$$

(the same result as in part (i)).

If $r_2 = 1/p$, $\sigma_p < r_1 < 1/p$, $p \geq 1$ and $q \geq 1$ then we get

$$\operatorname{tr} S^{r_1,r_2}_{p\,q} B(\mathbb{R}^2) = B^{(r_1,\alpha)}_{p\,q}(\mathbb{R}),$$

where $\alpha = 1/q - 1 \leq 0$ and where $B^{(r_1,\alpha)}_{p\,q}(\mathbb{R})$ denotes a Besov space with generalized smoothness which is defined as in Definition 2.6, formulas (2.13)–(2.15) with $2^{jr_1}(j+1)^\alpha$ in place of $2^{jr_1}$. We refer to [28] for a systematic treatment in the spirit of sections 2 and 3. Plainly, it is a larger space because of $\alpha < 0$.

If $r_2 = 1/p$, $\sigma_p < r_1 < 1/p$, $p \leq 1$ and $p \leq q$ then we have the embedding

$$\operatorname{tr} S^{r_1,r_2}_{p\,q} B(\mathbb{R}^2) \hookrightarrow B^{(r_1,\beta)}_{p\,q}(\mathbb{R}),$$

where $\beta = 1/q - 1/p \leq q$. It is not known whether the result is optimal or not.

As an example let us compare the traces of $H^1_2(\mathbb{R}^2)$ and $S^{1/2,1/2}_2 H(\mathbb{R}^2)$. We have

$$\operatorname{tr} H^1_2(\mathbb{R}^2) = B^{1/2}_{2\,2} B(\mathbb{R}) = H^{1/2}_2(\mathbb{R}),$$

whereas

$$\operatorname{tr} S^{1/2,1/2}_2 H(\mathbb{R}^2) = B^{(1/2,-1/2)}_{2\,2} B(\mathbb{R}) \supset H^{1/2}_2(\mathbb{R}).$$

**Theorem 5.4** (Traces, $F$-spaces). *Let $0 < p < \infty$, $q \le \infty$, $r_1, r_2 > 0$ and $\sigma_{p\,q} := (1/\min(p,q) - 1)_+$.*

(i) *If $r_2 \ge r_1 > \sigma_{p\,q}$ and $r_2 > 1/p$ then*

$$\operatorname{tr} S^{r_1,r_2}_{p\,q}F(\mathbb{R}^2) = F^{r_1}_{p\,q}(\mathbb{R}).$$

(ii) *If $r_1 \ge r_2 > \sigma_{p\,q}$ and $r_1 > 1/p$ then*

$$\operatorname{tr} S^{r_1,r_2}_{p\,q}F(\mathbb{R}^2) = F^{r_2}_{p\,q}(\mathbb{R}).$$

(iii) *If $\sigma_{p\,q} < r_1 < 1/p$, $\sigma_{p\,q} < r_2 < 1/p$ and $r_1 + r_2 - 1/p > \sigma_{p\,q}$ then*

$$\operatorname{tr} S^{r_1,r_2}_{p\,q}F(\mathbb{R}^2) = B^{r_1+r_2-1/p}_{p\,p}(\mathbb{R}).$$

**Remark 5.5.** Remark 5.2 and Figures 13 and 14 apply also to the results of Theorem 5.4 with $F$ in place of $B$ and with $\sigma_{p\,q}$ in place of $\sigma_p$.

Again we want to discuss the borderline cases. Let $r_2 = 1/p$ and $\sigma_{p\,q} < r_1 < 1/p$. If $p \le 1$ and $p \le q < \infty$ then we have

$$\operatorname{tr} S^{r_1,r_2}_{p\,q}F(\mathbb{R}^2) = F^{r_1}_{p\,q}(\mathbb{R}).$$

If $q \le p \le 1$ then we get

$$\operatorname{tr} S^{r_1,r_2}_{p\,q}F(\mathbb{R}^2) = B^{r_1}_{p\,p}(\mathbb{R}).$$

If $1 < p \le q$ then we get the embedding

$$\operatorname{tr} S^{r_1,r_2}_{p\,q}F(\mathbb{R}^2) \hookrightarrow F^{(r_1,\alpha)}_{p\,q}(\mathbb{R}),$$

where $\alpha = 1/q - 1 < 0$ and where $F^{(r_1,\alpha)}_{p\,q}(\mathbb{R})$ denotes a Lizorkin-Triebel space with generalized smoothness which is defined as in Definition 2.6, formula (2.16) and (2.17) with $2^{jr_1}(j+1)^\alpha$ in place of $2^{jr_1}$. Plainly, it is a larger space because of $\alpha < 0$.

If $p > 1$ and $q < p < \infty$ then we get the embedding

$$\operatorname{tr} S^{r_1,r_2}_{p\,q}F(\mathbb{R}^2) \hookrightarrow B^{(r_1,\gamma)}_{p\,p}(\mathbb{R}),$$

where $\gamma = 1/p - 1 < 0$.

As an example we compare the trace of $S^{1,1}_{1\,2}F(\mathbb{R}^2)$ on the diagonal and the trace of $F^2_{1\,2}(\mathbb{R}^2)$. We have

$$\operatorname{tr} S^{1,1}_{1\,2}F(\mathbb{R}^2) = F^1_{1\,2}(\mathbb{R})$$

whereas

$$\operatorname{tr} F^2_{1\,2}(\mathbb{R}^2) = B^1_{1\,1}(\mathbb{R}) \subset F^1_{1\,2}(\mathbb{R}).$$

**5.2. Critical embeddings.** Recall our introductory remarks in Subsection 1.4. As we have pointed out in [42] and [43] Besov and Lizorkin-Triebel spaces with dominating mixed smoothness properties are are of great use to establish reduced Sobolev embeddings. We do not repeat the details given there. Here we concentrate on the so-called critical case and describe a new aspect related to multivariate rearrangements of functions. We consider the case $d = 2$, $r_1 = r_2$ and $m = n$. First we state sharp embeddings of Sobolev type within the scales of spaces $S^r_{p\,q}B$ and $S^r_{p\,q}F$.

**Proposition 5.6** (Sharp embeddings). *Let $0 < p < p_0 \le \infty$, $r_0 < r$ and $r - \frac{n}{p} = r_0 - \frac{n}{p_0}$.*

(i) *If $0 < q \le q_0 \le \infty$ then*

$$S^r_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n) \hookrightarrow S^{r_0}_{p_0\,q_0}B(\mathbb{R}^m \times \mathbb{R}^n). \tag{5.1}$$

(ii) *If $p_0 < \infty$, $0 < q \le \infty$ and $0 < q_0 \le \infty$ then*

$$S^r_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n) \hookrightarrow S^{r_0}_{p_0\,q_0}F(\mathbb{R}^m \times \mathbb{R}^n). \tag{5.2}$$

(iii) *If $p < \infty$ and $0 < q \le \infty$ then*

$$S^r_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n) \hookrightarrow S^{r_0}_{p_0\,p}B(\mathbb{R}^m \times \mathbb{R}^n). \tag{5.3}$$

A proof of (5.1) and (5.2) can be found in [63] (see also [60], [61]). For (5.3) we refer to [43] and [64]. The sharpness of these embeddings can be obtained combining the isotropic case (see [68]) and (2.20). Of peculiar interest here is the case $r_0 = 0$. Taking into account Proposition 2.2 and choosing $r_0 = 0$, $q_0 = 2$ and $p_0 > 1$ in (5.2) we obtain the Sobolev embedding

$$S^r_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n) \hookrightarrow S^0_{p_0\,2}F(\mathbb{R}^m \times \mathbb{R}^n) = L_{p_0}(\mathbb{R}^{2n}) \;\text{ for }\; r = n\Big(\frac{1}{p} - \frac{1}{p_0}\Big). \tag{5.4}$$

In this case $p_0$ corresponds to Sobolev's critical index for the embedding of isotropic Lizorkin-Triebel spaces (which include the classical Sobolev spaces) on $\mathbb{R}^{2n}$ with smoothness $2r$. Namely, we have also

$$F^{2r}_{p\,q}(\mathbb{R}^{2n}) \hookrightarrow F^0_{p_0\,2}(\mathbb{R}^{2n}) = L_{p_0}(\mathbb{R}^{2n}) \quad \text{for} \quad r = n\Big(\frac{1}{p} - \frac{1}{p_0}\Big). \tag{5.5}$$

The question of interest here is what happens in the limiting case $p_0 = \infty$ which is not covered by (5.4) and (5.5), respectively. In this so-called critical case $r = n/p$ the following embeddings hold true:

$$S^{n/p}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^{2n}) \quad \text{iff} \quad 0 < p \le \infty,\; q \le 1, \tag{5.6}$$

$$S^{n/p}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^{2n}) \quad \text{iff} \quad 0 < p \le 1,\; 0 < q \le \infty. \tag{5.7}$$

Again, the same result holds true for $B^{2n/p}_{p\,q}(\mathbb{R}^{2n})$ and $F^{2n/p}_{p\,q}(\mathbb{R}^{2n})$. The "if-parts" of (5.6) and (5.7) follow from (5.1) with $q \leq 1$ and (5.3) with $p \leq 1$ as well as the embeddings

$$S^{0}_{\infty\,1}B(\mathbb{R}^m \times \mathbb{R}^n) \hookrightarrow C(\mathbb{R}^{2n}) \hookrightarrow L_\infty(\mathbb{R}^{2n})$$

which can be deduced from Definition 2.7, formula (2.18). The sharpness is again a consequence of the isotropic case (see [68]) and (2.20). In all other cases the spaces $S^{n/p}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n)$, $S^{n/p}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n)$, $B^{2n/p}_{p\,q}(\mathbb{R}^{2n})$ and $F^{2n/p}_{p\,q}(\mathbb{R}^{2n})$ contain unbounded functions. The task is to characterize the unboundedness of functions belonging to the spaces on the line $r = n/p$ in the $(r, 1/p)$-diagram (see Figure 7, Remark 2.6) for $1 < q \leq \infty$ in case of $S^{n/p}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n)$ and $1 < p < \infty$ in case of $S^{n/p}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n)$, respectively.

The same situation appears for isotropic spaces $B^{2n/p}_{p\,q}(\mathbb{R}^{2n})$ if $1 < q \leq \infty$ and $F^{2n/p}_{p\,q}(\mathbb{R}^{2n})$ if $1 < p < \infty$. The question of optimal target spaces for embeddings for isotropic (fractional) Sobolev spaces has been intensively studied. First results have been proved by Yudovich [99], Pohozaev [58] and Trudinger [91]. Later on the results have been extended and improved by Strichartz [73], Hansson [35] and Brezis, Wainger [13]. In particular, we are lead to exponential Orlicz spaces and (Lorentz-)Zygmund spaces as target spaces. A survey of these results can be found in Triebel [89, Section 11]. A final solution in terms of growth envelopes has been found by Haroske and Triebel (see [89, Section 12] and [36]). The idea is to measure the unboundedness of functions by means of the behaviour of its non-increasing rearrangement near to zero. In contrast to (5.4) and (5.5) where we have the same target spaces (we did not consider more refined Sobolev embeddings into Lorentz spaces as it has been done in Kolyada [40] or in [36] and [89] in the (an)isotropic case) the situation will turn out to be different in the limiting case. This has been observed in [42]. In the following we present new results from [44] where it is shown that multivariate rearrangements are in fact better adapted to spaces with dominating mixed smoothness. Let us introduce some notation. For a measurable function $f : \mathbb{R}^N \to \mathbb{C}$ the non-increasing rearrangement of $f$ is the function

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}, \quad t \in (0, \infty),$$

where

$$\mu_f(\lambda) = |\{x \in \mathbb{R}^N : |f(x)| > \lambda\}|, \quad \lambda > 0.$$

If $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a measurable function we put

$$(R_1 f)(s, y) = [f(\cdot, y)]^*(s), \quad s > 0,\ y \in \mathbb{R}^n,$$
$$(R_2 f)(x, t) = [f(x, \cdot)]^*(t), \quad x \in \mathbb{R}^n,\ t > 0,$$

and define the bivariate (iterated) non-increasing rearrangement of $f$ by

$$(R_{12})f(s,t) = (R_2 \circ R_1 f)(s,t) = [(R_1 f)(s,\cdot)]^*(t), \quad s,t>0.$$

For basic properties of $f^*$ and $(R_{12})f$ we refer to Bennett, Sharpley [9] or Ziemer [100] and Blozinski [12], respectively. The utility of multivariate rearrangements in connection with embeddings of Sobolev type has been discovered by Kolyada [41]. He studied embeddings of anisotropic Sobolev and Besov spaces in the non-limiting situation as in (5.5) to get refinements in terms of multivariate Lorentz spaces. Because of the lack of subadditivity of the above operators it is sometimes more convenient to work with their averages. Let

$$(A_1 f)(s,y) = \frac{1}{s}\int_0^s (R_1 f)(u,y)\,du, \quad s>0,\ y\in\mathbb{R}^n,$$
$$(A_2 f)(x,t) = \frac{1}{t}\int_0^t (R_2 f)(x,v)\,dv, \quad x\in\mathbb{R}^n,\ t>0,$$

and define the mulivariate average of $f$ by

$$\begin{aligned}(A_{12}f)(s,t) &= (A_2\circ A_1 f)(s,t)\\ &= \frac{1}{st}\int_0^t R_2\Big(\int_0^s (R_1 f)(u,\cdot)\,du\Big)(v)\,dv, \quad s,t>0.\end{aligned}$$

Plainly, we have $(R_{12}f)(s,t) \le (A_{12}f)(s,t)$ for all $s,t$. Our main result reads as follows.

**Theorem 5.6.** (i) *Let* $0<p<\infty$ *and* $1<q<\infty$. *Then*

$$\int_0^1\int_0^1 \Big[\frac{(A_{12}f)(s,t)}{(\log e/s)(\log e/t)}\Big]^q \frac{ds}{s}\,\frac{dt}{t} \le c\|f \mid S^{n/p}_{p\,q}B(\mathbb{R}^m\times\mathbb{R}^n)\|^q \tag{5.8}$$

*and if* $1<q\le\infty$

$$\sup_{0<s\le 1}\ \sup_{0<t\le 1} \frac{(A_{12}f)(s,t)}{[(\log e/s)(\log e/t)]^{1/q'}} \le c\|f \mid S^{n/p}_{p\,q}B(\mathbb{R}^m\times\mathbb{R}^n)\|. \tag{5.9}$$

(ii) *Let* $1<p<\infty$ *and* $0<q\le\infty$. *Then*

$$\int_0^1\int_0^1 \Big(\frac{(A_{12}f)(s,t)}{(\log e/s)(\log e/t)}\Big)^p \frac{ds}{s}\,\frac{dt}{t} \le \|f \mid S^{n/p}_{p\,q}F(\mathbb{R}^m\times\mathbb{R}^n)\|^p. \tag{5.10}$$

Note that part (ii) can be reduced to part (i) using the sharp embeddings (5.3). The proof of inequality (5.9) is based on atomic the characterization established in Theorem 4.6 which permits a reduction of the integral inequalities to corresponding problems in sequence spaces which are easier to handle. This approach has been elaborated in the isotropic setting by TRIEBEL [89] and HAROSKE [36]. Let us discuss the sharpness of estimates (5.8)–(5.10). Here the concept of growth envelopes introduced in [89] and studied in detail in [36] turns out to be very useful. We are interested in the quantities

$$\sup\{(R_{12})f(s,t) : \|f \mid S^{n/p}_{p\,q}A(\mathbb{R}^n \times \mathbb{R}^n)\| \leq 1\}$$

and

$$\sup\{f^*(t) : \|f \mid S^{n/p}_{p\,q}A(\mathbb{R}^n \times \mathbb{R}^n)\| \leq 1\},$$

where $A$ stands for $B$ or $F$. The latter one coincides with the growth envelope function as defined in [89], Section 12, and [36]. The first one is its multivariate (bivariate) counterpart.

**Corollary 5.8.** *Let $0 < p < \infty$, $1 < q < \infty$ and $1/q + 1/q' = 1$. Then*

$$\begin{aligned} \sup\{(R_{12})f(s,t) : \|f \mid S^{n/p}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n)\| \leq 1\} \\ \asymp [(\log e/s)(\log e/t)]^{1/q'}, \quad 0 < s,t < 1. \end{aligned} \tag{5.11}$$

*An analogous result holds true for $S^{n/p}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n)$ in the case $1 < p < \infty$ and $0 < q \leq \infty$ with $p'$ in place of $q'$ on the right-hand side.*

Moreover, we cannot improve the estimates (5.9) and (5.11) with respect to the exponent $q$ and $p$ in the integrals on the left-hand sides in the following sense.

**Proposition 5.9.** (i) *Let $0 < p < \infty$ and $1 < q \leq \infty$. If there exists $c > 0$ such that*

$$\int_0^1 \int_0^1 \Big[\frac{(R_{12}f)(s,t)}{(\log \frac{e}{s})^{1/q'}(\log \frac{e}{t})^{1/q'}}\Big]^u \frac{ds}{s\log\frac{e}{s}}\,\frac{dt}{t\log\frac{e}{t}} \leq c\|f \mid S^{n/p}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n)\|^u$$

*for all $f \in S^{n/p}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n)$, then $q \leq u$.*

(ii) *Let $1 < p < \infty$ and $0 < q \leq \infty$. If there exists $c > 0$ such that*

$$\int_0^1 \int_0^1 \Big[\frac{(R_{12}f)(s,t)}{(\log \frac{e}{s})^{1/p'}(\log \frac{e}{t})^{1/p'}}\Big]^u \frac{ds}{s\log\frac{e}{s}}\,\frac{dt}{t\log\frac{e}{t}} \leq c\|f \mid S^{n/p}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n)\|^u$$

*for all $f \in S^{n/p}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n)$, then $p \leq u$.*

It means that in Theorem 5.7 we have established the local multivariate growth envelopes for the spaces $S^{n/p}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n)$ and $S^{n/p}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n)$, respectively. More information about the concept of growth envelopes can be found in [89] and [36]. It is natural to ask what can be said about the growth envelope with respect to the usual non-increasing rearrangement $f^*$.

**Theorem 5.10.** (i) *Let* $1 \le p < \infty$, $1 < q < \infty$ *and let* $f \in S^{n/p}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n)$ *be supported in* $\Omega_1 \times \Omega_2 \subset \mathbb{R}^n \times \mathbb{R}^n$, $|\Omega_1| = |\Omega_2| = 1$. *Then*

$$\sup\{f^*(t) : \|f \mid S^{n/p}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n)\| \le 1\} \asymp \left(\log \frac{e}{t}\right)^{2/q'} \tag{5.12}$$

*for all* $t$, $0 < t \le 1$, *and*

$$\int_0^1 \left[\frac{f^*(t)}{(\log e/t)^2}\right]^q \frac{dt}{t} \le c\|f \mid S^{n/p}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n)\|^q. \tag{5.13}$$

(ii) *Let* $1 < p < \infty$ *and let* $f \in S^{n/p}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n)$ *be supported in* $\Omega_1 \times \Omega_2 \subset \mathbb{R}^n \times \mathbb{R}^n$, $|\Omega_1| = |\Omega_2| = 1$. *Then*

$$\sup\{f^*(t) : \|f \mid S^{n/p}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n)\| \le 1\} \asymp \left(\log \frac{e}{t}\right)^{2/p'} \tag{5.14}$$

*for all* $t$, $0 < t \le 1$, *and*

$$\int_0^1 \left[\frac{f^*(t)}{(\log e/t)^2}\right]^p \frac{dt}{t} \le c\|f \mid S^{n/p}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n)\|^p. \tag{5.15}$$

**Remark 5.11.** Recall the non-limiting Sobolev embeddings (5.4) and (5.6) for $F$-spaces (the same holds true for $B$-spaces) and the coincidence of the target spaces in isotropic and the related dominating mixed case. The results of Theorem 5.10 allow us to compare the measure of unboundedness for functions in the spaces $B^{2n/p}_p(\mathbb{R}^{2n})$ and $S^{n/p}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n)$ as well as in $F^{2n/p}_p(\mathbb{R}^{2n})$ and $S^{n/p}_{p\,q}F(\mathbb{R}^m \times \mathbb{R}^n)$. This turns out to be different in these cases. The following sharp results for isotropic spaces are known from the theory of the growth envelopes (see [36], [89]). If $1 < p < \infty$ and $0 < q \le \infty$ then for small $\varepsilon$,

$$\sup\{f^*(t) : \|f \mid F^{2n/p}_{p\,q}(\mathbb{R}^{2n})\| \le 1\} \asymp |\log t|^{1/p'}, \quad 0 < t < \varepsilon, \tag{5.16}$$

and

$$\int_0^\varepsilon \left[\frac{f^*(t)}{|\log t|}\right]^p \frac{dt}{t} \le c\|f \mid F^{2n/p}_p(\mathbb{R}^{2N})\|^p. \tag{5.17}$$

Moreover, if $0 < p < \infty$ and $1 < q \leq \infty$, then

$$\sup\{f^*(t) : \|f \mid B^{2n/p}_{p,q}(\mathbb{R}^{2n})\| \leq 1\} \asymp |\log t|^{1/q'} \tag{5.18}$$

and

$$\int_0^{\varepsilon} \Big[\frac{f^*(t)}{|\log t|}\Big]^q \frac{dt}{t} \leq c\|f \mid B^{2n/p}_{p,\,q}(\mathbb{R}^{2n})\|^q. \tag{5.19}$$

We point out the interesting appearance of different exponents: $2/p'$ and $2/q'$ in (5.12), (5.14), and $1/p'$ and $1/q'$ in (5.16), (5.18). As well we have $(\log e/t)^2$ on the left-hand of (5.13), (5.15) in place of $|\log t|$ in (5.17) and (5.19), respectively. Taking into account the embedding $B^{2n/p}_{p,q}(\mathbb{R}^{2n}) \hookrightarrow S^{n/p}_{p\,q}B(\mathbb{R}^m \times \mathbb{R}^n)$ (similarly for $F$-spaces) we see that the behaviour of functions near to singularities might be worse in the larger spaces. In other words we have a larger Lorentz-Orlicz space as target space for the related spaces with dominating mixed smoothness.

The above results enable us to study limiting cases of reduced Sobolev embeddings as discussed in Subsection 1.4. We give an example. The reduced Sobolev space $W^{\mathcal{M}}_p(\mathbb{R}^4)$ introduced in Example 1.10 coincides with $S^1_pH(\mathbb{R}^2 \times \mathbb{R}^2) = S^1_{p\,2}F(\mathbb{R}^2 \times \mathbb{R}^2)$. The limiting case is achieved at $p = 2$. Then, for example,

$$\sup\{f^*(t) : \|f \mid W^{\mathcal{M}}_2(\mathbb{R}^4)\| \leq 1\} \asymp \Big(\log \frac{e}{t}\Big)^2$$

in contrast to

$$\sup\{f^*(t) : \|f \mid W^2_2(\mathbb{R}^4)\| \leq 1\} \asymp \log \frac{e}{t}.$$

**5.3. Entropy numbers.** Recall our discussions in Subsection 1.3 with respect to high-dimensional approximations and phenomena. Now, we consider the general case of spaces of functions defined on $\mathbb{R}^d$, $d > 1$ and the splitting $\mathbb{R}^d = \mathbb{R} \times \cdots \times \mathbb{R}$. The spaces $S^r_{p\,q}B(\mathbb{R}^d)$ and $S^r_{p\,q}F(\mathbb{R}^d)$ are defined as in Definition 2.7 for $r_1 = \cdots = r_d$, where the extension from $d = 2$ to arbitrary $d$ is obvious. The spaces $S^r_{p\,2}F(\mathbb{R}^d)$ coincide with the fractional Sobolev spaces with dominating mixed derivatives $S^r_pH(\mathbb{R}^d)$ introduced in Definition 1.5 if $1 < p < \infty$. Of course, the remarks in Subsection 1.6 apply also to the general $d$-dimensional case. We deal with spaces on domains and study the asymptotic behaviour of entropy numbers of compact embeddings within these spaces with respect to the interrelations of dimension and dominating mixed smoothness. We follow Vybíral [96] where a unified approach is elaborated which reduces the problem to the investigation of compact embeddings of sequence spaces. Recall that we have discussed the correspondence of function spaces with dominating mixed smoothness of Besov-Lizorkin-Triebel spaces and sequence spaces in Subsections 4.2 and 4.3.

First we define function spaces on domains by restriction of functions defined on $\mathbb{R}^d$.

**Definition 5.12** (Spaces on domains)**.** Let $\Omega$ be an arbitrary bounded domain in $\mathbb{R}^d$. Let $A$ stand for $B$ or $F$. Then we put

$$S^r_{p\,q}A(\Omega) = \{f \in D'(\Omega) : \exists g \in S^r_{p\,q}A(\mathbb{R}^d) \text{ with } g\big|_\Omega = f\}$$

and

$$\|f \mid S^r_{p\,q}A(\Omega)\| = \inf \|g \mid S^r_{p\,q}A(\mathbb{R}^d)\|,$$

where the infimum is taken over all $g \in S^r_{p\,q}A(\mathbb{R}^d)$ such that its restriction $g\big|_\Omega$ to $\Omega$ coincides with $f$ in the space of distributions $D'(\Omega)$.

**Proposition 5.13** (Compactness of embeddings)**.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain. Let $0 < p_1, p_2, q_1, q_2 \leq \infty$ with $p_1, p_2 < \infty$ in the $F$-case, and let $r_1, r_2 \in \mathbb{R}$. If $A \in \{B, F\}$ and $\tilde{A} \in \{B, F\}$ then the embedding*

$$\mathrm{id} : S^{r_1}_{p_1\,q_1}A(\Omega) \to S^{r_2}_{p_2\,q_2}\tilde{A}(\Omega)$$

*is compact if and only if*

$$r_1 - r_2 - \Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+ > 0. \tag{5.20}$$

If (5.20) is satisfied it makes sense to study the degree of compactness by the asymptotic behaviour of entropy numbers.

**Definition 5.14** (Entropy numbers)**.** Let $m \in \mathbb{N}$ and let $E$, $G$ be quasi Banach spaces. Let us denote the unit balls in $E$ and $G$ by $U_E$ and $U_G$, respectively. If $T : E \to G$ is a linear and bounded operator then we put

$$e_m(T) := \inf\Big\{\varepsilon : \exists g_1, \ldots, g_{2^{m-1}} \in G,\ T(U_F) \subset \bigcup_{j=1}^{2^{m-1}} (g_j + \varepsilon U_G)\Big\}.$$

It is well-known that the operator $T$ is compact if an only if $\lim\limits_{m\to\infty} e_m = 0$. For more information on entropy numbers and related quantities such as approximation numbers and different kinds of $s$-numbers we refer to Pietsch [57], Carl, Stephani [17] and Edmunds, Triebel [27].

**Remark 5.15.** There is a close connection between entropy numbers and eigenvalues. This has been discovered by Carl [16]. If $T : E \to E$ is a compact operator and if $\lambda_m(T)$, $|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots > 0$ are its eigenvalues, then Carl's inequality

$$|\lambda_m(T)| \leq \sqrt{2}\, e_m(T), \quad m \in \mathbb{N}$$

holds. For the use of entropy numbers and related quantities in spectral theory we refer also to König [39]. Moreover, there is a strong connection of compactness of embeddings in different types of function spaces and the compactness of related operators. In this respect we refer to Edmunds, Triebel [27] and Triebel [88], [89]. Let us also mention (see also Subsection 1.3) the close connection to multivariate approximation and optimal reconstruction of functions from a finite number of values. In the context of isotropic Besov-Lizorkin-Triebel spaces this has been studied recently in the framework of so-called sampling numbers (see [90] and the references given there).

The main result concerning entropy numbers of embeddings in spaces with dominating mixed smoothness reads as follows ([96, Theorem 4.11]).

**Theorem 5.16** (Entropy numbers of compact embeddings)**.** *Let $p_i, q_i, r_i$, $i = 1, 2$, and $\Omega$ be as in Proposition 5.13. Suppose that* (5.20) *is satisfied.*

(i) *If $A \in \{B, F\}$ and $\tilde{A} \in \{B, F\}$ then*

$$e_m\big(\mathrm{id} : S^{r_1}_{p_1\,q_1}A(\Omega) \to S^{r_2}_{p_2\,q_2}\tilde{A}(\Omega)\big) \geq c m^{r_2-r_1}(\log m)^{(d-1)\left(r_1-r_2+\frac{1}{q_2}-\frac{1}{q_1}\right)_+}$$

*for $m \geq 2$ with a constant $c$ independent of $m$.*

(ii) *If $A = B$ or if $A = F$ and if*

$$r_1 - r_2 + \Big(\frac{1}{q_2} - \frac{1}{q_1}\Big) > 0 \tag{5.21}$$

*then*

$$e_m\big(\mathrm{id} : S^{r_1}_{p_1\,q_1}A(\Omega) \to S^{r_2}_{p_2\,q_2}A(\Omega)\big) \leq c\Big(\frac{\log^{d-1} m}{m}\Big)^{r_1-r_2}(\log m)^{(d-1)\left(\frac{1}{q_2}-\frac{1}{q_1}\right)}$$

*for $m \geq 2$ with a constant $c$ independent of $m$.*

(iii) *If $A = B$ or if $A = F$ and if*

$$r_1 - r_2 + \Big(\frac{1}{q_2} - \frac{1}{q_1}\Big) \leq 0 \tag{5.22}$$

*then*

$$e_m\big(\mathrm{id} : S^{r_1}_{p_1\,q_1}A(\Omega) \to S^{r_2}_{p_2\,q_2}A(\Omega)\big) \le c_\varepsilon m^{-(r_1-r_2)}(\log m)^\varepsilon$$

*for* $m \ge 2$ *and* $\varepsilon > 0$ *with a constant* $c_\varepsilon$ *independent of* $m$.

(iv) *If* $A \in \{B, F\}$ *and* $\tilde{A} \in \{B, F\}$ *and if*

$$r_1 - r_2 + \Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+ > \Big(\frac{1}{q_1} - \frac{1}{\min(p_1, p_2)}\Big)_+ + \Big(\frac{1}{p_2} - \frac{1}{q_2}\Big)_+ \tag{5.23}$$

*then*

$$e_m\big(\mathrm{id} : S^{r_1}_{p_1\,q_1}A(\Omega) \to S^{r_2}_{p_2\,q_2}\tilde{A}(\Omega)\big) \le c\Big(\frac{\log^{d-1} m}{m}\Big)^{r_1-r_2}(\log m)^{(d-1)\left(\frac{1}{q_2}-\frac{1}{q_1}\right)}$$

*for* $m \ge 2$ *with a constant* $c$ *independent of* $m$.

A new and somewhat surprising effect is the dependence of the interrelation of parameters $r_i$ and $q_i$ as we can find in (5.21) and (5.22).

**Corollary 5.17.** (i) *Assume that* (5.20) *and* (5.21) *hold. Then we have*

$$e_m\big(\mathrm{id} : S^{r_1}_{p_1\,q_1}A(\Omega) \to S^{r_2}_{p_2\,q_2}A(\Omega)\big) \asymp \Big(\frac{\log^{d-1} m}{m}\Big)^{r_1-r_2}(\log m)^{(d-1)\left(\frac{1}{q_2}-\frac{1}{q_1}\right)}.$$

(ii) *Assume that* (5.20) *and* (5.22) *hold. Then we have*

$$cm^{-(r_1-r_2)} \le e_m\big(\mathrm{id} : S^{r_1}_{p_1\,q_1}A(\Omega) \to S^{r_2}_{p_2\,q_2}A(\Omega)\big) \le c_\varepsilon m^{-(r_1-r_2)}(\log m)^\varepsilon$$

*for* $m \ge 2$ *and* $\varepsilon > 0$ *with constants* $c$ *and* $c_\varepsilon$ *independent of* $m$.

**Remark 5.18.** Part (ii) of Corollary 5.17 is less satisfactory. It is not clear whether the logarithmic factor on the left-hand side can be removed in general.

Moreover, condition (5.23) part (iv) of Theorem 5.16 does not seem to be natural.

Finally, let us state state two results for embeddings of fractional Sobolev spaces and Nikol'skii spaces with dominating mixed smoothness which can be deduced as special cases if we choose $r_2 = 0$ and put $p_1 = p$, $p_2 = q$. Let $0 < p, q < \infty$, $r > \big(\frac{1}{p} - \frac{1}{q}\big)_+$. Then

$$e_m\big(\mathrm{id} : S^{r}_{p\,2}F(\Omega) \to S^{0}_{q\,2}F(\Omega)\big) \asymp \Big(\frac{\log^{d-1} m}{m}\Big)^{r},$$

in particular,

$$e_m\big(\mathrm{id} : S^r_p H(\Omega) \to L_q(\Omega)\big) \asymp \left(\frac{\log^{d-1} m}{m}\right)^r, \quad 1 < p, q < \infty.$$

Let $0 < p < \infty$, $2 \le q < \infty$, $r > \left(\frac{1}{p} - \frac{1}{2}\right)_+$. Then

$$e_m\big(\mathrm{id} : S^r_{p\,\infty} B(\Omega) \to L_q(\Omega)\big) \asymp \left(\frac{\log^{d-1} m}{m}\right)^r (\log^{d-1} m)^{1/2}.$$

Let us compare these results with the behaviour and the dependence on the dimension $d$ for classical isotropic spaces with smoothness $r$. In that case we have

$$e_m\big(\mathrm{id} : F^r_{p\,2}(\Omega) \to L_q(\Omega)\big) \asymp e_m\big(\mathrm{id} : B^r_{p\,\infty}(\Omega) \to L_q(\Omega)\big) \asymp m^{-r/d}$$

if $r > d\big(\frac{1}{p} - \frac{1}{q}\big)_+$.

Note that we do not get satisfying results for embeddings into $L_1(\Omega)$and $L_\infty(\Omega)$. A detailed discussion and a comparison with results obtained by Belinsky [10], Dinh Dung [24], [25], and Temlyakov [78] by different methods in the periodic case can be found in [96], Subsection 4.6.

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