THE BRASCAMP–LIEB INEQUALITIES: RECENT DEVELOPMENTS

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Abstract. We discuss recent progress on issues surrounding the Brascamp–Lieb inequalities.

1. Introduction

In these notes we intend to survey some recent developments in the area of Brascamp–Lieb inequalities which are due mainly to Bennett, Christ, Tao and the present author ([BCCT1], [BCCT2]). The notes are intended to be informal and expository, and there are exercises throughout designed to engage the reader's attention – as befits a NAFSA Spring School. In every case, the reader should consult the published works [BCCT1] and [BCCT2] for precise statements, details and attributions. It is hoped that the present notes might provide an introduction to these papers, especially [BCCT1]. It is to be emphasised that we make no attempt here to document accurately the history of the subject, nor the very important contributions of other authors in this field. We refer again to [BCCT1] and [BCCT2] for matters of this nature.

We begin by setting the scene with some inequalities which are familiar to all analysts. The setting we take is of euclidean space, although some of the inequalities clearly live in more general settings.

- Hölder's inequality. Let $\sum_j p_j = 1$ and $f_j \in L^1(\mathbb{R}^n)$. Then $\prod\limits_j f_j^{p_j} \in L^1$ and

$$\int \prod_j f_j^{p_j} \leq \prod_j \Big(\int f_j \Big)^{p_j}.$$

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- The Loomis–Whitney inequality [LW]. Let Π_j denote orthogonal projection onto the hyperplane perpendicular to the x_j axis. If $f_j \in L^1(\mathbb{R}^{n-1})$ then $\prod_j f_j(\Pi_j \cdot\,)^{1/(n-1)} \in L^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \prod_j f_j(\Pi_j x)^{1/(n-1)} dx \leq \prod_j \Big(\int_{\mathbb{R}^{n-1}} f_j\Big)^{1/(n-1)}.$$

- Beckner's sharp Young inequality [Be]. Let $f_j \in L^1(\mathbb{R}^k)$, $1 \leq j \leq 3$ and $\sum_{j=1}^3 p_j = 2$. Then

$$\int_{\mathbb{R}^k \times \mathbb{R}^k} f_1^{p_1}(x) f_2^{p_2}(x-y) f_3^{p_3}(y)\, dxdy \leq \left(\prod_{j=1}^{3} \frac{(1-p_j)^{1-p_j}}{p_j^{p_j}}\right)^{k/2} \prod_{j=1}^{3} \Big(\int_{\mathbb{R}^k} f_j\Big)^{p_j}.$$

Remark. The parameter p_j in each of these inequalities corresponds to the *usual* $1/p_j$. This will simplify our notation significantly in what follows.

To set these inequalities in the framework we wish to consider, we proceed as follows. Let $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ be linear maps and p_j non-negative exponents $(1 \leq j \leq m)$. We consider inequalities of the form

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(B_j x)^{p_j} dx \leq C \prod_{j=1}^{m} \Big(\int_{\mathbb{R}^{n_j}} f_j\Big)^{p_j} \tag{1}$$

which we shall call Brascamp–Lieb inequalities after Brascamp and Lieb who first considered them, established them in certain cases and proposed their further study in [BL].

The three examples above are recovered as follows:

- Hölder's inequality. Take $n_j = n$, $B_j = I_n$, $\sum_j p_j = 1$.
- The Loomis–Whitney inequality. Take $n_j = n-1$, $B_j =$ projection onto $e_j^\perp$, $p_j = 1/(n-1)$.
- Young's convolution inequality. Take $m = 3$, $n_j = k$, $n = 2k$, $B_1(x,y) = x$, $B_2(x,y) = x-y$, $B_3(x,y) = y$, $\sum_{j=1}^3 p_j = 2$.

We shall always be interested in the *sharp* value of the constant in (1) and $C = C\{B_i, p_i\}$ is always interpreted as the best constant in (1).

Immediately, several natural questions arise:

Questions

- What is $C\{B_i, p_i\}$?
- When is $C\{B_i, p_i\}$ *finite*?
- When is it achieved, i.e. when do there exist f_j such that (1) holds with equality?
- If it is achieved, is there "uniqueness" of extremals in any sense?
- What is the **structure** of the Brascamp–Lieb inequalities (1)?

A Partial Answer:

An expression for $C\{B_i, p_i\}$ is given by Lieb's theorem:

Theorem ([L]). *The Brascamp–Lieb inequality* (1) *is exhausted by gaussians.*

To explain what this means more precisely, let A_j be a positive definite $n_j \times n_j$ matrix. Let

$$f_j(x) = \exp(-\pi\langle A_j x, x\rangle).$$

Then the left hand side of (1) is

$$\frac{1}{\det\left(\sum_{j=1}^m p_j B_j^* A_j B_j\right)^{1/2}}$$

while the right hand side is

$$\prod_{j=1}^m (\det A_j)^{-p_j/2}.$$

So obviously

$$C\{B_i, p_i\} \geq \sup_{A_j > 0} \frac{\prod_{j=1}^m (\det A_j)^{p_j/2}}{\det\left(\sum_{j=1}^m p_j B_j^* A_j B_j\right)^{1/2}} := G\{B_i, p_i\}.$$

The (highly nontrivial) content of Lieb's theorem [L] is that $C\{B_i, p_i\} = G\{B_i, p_i\}$. F. Barthe gave a different proof of Lieb's theorem in [Bar], and we shall give a further proof at the end of these notes.

But what is not obvious is under which circumstances this common value of $C\{B_i, p_i\}$ and $G\{B_i, p_i\}$ is finite. This we turn to next.

2. Finiteness

We first look at necessary conditions for finiteness of $C\{B_i, p_i\} = G\{B_i, p_i\}$.

Necessary conditions for finiteness of $G\{B_i, p_i\}$:

- If there is a non-zero $\xi \in \bigcap_j \ker B_j$, then

$$\Big\langle \sum_j p_j B_j^* A_j B_j \xi, \xi \Big\rangle = \sum_j p_j \langle A_j B_j \xi, B_j \xi \rangle = 0,$$

  whence $\sum_j p_j B_j^* A_j B_j$ is not invertible. Thus:

$$\bigcap_j \ker B_j = \{0\}.$$

- If some B_j is *not* surjective, take f_j to be a smooth gaussian bump living on a slight fattening of $\operatorname{im} B_j$. Thus $\int f_j$ will be very small while the left hand side of (1) will not be small. Thus:

$$\text{Each } B_j \text{ is surjective.}$$

- Perform the scaling $f_j(\,\cdot\,) \to f_j(\lambda\,\cdot\,)$. For the powers of λ to cancel we require

$$\sum_j p_j n_j = n.$$

Exercise 1. Check in detail the necessity of these three conditions. We shall call these the **standard** necessary conditions for finiteness of $G\{B_i, p_i\}$.

What about **sufficiency**? The first guess would be that the standard necessary conditions are sufficient for finiteness of $G\{B_i, p_i\}$. But this is not so, because of the Loomis–Whitney example: the condition $\sum_{j=1}^n p_j = n/(n-1)$ defines a hyperplane containing the point $(1/(n-1), \dots, 1/(n-1))$ which is the *only* point for which the constant is finite.

Exercise 2. Check that $(1/(n-1), \dots, 1/(n-1))$ is the *only* point for which the constant in the Loomis–Whitney inequality is finite.

Theorem 1 ([BCCT2], [BCCT1], [V1])**.** *The best constant $C\{B_i, p_i\}$ in the Brascamp–Lieb inequality* (1) *is finite if for all subspaces $V \in \mathrm{Latt}(\ker B_j)$,*

$$\dim V \le \sum_j p_j \dim(B_j V) \tag{2}$$

and

$$\sum_j p_j n_j = n. \tag{3}$$

Conversely, if $G\{B_i, p_i\}$ *is finite, then* (3) *holds, and* (2) *holds for all subspaces* V *of* $\mathbb{R}^n$*.*

Here, $\mathrm{Latt}(V_j)$ is the lattice of subspaces of a vector space generated by the subspaces V_j, i.e. the smallest collection of subspaces containing each V_j which is closed under intersections and subspace sums.

So (3) and (2) for all $V \in \mathrm{Latt}(\ker B_j) \implies C\{B_i, p_i\} < \infty \implies G\{B_i, p_i\} < \infty \implies$ (3) and (2) for all subspaces V of $\mathbb{R}^n$.

Exercise 3. By taking $V = \mathbb{R}^n$ in (2) show that each B_j must be surjective, and by taking $V = \bigcap_j \ker B_j$ show that $\bigcap_j \ker B_j = \{0\}$. (Thus we recover the standard necessary conditions.)

Exercise 4. By taking $V = \ker B_j$ in (2) show that $p_j \leq 1$ is necessary for finiteness of $G\{B_i, p_i\}$. (This we had not noticed previously.)

The necessity in Theorem 1 is easy: test on f_j a gaussian associated to an ε-neighbourhood of the unit ball of $B_j V$ and let $\varepsilon \to 0$.

Exercise 5. Complete the details of the necessity argument.

Here, although we do not do so systematically, it is appropriate to mention some historical antecedents of Theorem 1. Apart from Hölder's inequality and the Loomis–Whitney inequality [LW], there are papers of Calderón (1976) [C] and Finner (1992) [F] giving combinatorial versions of Theorem 1; in the rank-one case (see below) there are also papers of Barthe (1998) [Bar] (with a different formulation) and Carlen–Lieb–Loss (2004) [CLL].

But is Theorem 1 really an improvement over Lieb's theorem in terms of it being easier to verify in any given case? That is, is it easier to check that the hypotheses of Theorem 1 are verified than to check that the quantity defining $G\{B_i, p_i\}$ is finite?

In the first place, we "only" have to check (2) over all subspaces in $\mathrm{Latt}(\ker B_j)$ rather than calculating the quantity in the definition of G for all $n_j \times n_j$ positive definite matrices A_j. But in general it is an unsolved problem (as far as I am aware) as to whether the subspace lattice generated by a finite number of subspaces of a finite dimensional vector space is itself finite.

On the other hand, there are only finitely many *different* conditions

$$\dim V \leq \sum_j p_j \dim(B_j V)$$

as V varies over all subspaces of $\mathbb{R}^n$. These finitely many conditions thus identify the set

$$K = \{\mathbf{p} = (p_1, \dots, p_n) : (1) \text{ holds with finite constant}\}$$

as a convex polytope contained in $[0,1]^m \cap \{\mathbf{p} : \sum p_j n_j = n\}$.

Effectively these conditions describe the *faces* of the polytope K. A previous approach by F. Barthe in the rank one case characterised instead the *extreme points* of K in a quite explicit way, see below. Valdimarsson [V1] has obtained extreme point characterisations in the rank two, mixed rank one and two and co-rank one cases, and has given an algorithm for "knowing when to stop" checking conditions (2) and being sure one has the full list.

Rank one case (F. Barthe (1998), [Bar])**.** This is when each B_j maps $\mathbb{R}^n$ onto $\mathbb{R}^1$. Thus B_j is given by $B_j(x) = \langle x, v_j \rangle$ for a certain vector v_j in $\mathbb{R}^n$.

Exercise 6. (i) If $S \subseteq \{1, 2, \dots, m\}$, $|S| = n$ and $\{v_j : j \in S\}$ is a basis for $\mathbb{R}^n$, show that

$$\int_{\mathbb{R}^n} \prod_{j \in S} f_j(B_j x)\, dx = \det W \prod_j \int_{\mathbb{R}} f_j$$

where W is the linear change of variables such that $W v_j = e_j$.

(ii) Let $\mathcal{S}$ be the collection of subsets $S \subseteq \{1, 2, \dots, m\}$ such that $\{v_j : j \in S\}$ is a basis for $\mathbb{R}^n$. Show that the constant C in the rank-one Brascamp–Lieb inequality is finite if $\mathbf{p} = (p_1, p_2, \dots, p_m)$ lies in the convex hull K of χ_S as S ranges over $\mathcal{S}$.

(iii) (Harder) Show that the converse to (ii) is true.

We now turn to the proof of the sufficiency part of Theorem 1. We break the argument into several steps.

- By multilinear interpolation, it is enough to show that the constant in the Brascamp–Lieb inequality (1) is finite at extreme points of

$$\bigcap_{V \in \mathcal{L}} \Big\{\mathbf{p} : \dim V \le \sum_j p_j \dim(B_j V)\Big\} \cap \{\mathbf{p} \ge 0\} \cap \{\mathbf{p} : \mathbf{p} \cdot \mathbf{n} = n\}$$

  where $\mathcal{L} = \mathrm{Latt}(\ker B_j)$.
- Except when some $p_i = 0$ (which is handled by induction on the degree of multilinearity m), for $\mathbf{p}$ to be an extreme point there must be a proper subspace $V \in \mathcal{L}$ such that

$$\dim V = \sum_j p_j \dim(B_j V).$$

Such a subspace is called a **critical subspace** for $\{B_i, p_i\}$ and is used to reduce the problem to a lower-dimensional version of itself.

- If $\{B_i\}$ are given and U is now *any* subspace of $\mathbb{R}^n$ then there are **lower-dimensional** $\tilde{B}_j$, $\tilde{\tilde{B}}_j$ such that

$$C\{B_i, p_i\} \leq C\{\tilde{B}_i, p_i\} C\{\tilde{\tilde{B}}_i, p_i\}.$$

Indeed, define

$$\begin{aligned}
&\tilde{B}_j : U \to B_j U, \quad x \mapsto B_j x,\\
&\tilde{\tilde{B}}_j : U^\perp \to (B_j U)^\perp, \quad x \mapsto \Pi_{(B_j U)^\perp} B_j x,\\
&\Gamma_j : U^\perp \to B_j U, \quad x \mapsto \Pi_{B_j U} B_j x,
\end{aligned}$$

where $\Pi_{(B_j U)^\perp}$ and $\Pi_{B_j U}$ denote the orthogonal projection onto the relevant spaces.

Then, we can calculate as follows, (with the dependence on p_j in $C\{B_j, p_j\}$ suppressed):

$$\begin{aligned}
\int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{p_j}(B_j x)\, dx &= \int_{U^\perp} \int_U \prod_{j=1}^m f_j^{p_j}(\tilde{B}_j \tilde{x} + B_j \tilde{\tilde{x}})\, d\tilde{x} d\tilde{\tilde{x}}\\
&\leq C\{\tilde{B}_j\} \int_{U^\perp} \prod_{j=1}^m \left(\int_{B_j U} f_j(\tilde{y} + B_j \tilde{\tilde{x}})\, d\tilde{y} \right)^{p_j} d\tilde{\tilde{x}}\\
&= C\{\tilde{B}_j\} \int_{U^\perp} \prod_{j=1}^m \left(\int_{B_j U} f_j(\tilde{y} + \Gamma_j \tilde{\tilde{x}} + \tilde{\tilde{B}}_j \tilde{\tilde{x}})\, d\tilde{y} \right)^{p_j} d\tilde{\tilde{x}}\\
&= C\{\tilde{B}_j\} \int_{U^\perp} \prod_{j=1}^m \left(\int_{B_j U} f_j(\tilde{y} + \tilde{\tilde{B}}_j \tilde{\tilde{x}})\, d\tilde{y} \right)^{p_j} d\tilde{\tilde{x}}\\
&\leq C\{\tilde{B}_j\} C\{\tilde{\tilde{B}}_j\} \prod_{j=1}^m \left(\int_{B_j U^\perp} \int_{B_j U} f_j(\tilde{y} + \tilde{\tilde{y}})\, d\tilde{y} d\tilde{\tilde{y}} \right)^{p_j}\\
&= C\{\tilde{B}_j\} C\{\tilde{\tilde{B}}_j\} \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j(y)\, dy \right)^{p_j}.
\end{aligned}$$

Let us explain the arguments used in this chain of inequalities. We have used for the first inequality that for almost any $\tilde{\tilde{x}} \in U^\perp$ the tuple $(f_j(\cdot + B_j \tilde{\tilde{x}}))$ consists of non-negative integrable functions defined on $B_j U$

and we can therefore use the Brascamp–Lieb inequality for $\{\tilde{B}_j, p_j\}$. For the next equality we use the definitions of Γ_j and $\tilde{\tilde{B}}_j$, and for the one below that we use the translation invariance of the inner integral and the fact that $\Gamma_j x \in B_j U$ for any $x \in U^\perp$. For the second inequality we use the fact that for any j the inner integral defines a non-negative function of $\tilde{\tilde{B}}_j \tilde{\tilde{x}}$ with domain $(B_j U)^\perp$ and we can therefore use the Brascamp–Lieb inequality for the datum $\{\tilde{\tilde{B}}_j, p_j\}$.

- If we take $U = V$ to be a **critical** subspace, then the condition

$$\dim W \leq \sum_j p_j \dim(B_j W) \quad \text{for all} \quad W \in \mathrm{Latt}(\ker B_j)$$

is inherited by $\{\tilde{B}_j, p_j\}$ and $\{\tilde{\tilde{B}}_j, p_j\}$.

Exercise. 7 Check this.

Gathering things together, we have a proof of sufficiency by induction. Formally, we do a double induction on m and then n. For the case $m = 1$ the subspace lattice is just $\ker B_1$ and of course we need this to be $\{0\}$ and $p_1 = 1$. Assuming inductively that the result is true for some fixed $m-1$, and all n, we want to show it is true for m and all n. For this m, and $n = 1$, we need each $\ker B_j = \{0\}$ (i.e. each $B_j \neq 0$), and things reduce to Hölder's inequality. Now assume that the result is true for this m and $n = 1, 2, \ldots, n_0 - 1$. Using the main argument presented above with underlying dimension n_0, one possibility leads to a case from the previous level of multilinearity $m-1$, and the other leads to a critical subspace of dimension less than n_0 for which we may assume inductively that the corresponding $C\{\tilde{B}_i, p_i\}$ and $C\{\tilde{\tilde{B}}_i, p_i\}$ are finite, hence also $C\{B_i, p_i\}$ is finite.

Exercise 8. Convince yourself that the subspace lattice generated by the $\{\ker B_j\}$ is the smallest class of subspaces on which to assume (2) in order for the proof to work.

The original version of the theorem was stated in [BCCT1] and [BCCT2] assuming (2) for *all* subspaces V of $\mathbb{R}^n$. The variant presented here is from Valdimarsson's thesis [V1].

3. Factorisation and structure

In reading the arguments presented above, we naturally ask ourselves to what extent have we **factorised** the Brascamp–Lieb problem over critical subspaces? The next theorem gives a partial answer.

Theorem 2 ([BCCT1], [BCCT2])**.** *If V is critical, and $\tilde{B}_j$ and $\tilde{\tilde{B}}_j$ are as in the previous section, then*

$$C\{B_i, p_i\} = C\{\tilde{B}_i, p_i\} C\{\tilde{\tilde{B}}_i, p_i\}$$

and

$$G\{B_i, p_i\} = G\{\tilde{B}_i, p_i\} G\{\tilde{\tilde{B}}_i, p_i\}.$$

The "$\leq$" inequality for G is not so obvious (but is not hard either, and does not need criticality, see [BCCT1], towards the end of Section 4). For C we have just done it.

For the inequalities "$\geq$" the obvious strategy is to take tensor products of lower dimensional test functions approximating the respective suprema. (Note that we may assume (3) as otherwise there is no content.)

Here we come across a subtlety: even if the two problems $\{\tilde{B}_i, p_i\}$ and $\{\tilde{\tilde{B}}_i, p_i\}$ admit (for example) gaussian extremals, it may *not* be the case that tensor products of these give *exact* gaussian extremals for $\{B_i, p_i\}$.

The reason for this is, the way things are set up, we can represent the operator B_j by the matrix

$$\begin{pmatrix} \tilde{\tilde{B}}_j & 0 \\ \Gamma_j & \tilde{B}_j \end{pmatrix}$$

acting on a column vector with components in $V^\perp$ followed by V. The presence of Γ_j is an obstruction to exact factorisation. Let us see why this is so.

It is very easy to check that if $\{A_i\}$ are positive definite matrices giving gaussian extremals for (1) then so are $\{\lambda A_i\}$ for all $\lambda > 0$. Thus if $\tilde{A}_i$ and $\tilde{\tilde{A}}_i$ are gaussian extremals for the problems $\{\tilde{B}_i, p_i\}$ and $\{\tilde{\tilde{B}}_i, p_i\}$, it is therefore very natural to test on

$$A_j := \begin{pmatrix} \lambda \tilde{\tilde{A}}_j & 0 \\ 0 & \mu \tilde{A}_j \end{pmatrix}$$

(where λ and μ are arbitrary positive scalars) to obtain

$$G\{B_j, p_j\} \geq \left(\frac{\prod_j (\det A_j)^{p_j}}{\det \left(\sum_j p_j B_j^* A_j B_j \right)} \right)^{1/2} = \left(\frac{\prod_j (\det \tilde{\tilde{A}}_j)^{p_j} \prod_j (\det \tilde{A}_j)^{p_j}}{\det R} \right)^{1/2}$$

where

$$R = \begin{pmatrix} \sum p_j \tilde{\tilde{B}}_j^* \tilde{\tilde{A}}_j \tilde{\tilde{B}}_j + \frac{\mu}{\lambda} \sum p_j \Gamma_j^* \tilde{A}_j \Gamma_j & \left(\frac{\mu}{\lambda}\right)^{1/2} \sum p_j \Gamma_j^* \tilde{A}_j \tilde{B}_j \\ \left(\frac{\mu}{\lambda}\right)^{1/2} \sum p_j \tilde{B}_j^* \tilde{A}_j \Gamma_j & \sum p_j \tilde{B}_j^* \tilde{A}_j \tilde{B}_j \end{pmatrix}.$$

So in the limit as $\frac{\mu}{\lambda}$ tends to zero,

$$\left(\frac{\prod_j (\det A_j)^{p_j}}{\det\left(\sum_j p_j B_j^* A_j B_j\right)}\right)^{1/2}$$

approaches

$$\left(\frac{\prod_j (\det \tilde{\tilde{A}}_j)^{p_j}}{\det\left(\sum_j p_j \tilde{\tilde{B}}_j^* \tilde{\tilde{A}}_j \tilde{\tilde{B}}_j\right)}\right)^{1/2} \left(\frac{\prod_j (\det \tilde{A}_j)^{p_j}}{\det\left(\sum_j p_j \tilde{B}_j^* \tilde{A}_j \tilde{B}_j\right)}\right)^{1/2}$$

which is as close to $G\{\tilde{B}_j, p_j\}\, G\{\tilde{\tilde{B}}_j, p_j\}$ as we desire.

Exercise 9. (i) Carry out the details of the above calculation. Note that criticality enters by making certain powers of λ, μ arising equal to 0. (You will also need to use (3).)

(ii) Do a similar calculation to establish the lower bounds for $C\{B_j, p_j\}$.

(iii) Assume (2) and (3) hold. Show that V is critical for $\{B_j, p_j\}$ if and only if the constant in (1) for datum

$$\begin{pmatrix} \tilde{\tilde{B}}_j & 0 \\ \lambda \Gamma_j & \tilde{B}_j \end{pmatrix}$$

is independent of $\lambda > 0$. (Replace f_j by $f_j(\alpha\,\cdot\,, \beta\,\cdot\,)$.)

The upshot of all this is that the factorisation is in general "analytic" rather than "algebraic" as we need to take limits to make the effect of Γ_j disappear.

On the other hand, if Γ_j happens to be 0 for all j, that is $B_j V^\perp \subseteq (B_j V)^\perp$ for all j, then the problem $\{B_i, p_i\}$ factorises algebraically over the critical subspace V into two **orthogonal** subproblems (in a way which is independent of the $\{p_i\}$); tensor products of any extremals for $\{\tilde{B}_i, p_i\}$ and $\{\tilde{\tilde{B}}_i, p_i\}$ are now indeed extremals for $\{B_i, p_i\}$. We do not need to employ λ, μ, and we do not need the (p-dependent) notion of criticality to make powers disappear in the above calculation.

In this case, by Exercise 8 (iii), V will be automatically critical for $\{B_i\}$ for all values of p such that (2) and (3) hold.

Moreover, in this situation the roles of V and $V^\perp$ are completely interchangeable, and so $V^\perp$ is likewise a critical subspace for $\{B_i\}$ for all values of p such that (2) and (3) hold.

In fact it is this property of a critical subspace V possessing a complementary critical subspace which characterises when we have exact algebraic factorisation – once we have also taken into account the affine invariance of the whole problem. We first need a definition:

Definition. An **ultracritical subspace** for $\{B_j\}$ is a subspace V of $\mathbb{R}^n$ such that there exists a complementary subspace W for V such that, for all j, B_jV and B_jW are complementary in $\mathbb{R}^{n_j}$.

Notice that this is symmetric in V, W. The terminology "ultracritical" is justified by:

Proposition ([BCCT1])**.** *Fix $\{B_j, p_j\}$ such that* (2) *and* (3) *hold. The following conditions on a subspace V of $\mathbb{R}^n$ are equivalent:*

(i) *V is ultracritical;*
(ii) *V is critical for $\{B_j, p_j\}$ and possesses a complementary space which is also critical for $\{B_j, p_j\}$;*
(iii) *V is critical for $\{B_j, p_j\}$ and there exist linear changes of variables of $\mathbb{R}^n$ and $\mathbb{R}^{n_j}$ after which the corresponding Γ_j are all zero.*

For now we note that according to all this, if V is ultracritical and if the problems $\{\tilde{B}_i, p_i\}$ and $\{\tilde{\tilde{B}}_i, p_i\}$ have extremals for C or G, then $\{B_i, p_i\}$ has a family of corresponding extremals indexed by *two* real parameters.

It also raises the possibility that there may be no extremals to a problem which has a critical subspace but which is not ultracritical. We shall see in Section 5 that this is indeed the case.

It is in the problem of understanding the existence of gaussian extremals that the importance of the distinction between ultracritical and merely critical subspaces lies. For this problem one can develop an appropriate structural theory, under the assumption that $C\{B_i, p_i\} < \infty$, i.e. by Theorem 1, that (2) and (3) hold.

Structural perspective on Brascamp–Lieb inequalities with data $\{B_i, p_i\}$ (satisfying (2), (3)).

We describe this structure algorithmically.

- First seek ultracritical subspaces (independent of $\{p_i\}$) until none are left.
- Next seek critical (but non-ultracritical) subspaces inside the ultracritical ones.
- Repeat the first step for the orthogonal complements of such critical subspaces; enter a loop which halts when no further critical subspaces are to be found.
- If we have (a 1-parameter family of) gaussian extremals for each of a pair of "ultracritical components", then the original problem has a two-parameter family of gaussian extremals.

Exercise 10. Check that the third bullet point in the above "algorithm" is really needed.

It is amusing to note that one can use quiver theory to study the uniqueness (up to certain equivalence relations) of the above decomposition, but this will be of no relevance for us. A more formal approach to this structural perspective is given in [BCCT1].

4. Gaussian extremals

In this section we begin to study whether or not the Brascamp–Lieb inequality (1) possesses gaussian extremals of the form $f_j(x) = \exp(-\pi\langle A_j x, x\rangle)$, and related question of whether the expression

$$G\{B_i, p_i\} := \sup_{A_j>0} \frac{\prod_{j=1}^m (\det A_j)^{p_j/2}}{\det(\sum_{j=1}^m p_j B_j^* A_j B_j)^{1/2}}$$

has extremisers. It turns out that these questions are equivalent, and we begin with a characterisation of such extremals $\{A_j\}$.

We recall that the three standard necessary conditions for finiteness of $C\{B_j, p_j\}$ or $G\{B_i, p_i\}$ are: each B_j surjective, $\cap_j \ker B_j = \{0\}$ and (3).

Theorem 3 ([BCCT1]). *Suppose $\{B_i, p_i\}$ satisfy the three standard necessary conditions. Let $\{A_i\}$ be positive definite $n_i \times n_i$ matrices. The following are equivalent:*

(i) $\{\exp(-\pi\langle A_j x, x\rangle)\}$ *gives an extremal for* (1);
(ii) $\{A_j\}$ *gives an extremal for*

$$\sup_{A_j>0} \frac{\prod_{j=1}^m (\det A_j)^{p_j/2}}{\det\big(\sum_{j=1}^m p_j B_j^* A_j B_j\big)^{1/2}} := G\{B_i, p_i\};$$

(iii) $M := \sum p_j B_j^* A_j B_j$ *is invertible and* $M \geq B_l^* A_l B_l$ *for all* l;
(iv) $B_j M^{-1} B_j^* = A_j^{-1}$ *for all* j.

The scheme of the proof is (iii) $\Longrightarrow$ (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii).

That (i) $\Longrightarrow$ (ii) is trivial, (ii) $\Longrightarrow$ (iv) is a variational argument, and (iv) $\Longrightarrow$ (iii) is linear algebra. See below for these implications. The main part of the argument is (iii) $\Longrightarrow$ (i), and for this several approaches are available. One of these is via heat flow (cf. Carlen, Lieb and Loss, [CLL]) and is presented in [BCCT1]. However, we have chosen a different route for these notes, taken from [BCT]. Note that (ii) $\Longrightarrow$ (i) is *not* obvious.

Exercise 11. (i) Give a simple direct proof of BECKNER's sharp Young inequality [Be] by taking $A_j = (p_j(1-p_j))^{-1} I_k$ and verifying condition (iv) of Theorem 3.

(ii) Prove KEITH BALL's Geometric Brascamp–Lieb inequality [Ball], [Bar]: Let E_j be linear subspaces of $\mathbb{R}^n$ and let $P_j : \mathbb{R}^n \to \mathbb{R}^n$ be orthogonal projection onto E_j. If $\sum_{j=1}^m p_j P_j = I_n$, then

$$\int \prod_{j=1}^m f_j(P_j x)^{p_j}\, dx \leq \prod_{j=1}^m \left(\int_{E_j} f_j \right)^{p_j},$$

and the standard gaussians $\exp(-\pi|x|^2)$ are extremals.

Proof of (ii) $\Longrightarrow$ (iv). Suppose that $\{A_j\}$ forms a local maximum for the functional defining G. Then M and each A_j are invertible. Fix j, and let $x \in \mathbb{R}^{n_j} \setminus \{0\}$. Define the rank-one operator $x \otimes x$ by $(x \otimes x)(y) = \langle x, y \rangle x$. Note that for sufficiently small $|h|$, $A_j + hx \otimes x$ is still positive definite. Now replace A_j by $A_j + hx \otimes x$ and leave all the other A_l unchanged.

Then for sufficiently small $|h|$,

$$\frac{(\det(A_j + hx \otimes x))^{p_j}}{\det(\sum_i p_i B_i^* A_i B_i + p_j B_j^* hx \otimes x B_j)} \leq \frac{(\det A_j)^{p_j}}{\det(\sum_i p_i B_i^* A_i B_i)}.$$

So we differentiate the left hand side of the previous inequality with respect to h and then set $h = 0$ to obtain

$$\langle A_j^{-1} x, x \rangle = \langle B_j M^{-1} B_j^* x, x \rangle.$$

(Here we have used the elementary fact that $\det(A + hx \otimes x) = \det(A) + h\langle \tilde{A}x, x \rangle$ where $\tilde{A}$ denotes the adjugate matrix of A.) So (iv) holds.

Proof of (iv) $\Longrightarrow$ (iii). Suppose that M and A are invertible $n \times n$ and $q \times q$ matrices respectively, where $q \leq n$, and that $BM^{-1}B^* = A^{-1}$. We claim that $M \geq B^*AB$. To see this, let $\langle x, y \rangle_M = \langle Mx, x \rangle$, and observe that $M^{-1}B^*AB$ is a self-adjoint projection on $\mathbb{R}^n$ with respect to the inner product $\langle \cdot\,, \cdot \rangle_M$. Thus $\|M^{-1}B^*ABx\|_M \leq \|x\|_M$ for all x. Now

$$\begin{aligned} \langle B^*ABx, x \rangle &= \langle M^{-1}B^*ABx, x \rangle_M \\ &\leq \|x\|_M \|M^{-1}B^*ABx\|_M \\ &\leq \|x\|_M^2 \\ &= \langle Mx, x \rangle. \end{aligned}$$

The claim follows and so does the implication (iv) $\Longrightarrow$ (iii).

Now we turn to the main implication (iii) $\Longrightarrow$ (i), which is that $M = \sum B_j^* A_j B_j$ invertible and $M \geq B_l^* A_l B_l$ for all l implies that $\exp(-\pi\langle A_j x, x\rangle)$ gives an extremal for (1). This statement will follow from a more general result which we now describe.

If A is a positive semidefinite $n \times n$ real matrix, we say that a real-valued function f is **of class** A if it is the convolution of the centred gaussian $\exp(-\pi\langle Ax, x\rangle)$ with a positive finite measure on $\mathbb{R}^n$.

Exercise 12. Show that $\exp(-\pi\langle Qx, x\rangle)$ where $Q \leq A$ in the sense of positive definite matrices is of class A.

The *standard* gaussian of class A is defined to be $\exp(-\pi\langle Ax, x\rangle)$.

Now let $1 \leq i \leq m$. Let A_i be a positive semidefinite real matrix, μ_i a positive finite measure on $\mathbb{R}^n$ and $p_i > 0$. Let $p = (p_1, \dots, p_m)$. If $t \geq 0$ define

$$f_i(x,t) = \int \exp(-\pi\langle A_i(x - tv), (x - tv)\rangle)\, d\mu_i(v).$$

(Thus $f_i(x, 1)$ is a typical function of class A_i.) Finally define

$$F(x,t)^p = \prod_{i=1}^{m} f_i(x,t)^{p_i}.$$

Theorem 4 ([BCT]). *Suppose that A_i, μ_i, p_i and F^p are as above. If $\sum_i p_i A_i \geq A_l$ for all l and $\sum_i p_i A_i$ is invertible, then $\int F(x,t)^p\, dx$ is a decreasing function of t.*

We can translate Theorem 4 into the language of solutions to heat equations:

Proposition 2. *Suppose A_i are positive definite $n \times n$ matrices. Let $\triangle_{A_i^{-1}} = \nabla \cdot A_i^{-1} \nabla$. Suppose that u_i are functions on $\mathbb{R}^n \times (0, \infty)$ satisfying*

$$\triangle_{A_i^{-1}} u_i = \partial_s u_i$$

with initial data positive finite measures. If $\sum_i p_i A_i \geq A_l$ for all l, then

$$\int s^{-n/2} \prod_i \left[u_i(x,s) s^{n/2}\right]^{p_i} dx$$

is increasing in s.

Taking $m = 1$ and $A = I$, we draw the conclusion that for solutions u to the heat equation with nonnegative integrable initial data, when $p \geq 1$ the quantity

$$s^{n(p-1)/2} \int_{\mathbb{R}^n} u(x,s)^p\, dx$$

is increasing in s. This appears to be a new result. The corresponding question for harmonic functions, that is, solutions to Laplace's equation on the upper half space seems to be open.

How does Theorem 4 give (iii) $\Longrightarrow$ (i) in Theorem 3?

Take f_j to be of class A_j. So for $y \in \mathbb{R}^{n_j}$ we have

$$f_j(y) = (\det A_j)^{1/2} \int_{\mathbb{R}^{n_j}} \mathrm{e}^{-\pi\langle A_j(y-w),(y-w)\rangle}\, d\mu_j(w)$$

for some finite non-negative measure μ_j on $\mathbb{R}^{n_j}$. For each j we define a measure $\widetilde{\mu}_j$ on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} \phi\, d\widetilde{\mu}_j = \int_{\mathbb{R}^{n_j}} \phi\left(B_j^*(B_jB_j^*)^{-1}w\right) d\mu_j(w),$$

(here we have used the surjectivity of B_j) and observe that for $x \in \mathbb{R}^n$,

$$f_j(B_jx) = (\det A_j)^{1/2} \int_{\mathbb{R}^n} \mathrm{e}^{-\pi\langle B_j^*A_jB_j(x-v),(x-v)\rangle}\, d\widetilde{\mu}_j(v).$$

Hence

$$\int \prod f_j^{p_j}(B_jx)\, dx = \left(\prod (\det A_j)^{p_j}\right)^{1/2} \int_{\mathbb{R}^n} \prod g_j(x,1)^{p_j}\, dx,$$

where for each $t \in \mathbb{R}$,

$$g_j(x,t) = \int_{\mathbb{R}^n} \mathrm{e}^{-\pi\langle B_j^*A_jB_j(x-tv),(x-tv)\rangle}\, d\widetilde{\mu}_j(v).$$

Now, by Theorem 4, (using $\cap \ker B_j = \{0\}$ to check invertibilty in the hypothesis!),

$$\int \prod_j f_j^{p_j}(B_jx)\, dx \le \left(\prod_j (\det A_j)^{p_j}\right)^{1/2} \int_{\mathbb{R}^n} \prod_j g_j(x,0)^{p_j}\, dx.$$

The result now follows for f_j of class A_j on observing that

$$\int_{\mathbb{R}^n} \prod_j g_j(x,0)^{p_j}\, dx = \det\left(\sum_j p_jB_j^*A_jB_j\right)^{-1/2} \prod_j \|\widetilde{\mu}_j\|^{p_j},$$

and $\|\widetilde{\mu}_j\| = \|\mu_j\| = \int f_j$ for all j.

Finally, in the presence of the scaling condition (3) we can simply drop the hypotheis f_j of class A_j by a limiting argument.

Now we turn to the **proof of Theorem 4**. It is this proof which turns out to be **essential** in the study of the multilinear Kakeya maximal function [BCT]. (The multilinear Kakeya maximal function was discussed in the NAFSA lectures, but we have decided to omit it from the written version presented here.)

We begin by considering the case when $p \in \mathbb{N}^m$. Then the quantity $Q_p(t)$ can be expanded as

$$\int_{\mathbb{R}^n}\int_{(\mathbb{R}^n)^{p_1}} \cdots \int_{(\mathbb{R}^n)^{p_m}} \mathrm{e}^{-\pi \sum_{j=1}^m \sum_{k=1}^{p_j} \langle A_j(x-v_{j,k}t),(x-v_{j,k}t)\rangle} \prod_{j=1}^m \prod_{k=1}^{p_j} d\mu_j(v_{j,k})\, dx.$$

On completing the square we find that

$$\sum_{j=1}^m \sum_{k=1}^{p_j} \langle A_j(x-v_{j,k}t),(x-v_{j,k}t)\rangle = \langle A_*(x-\overline{v}t),(x-\overline{v}t)\rangle + \delta t^2,$$

where $A_* := \sum_{j=1}^m p_j A_j$ is a positive definite matrix,

$$\overline{v} := A_*^{-1} \sum_{j=1}^m A_j \sum_{k=1}^{p_j} v_{j,k}$$

is the weighted average velocity, and δ is the weighted variance of the velocity,

$$\delta := \sum_{j=1}^m \sum_{k=1}^{p_j} \langle A_j v_{j,k}, v_{j,k}\rangle - \langle A_*\overline{v}, \overline{v}\rangle.$$

Using translation invariance in x, we have

$$Q_p'(t) =$$
$$-2\pi t \int_{\mathbb{R}^n}\int_{(\mathbb{R}^n)^{p_1}} \cdots \int_{(\mathbb{R}^n)^{p_m}} \delta \prod_{j=1}^m \prod_{k=1}^{p_j} \mathrm{e}^{-\pi\langle A_j(x-v_{j,k}t),(x-v_{j,k}t)\rangle}\, d\mu_j(v_{j,k}) dx,$$

(and since $\delta \geq 0$ we have result for $p_j \in \mathbb{N}$). Let $\mathbf{v}_j$ be v_j regarded as a random variable associated to the probability measure

$$\frac{\mathrm{e}^{-\pi\langle A_j(x-v_jt),(x-v_jt)\rangle}\, d\mu_j(v_j)}{f_j(t,x)},$$

and let $\mathbf{v}_{j,1}, \ldots, \mathbf{v}_{j,p_j}$ be p_j independent samples of these random variables (with the $\mathbf{v}_{j,k}$ being independent in both j and k). Then

$$Q_p'(t) = -2\pi t \int_{\mathbb{R}^n} \mathbf{E}(\delta) \prod_{j=1}^{m} f_j(t,x)^{p_j}\, dx$$

where δ is now considered a function of the $\mathbf{v}_{j,k}$, and $\mathbf{E}(\cdot)$ denotes probabilistic expectation.

Recalling that

$$\delta := \sum_{j=1}^{m} \sum_{k=1}^{p_j} \langle A_j v_{j,k}, v_{j,k} \rangle - \langle A_* \overline{v}, \overline{v} \rangle$$

and using linearity of expectation we have

$$\mathbf{E}(\delta) = \sum_{j=1}^{m} \sum_{k=1}^{p_j} \mathbf{E}(\langle A_j \mathbf{v}_{j,k}, \mathbf{v}_{j,k} \rangle) - \mathbf{E}(\langle A_* \overline{\mathbf{v}}, \overline{\mathbf{v}} \rangle),$$

(where $\overline{\mathbf{v}}$ is $\overline{v} := A_*^{-1} \sum_{j=1}^{m} A_j \sum_{k=1}^{p_j} v_{j,k}$ regarded as a random variable).

By symmetry, the first term is $\sum_{j=1}^{m} p_j \mathbf{E}(\langle A_j \mathbf{v}_j, \mathbf{v}_j \rangle)$.

The second term is

$$\begin{aligned}
\mathbf{E}(\langle A_* \overline{\mathbf{v}}, \overline{\mathbf{v}} \rangle) &= \mathbf{E}\Big(\Big\langle A_*^{-1} \sum_{j=1}^{m} \sum_{k=1}^{p_j} A_j \mathbf{v}_{j,k}, \sum_{j'=1}^{m} \sum_{k'=1}^{p_{j'}} A_{j'} \mathbf{v}_{j',k'} \Big\rangle\Big) \\
&= \sum_{j=1}^{m} \sum_{j'=1}^{m} \sum_{k=1}^{p_j} \sum_{k'=1}^{p_{j'}} \mathbf{E}(\langle A_*^{-1} A_j \mathbf{v}_{j,k}, A_{j'} \mathbf{v}_{j',k'} \rangle) \\
&= \sum_{j=1}^{m} \sum_{j'=1}^{m} \sum_{k=1}^{p_j} \sum_{k'=1}^{p_{j'}} \mathbf{E}(\langle A_*^{-1} A_j \mathbf{v}_{j,k}, A_{j'} \mathbf{v}_{j',k'} \rangle) \\
&= \sum_{\substack{j=j' \\ k=k'}} + \sum_{(j,k)\neq(j',k')} \\
&= \sum_{j=1}^{m} p_j \mathbf{E}(\langle A_*^{-1} A_j \mathbf{v}_j, A_j \mathbf{v}_j \rangle) + \sum_{\substack{j \\ k\neq k'}} + \sum_{\substack{j\neq j' \\ k,k'}}.
\end{aligned}$$

When $(j,k) \neq (j',k')$, we can factorise the expectation using independence and symmetry to obtain

$$\begin{aligned}
\mathbf{E}(\langle A_* \overline{\mathbf{v}}, \overline{\mathbf{v}}\rangle) = &\sum_{j=1}^m p_j \mathbf{E}(\langle A_*^{-1} A_j \mathbf{v}_j, A_j \mathbf{v}_j\rangle) \\
&+ \sum_{j=1}^m p_j(p_j - 1)\langle A_*^{-1} A_j \mathbf{E}(\mathbf{v}_j), A_j \mathbf{E}(\mathbf{v}_j)\rangle \\
&+ \sum_{1 \le j \neq j' \le m} p_j p_{j'} \langle A_*^{-1} A_j \mathbf{E}(\mathbf{v}_j), A_{j'} \mathbf{E}(\mathbf{v}_{j'})\rangle.
\end{aligned}$$

So

$$\begin{aligned}
\mathbf{E}(\langle A_* \overline{\mathbf{v}}, \overline{\mathbf{v}}\rangle) = &\sum_{j=1}^m p_j \mathbf{E}(\langle A_*^{-1} A_j \mathbf{v}_j, A_j \mathbf{v}_j\rangle) \\
&- \sum_{j=1}^m p_j \langle A_*^{-1} A_j \mathbf{E}(\mathbf{v}_j), A_j \mathbf{E}(\mathbf{v}_j)\rangle \\
&+ \sum_{1 \le j, j' \le n} p_j p_{j'} \langle A_*^{-1} A_j \mathbf{E}(\mathbf{v}_j), A_{j'} \mathbf{E}(\mathbf{v}_{j'})\rangle,
\end{aligned}$$

and after a little more algebra we arrive at

$$\begin{aligned}
\mathbf{E}(\delta) = &\sum_{j=1}^m p_j \mathbf{E}(\langle (A_j - A_j A_*^{-1} A_j)\mathbf{v}_j, \mathbf{v}_j\rangle) \\
&- \sum_{j=1}^m p_j \langle (A_j - A_j A_*^{-1} A_j)\mathbf{E}(\mathbf{v}_j), \mathbf{E}(\mathbf{v}_j)\rangle \\
&+ \sum_{j=1}^m p_j \langle A_j \mathbf{E}(\mathbf{v}_j), \mathbf{E}(\mathbf{v}_j)\rangle \\
&- \sum_{j,j'} p_j p_{j'} \langle A_*^{-1} A_j \mathbf{E}(\mathbf{v}_j), A_{j'} \mathbf{E}(\mathbf{v}_{j'})\rangle \\
= &\sum_{j=1}^m p_j \mathbf{E}(\langle (A_j - A_j A_*^{-1} A_j)(\mathbf{v}_j - \mathbf{E}(\mathbf{v}_j)), (\mathbf{v}_j - \mathbf{E}(\mathbf{v}_j))\rangle \\
&+ \sum_{j=1}^n p_j \langle A_j(\mathbf{E}(\mathbf{v}_j) - \mathbf{E}(\overline{\mathbf{v}})), (\mathbf{E}(\mathbf{v}_j) - \mathbf{E}(\overline{\mathbf{v}}))\rangle.
\end{aligned}$$

Note that this last expression is nonnegative by hypothesis, makes sense for *all* nonnegative values of p_j and, after multiplication by $\det A_*$, is polynomial in p. In particular, as in the preceding arguments,

$$\mathbf{E}(\overline{\mathbf{v}}) = A_*^{-1} \sum_{j=1}^{m} p_j A_j \mathbf{E}(\mathbf{v}_j)$$

makes sense for all $p_j > 0$.

In summary, what we have done is show that for $p_j \in \mathbb{N}$, there is a formula

$$(\det A_*) Q_p'(t) = -2\pi t \int_{\mathbb{R}^n} G(p,t,x) \prod_j f_j(x,t)^{p_j}\, dx$$

where

- $G(\,\cdot\,,t,x)$ is defined for **all** $p_j > 0$,
- $G(\,\cdot\,,t,x)$ is polynomial,
- under the hypothesis of the theorem, $G \geq 0$.

So if we could see that the formula above remained valid for *all* $p_j > 0$, we would be finished. That this is so is a consequence of a simple uniqueness lemma:

Lemma 1 ([BCT]). *Let $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ be non-negative bounded measurable functions whose product $f_1 \ldots f_m$ is rapidly decreasing. Let $G : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be polynomial in its first variables $p = (p_1, \ldots, p_m)$, with measurable coefficients of polynomial growth in its second variables. If the identity*

$$\int_{\mathbb{R}^n} G(p,x) f_1(x)^{p_1} \cdots f_m(x)^{p_m}\, dx = 0$$

holds for all $p \in \mathbb{N}^m$, then it also holds for all $p \in (0,\infty)^m$.

Exercise 13. Prove this in the (already typical) case where $m = 1$ and G is a polynomial of degree 1. (Hint: Do so first for $p > 1$, when the function $t \mapsto t^p$ is approximable in the norm $\|\phi\|_\infty + \|\phi'\|_\infty$ by polynomials with zero constant term.)

It is an easy matter to find another suitable representation of $(\det A_*) Q_p'(t)$ with which to compare the one we constructed above in order to apply Lemma 1, and hence to conclude the proof of the theorem.

5. Existence and uniqueness of gaussian extremals, and Lieb's theorem again

Given data (B_j, p_j) we established in the last section a necessary and sufficient condition on A_j for the Brascamp–Lieb inequality (1) to be extremised by $\exp(-\pi\langle A_j x, x\rangle)$.

But the form of this condition does not make it clear whether such an extremiser exits, and, if it does, whether it is unique up to the trivial 1-parameter family given by λA_j where $\lambda > 0$. We *do* know that if B_j has an ultracritical subspace over which the two factors both possess gaussian extremals, then the original problem possesses a 2-parameter family of gaussian extremals. We also saw the possibility that it might be the case that when there are critical but not ultracritical subspaces, then there may not be gaussian extremals.

Theorem 5 ([BCCT1]). *Let $\{B_i, p_i\}$ be given. Assume that $C\{B_i, p_i\} < \infty$. Then gaussian extremals exist for* (1) *if and only if every critical subspace is ultracritical.*

One first proves (under (2), (3)) that if there is no critical subspace, then extremisers to $G\{B_i, p_i\}$ exist. The difficulty here is that the space of m-tuples of positive definite matrices in the definition of $G\{B_i, p_i\}$ is not compact. Non-existence of a critical subspace allows us to argue that this noncompact space may be replaced by a compact one, and so extremisers to $G\{B_i, p_i\}$ exist. This is the content of Proposition 3 below. Then by the (nontrivial) (ii) $\Longrightarrow$ (i) in Theorem 3, gaussian extremals for (1) exist. Tensor products of these give gaussian extremals for (1) when all critical subspaces are ultracritical.

Conversely, if there exist gaussian extremisers to (1), then using the (i) $\Longrightarrow$ (iv) of Theorem 3, one may argue that any critical subspace must be ultracritical. This is the content of Proposition 4 below.

Proposition 3. *Suppose we are given $\{B_i, p_i\}$, with each $p_i \geq 0$. If* (2) *and* (3) *hold for all subspaces V, then $G\{B_i, p_i\}$ is finite. If in addition $\{B_i, p_i\}$ has no critical subspace, then the supremum given by $G\{B_i, p_i\}$ is attained.*

The first part of the Proposition is already a consequence of Theorem 1. However, we need the argument which gives this in order to establish the second part. See Section 5 of [BCCT1].

Proof. Those p_j which are zero play no role in the quantity $G\{B_i, p_i\}$ and so we may assume that each $p_j > 0$. We begin with the first assertion.

Fix A_j, and let $M := \sum_j p_j B_j^* A_j B_j$. This matrix is self-adjoint and positive definite, so by choosing an appropriate orthonormal basis $e_1, \dots, e_n$

we may assume that $M = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ for some $\lambda_1 \geq \cdots \geq \lambda_n > 0$. Our task is thus to establish a bound of the form

$$\prod_j (\det A_j)^{p_j} \leq K^2 \prod_i \lambda_i$$

for some finite K.

Fix j, and let $\langle \cdot, \cdot \rangle_{A_j}$ be the positive definite inner product on $\mathbb{R}^{n_j}$ defined by

$$\langle x, y \rangle_{A_j} := \langle x, A_j y \rangle_{\mathbb{R}^{n_j}}.$$

Observe for each basis vector e_i that

$$\langle B_j e_i, B_j e_i \rangle_{A_j} = \langle e_i, B_j^* A_j B_j e_i \rangle_{\mathbb{R}^n} \leq \frac{1}{p_j} \langle e_i, M e_i \rangle_{\mathbb{R}^n} = \frac{\lambda_i}{p_j}.$$

In particular, by the triangle inequality we see that

$$\langle x, x \rangle_{A_j} = O(1)$$

for all x in the convex hull H of the vectors $B_j e_1 / \sqrt{\lambda_1}, \ldots, B_j e_n / \sqrt{\lambda_n}$, where the $O(1)$ is allowed to depend on the p_j, n_j and n (but not on the λ_j). Applying the linear transformation $x \mapsto A_j^{1/2} x$ to convert the inner product $\langle \cdot, \cdot \rangle_{A_j}$ into the usual one, we thus see that the Euclidean volume $\operatorname{vol}_{\mathbb{R}^{n_j}}(H)$ of H is bounded by

$$\operatorname{vol}_{\mathbb{R}^{n_j}}(H) = O(\det(A_j)^{-1/2}).$$

On the other hand, from elementary geometry we have

$$\operatorname{vol}_{\mathbb{R}^{n_j}}(H) \geq C^{-1} \Big| \bigwedge_{i \in I_j} B_j e_i / \sqrt{\lambda_i} \,\Big|$$

for any subset $I_j \subset \{1, \ldots, n\}$ of cardinality n_j. This gives us an upper bound

$$\det(A_j) \leq C \prod_{i \in I_j} \lambda_i / \Big| \bigwedge_{i \in I_j} B_j e_i \Big|^2. \tag{4}$$

Thus it will suffice to show that regardless of what the e_j and λ_j are, one can find sets $I_j \subset \{1, \ldots, n\}$ of cardinality $|I_j| = n_j$ such that

$$\prod_j \Big(\prod_{i \in I_j} \lambda_i / \Big| \bigwedge_{i \in I_j} B_j e_i \Big|^2 \Big)^{p_j} \leq C^{-1} K^2 \prod_i \lambda_i.$$

Since the λ_i are decreasing, it will suffice to establish the lower bound

$$\prod_j \Big|\bigwedge_{i\in I_j} B_j e_i\Big|^{p_j} \geq \frac{C^{1/2}}{K} \tag{5}$$

for some sets $I_j \subset \{1,\dots,n\}$ of cardinality $|I_j| = n_j$ which obey the additional inequalities

$$\sum_j p_j \, |I_j \cap \{k+1,\dots,n\}| \geq n-k \quad \text{for all } 1 \leq k < n \tag{6}$$

(the case $k=0$ is just (3)).

Indeed, if we can do this, then we only need to check that assuming $\lambda_1 \geq \cdots \geq \lambda_n > 0$,

$$\prod_j \prod_{i\in I_j} \lambda_i^{p_j} \leq \prod_{i=1}^n \lambda_i.$$

This is the same as

$$\prod_{i=1}^n \lambda_i^{\sum_{j:i\in I_j} p_j} \leq \prod_{i=1}^n \lambda_i.$$

Look at the power on the λ_n terms on the left hand side: $\sum_{j:n\in I_j} p_j = 1 + \big(\big(\sum_{j:n\in I_j} p_j\big) - 1\big)$. The power 1 we keep as a contribution to the right hand side, and we estimate $\lambda_n^{(\sum_{j:n\in I_j} p_j)-1}$ by $\lambda_{n-1}^{(\sum_{j:n\in I_j} p_j)-1}$, noting that by assumption $\big(\sum_{j:n\in I_j} p_j\big) - 1 \geq 0$. We continue in the same way with the λ_{n-1} terms, keeping a power 1 as a contribution to the right hand side and transferring a (nonnegative) power $\big(\sum_j p_j \, |I_j \cap \{n-1,n\}|\big) - 2$ onto the λ_{n-2} terms. We treat the powers of λ_{n-2}, λ_{n-3}, … , λ_2 in turn in the same way, arriving at λ_1 with a power of $1 + \big(\sum_j p_j \, |I_j \cap \{1,2,\dots,n\}|\big) - n = 1$ since $(\sum p_j n_j) - n = 0$.

It remains therefore to establish (5) under condition (6).

Observe that if an orthonormal system e_1, … , e_n is such that one can find sets I_j of cardinality $|I_j| = n_j$ obeying the inequalities (6), and such that $\bigwedge_{i\in I_j} B_j e_i \neq 0$ for all $1 \leq j \leq m$, then we will have the bound (5) for *some* finite $K < \infty$, and furthermore we can perturb this system by a small amount (keeping the sets I_j fixed) and still obtain the bound (5) for a uniform value of K. Since the space of all orthonormal bases is compact, we thus see that it now suffices to show that for each orthonormal system e_1, … , e_n, there exists I_j of cardinality $|I_j| = n_j$ obeying (6), such that the vectors $\{B_j e_i : i \in I_j\}$ span $\mathbb{R}^{n_j}$ for all $1 \leq j \leq m$.

In order to obey (6) it is desirable to have I_j consist of as large numbers as possible (basically this exploits the smaller eigenvalues λ_j as much as possible, and gives the best bound on $\det(A_j)$ in (4)). We shall thus select these I_j by a greedy algorithm. Namely, we set I_j equal to those indices i for which $B_j e_i$ is not in the linear span of $\{B_j e_{i'} : i < i' \leq n\}$ (thus for instance n will lie in I_j as long as $B_j e_n \neq 0$). Since the B_j are surjective, we see that $|I_j| = n_j$. To prove (6), we apply the hypothesis (2) with V equal to the span of $e_{k+1}, \ldots, e_n$, to obtain

$$\sum_j p_j \dim(B_j V) \geq n - k.$$

But by construction of I_j we see that $\dim(B_j V) = |I_j \cap \{k+1, \ldots, n\}|$, and (6) follows.

Now we turn to the constant in $G\{B_i, p_i\}$ being achieved when there is no critical subspace. This can be seen by an inspection of the preceding argument. For each fixed orthonormal basis $e_1, \ldots, e_n$, the above construction now gives a family of I_j for which we have strict inequality in (6):

$$\sum_j p_j \, |I_j \cap \{k+1, \ldots, n\}| > n - k \quad \text{for all } 1 \leq k < n.$$

This implies that not only is the bound (5) true, but also that we can improve upon this bound if the gap between the largest eigenvalue λ_1 and the smallest eigenvalue λ_n is sufficiently large. Indeed, all of the nonnegative powers $\left(\sum_j p_j \, |I_j \cap \{k+1, \ldots, n\}|\right) - (n-k)$ of the λ_{k+1} (with the exception of the last) in the argument above are now strictly positive.

Thus in order to locate an extremiser it suffices to work in the regime when $\lambda_1, \ldots, \lambda_n$ have bounded ratio; by scaling we may then take all of these eigenvalues to lie in a fixed compact set. The extremisation problem is now over a compact domain and so an extremiser will now necessarily exist, and the proof is complete. $\square$

Proposition 4. *Suppose for some $\{p_i\}$ that V is a critical subspace for $\{B_i, p_i\}$. Suppose that the problem $\{B_i, p_i\}$ has gaussian extremals. Then V is ultracritical for $\{B_i\}$.*

Proof. Suppose that A_j is an extremiser for G. Let $M := \sum_j p_j B_j^* A_j B_j$. Then Theorem 3 shows that M is invertible and

$$B_j M^{-1} B_j^* = A_j^{-1} \quad \text{for all } j.$$

In particular, if we define the self-adjoint matrices

$$P_j := M^{-1/2} B_j^* A_j B_j M^{-1/2}$$

on $\mathbb{R}^n$ then we see that P_j is the orthogonal projection onto the n_j-dimensional subspace $M^{-1/2} B_j^* \mathbb{R}^{n_j}$ of $\mathbb{R}^n$, and furthermore $\sum_j p_j P_j = I$. Writing $W := M^{1/2} V$, we thus see that criticality of V is equivalent to the assertion

$$\dim(W) = \sum_j p_j \dim(P_j W) \tag{7}$$

(noting that B_j^* is necessarily injective since B_j is surjective).

On the other hand, if π is the orthogonal projection from $\mathbb{R}^n$ to W, we see that $\mathrm{tr}(\pi) = \dim(W)$ and $\mathrm{tr}(P_j \pi) \leq \dim(P_j W)$ (the latter inequality follows because $P_j \pi$ is a contraction with range $P_j W$); since

$$\mathrm{tr}(\pi) = \mathrm{tr}\Big(\sum_j p_j P_j \pi\Big) = \sum_j p_j \,\mathrm{tr}(P_j \pi)$$

we deduce that (7) can only hold when $\mathrm{tr}(P_j \pi) = \dim(P_j W)$ for all j, which means that W is the direct sum of a space in $P_j \mathbb{R}^n$ and a space in $\ker P_j = (P_j \mathbb{R}^n)^\perp$. So $W = W \cap P_j \mathbb{R}^n \oplus W \cap \ker P_j$, and thus $P_j W = P_j(W \cap P_j \mathbb{R}^n) \oplus P_j(W \cap \ker P_j) = P_j(W \cap P_j \mathbb{R}^n) \subseteq W$ since P_j acts as the identity on $P_j \mathbb{R}^n$.

This basically shows that W and hence V is ultracritical, ending the proof. $\square$

Now we look at uniqueness.

If there are gaussian extremals, and if there are critical subspaces, we have just seen that they must be ultracritical. We can continue decomposing the problem until there are no critical subspaces remaining. On each minimal piece we have gaussian extremals by the Theorem 5 (or Proposition 3). Then the original problem will have a k-fold family of gaussian extremals where k is the number of subproblems we have decomposed into. So we have one half of:

Theorem 6. *Assume that gaussian extremals for* (1) *exist. Then they are unique (up to the 1-parameter family of scalars) if and only if there are no critical subspaces.*

The converse part uses Theorem 3 once again and some linear algebra. See [BCCT1, Section 9].

The matter of non-gaussian extremisers to (1) was left somewhat open in [BCCT1]. However, VALDIMARSSON has recently given a complete characterisation of all extremals to (1), see [V2].

Finally, we present a **proof of Lieb's theorem**:

Consider $\{B_i, p_i\}$. If $C\{B_i, p_i\} = \infty$ then either (2) or (3) fails by Theorem 1, hence $G\{B_i, p_i\} = \infty$ by (the proof of) the necessity in Theorem 1. So assume $C\{B_i, p_i\} < \infty$. Then

either

there is no critical subspace – in which case Theorem 5 gives existence of gaussian extremals to (1) so that $C\{B_i, p_i\} = G\{B_i, p_i\}$;

or

there is a critical subspace – so that

$$\begin{aligned} C\{B_i, p_i\} &\le C\{\tilde{B}_i, p_i\} C\{\tilde{\tilde{B}}_i, p_i\} = G\{\tilde{B}_i, p_i\} G\{\tilde{\tilde{B}}_i, p_i\} \\ &\le G\{B_i, p_i\} \le C\{B_i, p_i\}. \end{aligned}$$

Here we use the forward part of multiplicativity of C (i.e. the proof of Theorem 1), induction on the dimension and the easy part of multiplicativity of G which we presented above. The last inequality is obvious.

6. AFTERWORD

It is interesting to reflect upon the various proofs of Lieb's theorem which are now available. The first, [BL], (which did not apply in full generality), used rearrangement inequalites, in particular the Brascamp–Lieb–Luttinger rearrangement inequality [BLL]. BARTHE's proof [Bar] used optimal mass transportation, while our proof ([BCCT1] and the present notes) uses heat flow (as did CARLEN, LIEB and LOSS [CLL] in the rank one case). Each of these approaches uses a method to move mass in some general position to an equivalent mass in a position more suitable for direct analysis; despite this broad similarity they nevertheless seem to be distinct, each with their own advantages and disadvantages. However, it does not seem so easy to cast LIEB's original proof [L] within this framework.

In [BCCT1] Section 8, we give a further, perhaps more direct proof of Lieb's theorem than the one presented here. There, in a regularised situation, the problems with lack of compactness met in Theorem 5 are avoided completely.

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