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ON EMBEDDING THEOREMS

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Abstract. This paper is devoted to embedding theorems for classes of functions of several variables. One of our main objectives is to give an analysis of some basic embeddings as well as to study relations between them. We also discuss some methods in this theory that were developed in the last decades. These methods are based on non-increasing rearrangements of functions, iterated rearrangements, estimates of sections of functions, related mixed norms, and molecular decompositions.

1. Introduction

We will study spaces of functions defined in terms of L^p-norms of derivatives (Sobolev-type spaces) and spaces defined in terms of L^p-moduli of continuity (in particular, spaces of Besov-Nikol'skii and Lipschitz type). We emphasize that it is very important to include to these studies the limiting case $p = 1$. This case often requires special methods. In many estimates the proofs given for $p > 1$ are much easier than those for $p = 1$. It is clear that in such situations the constants in these estimates obtained by "easy" methods are not sharp. Therefore it is necessary to apply alternative methods that would cover simultaneously all values $p \geq 1$ (if the corresponding results are true for $p = 1$). The most known methods of this type are those related to the use of non-increasing rearrangements of functions. The systematic application of these methods in the Embedding Theory goes back to the works of Ul'yanov [64], [65].

In this paper we pay much attention to the estimates of rearrangements. We show that they enable us to obtain general results that include embeddings of spaces of Sobolev and Besov-Nikol'skii type. It was for the first time

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discovered in [10] that the relations between embeddings of these spaces are closely connected with optimal embedding constants. Later on, in [42] we showed that such constants can be readily derived from sharp estimates of non-increasing rearrangements in terms of moduli of continuity. These results concern the *isotropic* case. In the *anisotropic* case the situation is much more difficult and the results are not yet complete. An important open problem is to find general sharp estimates of rearrangements in terms of partial moduli of continuity.

One of the most remarkable properties of rearrangements is the variation reducing property. In 1951, PÓLYA and SZEGÖ proved the following theorem: the L^p-norm of the gradient of the symmetric rearrangement of a given function f does not exceed the L^p-norm of the gradient of f. Afterwards, fundamental estimates of the moduli of continuity of rearrangements in one variable were obtained in the middle of seventies by GARSIA and RODEMICH [24], OSWALD [54] and WIK [66]. Recently, CIANCHI [16] studied boundedness of the decreasing rearrangement operator in Besov spaces of higher order in the one-dimensional case. However, for functions of several variables, the known results are not complete, especially in the anisotropic case.

We shall consider also *iterated* (multivariate) rearrangements. The rough definition is the following. Given a function f on $\mathbb{R}^n$, we rearrange it non-increasingly first with respect to x_1, then with respect to x_2, and so on. As a result, we obtain a function on $\mathbb{R}^n_+$ that is non-increasing in each variable and equimeasurable with $|f|$. We denote it by $\mathcal{R}_{1,\dots,n}f$. Of course, we can change the order of variables which leads to a different function. We show that the use of iterated rearrangements enables one to simplify proofs and, at the same time, to obtain stronger results.

The most developed part of the embedding theory is devoted to the study of spaces defined by numerical parameters measuring smoothness. In this paper we consider the anisotropic fractional Sobolev spaces, the Besov-Nikol'skii spaces and the Lipschitz spaces and for these spaces we discuss Sobolev-type embeddings with the limiting exponent. Note that the anisotropic Lipschitz spaces inherit partly properties of Sobolev spaces and partly properties of Nikol'skii spaces. This is why the study of Lipschitz spaces is met with essential difficulties and leads to rather special results. However, we prove that sharp embeddings for Lipschitz spaces can be obtained as the limiting case of embeddings for Besov-Nikol'skii spaces.

We also discuss alternative statements of problems which are expressed not in terms of classes defined by smoothness exponents, but in terms of *individual* functions. Such problems are more general and may lead to essentially stronger results.

Our approach to the Sobolev-type inequalities is based on estimates of mixed norms. It was shown first by Gagliardo [22] and then by Fournier [21] that the integrability properties of a function of several variables can be controlled by the behavior of L^∞-norms of its $\widehat{x}_k$-sections. In [21], sharp estimates of Lorentz norms in terms of certain mixed norms were proved. These estimates immediately imply the Sobolev inequality and clarify the role of smoothness conditions in the embeddings of Sobolev spaces W_1^1. Similar results for the Sobolev spaces W_p^1 with $p > 1$ and for the anisotropic Besov spaces $B_p^{\alpha_1,\dots,\alpha_n}$ were obtained in our paper [41]. In a latter work, we introduced a more general scale of mixed norm spaces and studied some embeddings for these spaces.

The study of sections of functions leads also to the case when smoothness conditions are imposed on functions with respect to only one specific variable x_k. In a sense, this is a limiting case of anisotropic classes, when only one of the indices of smoothness is positive. This smoothness condition can be combined with conditions of other type. In [40] we studied embedding theorems and multiplicative inequalities of Gagliardo–Nirenberg type for the corresponding norms. We proved different norm inequalities for partial moduli of continuity with respect to a separate variable x_k, combining conditions on the "size" of a function and its smoothness in a given Lorentz norm with respect to *the same* variable. Applying these results, we obtained optimal constants in some known multiplicative inequalities. We considered also the case when a derivative belongs to the space L^1. In the latter case, along with estimates of rearrangements, we used the method of molecular decompositions due to Pełczyński and Wojciechowski [58].

In this paper we give an overview of the problems and results that have been briefly described above. We note that many of them are not new. However, they still generate important open problems and have interesting links with more recent results.

Only few statements in the paper are given with proofs. These statements were selected to show how the basic methods of rearrangements and iterated rearrangements work. We include also the proofs of some new, unpublished yet, results. The main of them are optimal estimates of Lorentz norms in terms of anisotropic Besov norms (Section 8). In a limit, these estimates give sharp embeddings of Lipschitz spaces.

2. Nonincreasing rearrangements

Denote by $S_0(\mathbb{R}^n)$ the class of all measurable and almost everywhere finite functions f on $\mathbb{R}^n$ such that, for each $y > 0$,

$$\lambda_f(y) \equiv |\{x \in \mathbb{R}^n : |f(x)| > y\}| < \infty.$$

A non-increasing rearrangement of a function $f \in S_0(\mathbb{R}^n)$ is a non-increasing function f^* on $\mathbb{R}_+ \equiv (0,+\infty)$ such that, for any $y>0$,

$$|\{t \in \mathbb{R}_+ : f^*(t) > y\}| = \lambda_f(y). \tag{2.1}$$

We shall assume in addition that the rearrangement f^* is left continuous on $(0,\infty)$. Under this condition it is defined uniquely by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) < t\} \quad (0 < t < \infty).$$

Besides, we have the equality

$$f^*(t) = \sup_{|E|=t} \inf_{x\in E} |f(x)|.$$

The following relation holds [63, Chap. 5]:

$$\sup_{|E|=t} \int_E |f(x)|\,dx = \int_0^t f^*(u)\,du. \tag{2.2}$$

In what follows we denote

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(u)\,du.$$

By (2.2), the operator $f \mapsto f^{**}$ is subadditive,

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t).$$

Moreover, this operator is bounded in L^p for $p>1$,

$$\|f^{**}\|_p \le \frac{p}{p-1}\|f\|_p \quad (1 < p \le \infty). \tag{2.3}$$

This inequality follows from the following Hardy lemma:

Lemma 2.1 ([63, p. 196])**.** *Let $\alpha > 0$ and $1 \le p < \infty$. Then for any non-negative measurable function φ on $(0,\infty)$,*

$$\left(\int_0^\infty \left(\int_0^t \varphi(u)\,du\right)^p t^{-\alpha-1}\,dt\right)^{1/p} \le \frac{p}{\alpha}\left(\int_0^\infty \big(t\varphi(t)\big)^p t^{-\alpha-1}\,dt\right)^{1/p}$$

and

$$\left(\int_0^\infty \left(\int_t^\infty \varphi(u)\,du\right)^p t^{\alpha-1}\,dt\right)^{1/p} \le \frac{p}{\alpha}\left(\int_0^\infty \big(t\varphi(t)\big)^p t^{\alpha-1}\,dt\right)^{1/p}. \tag{2.4}$$

The main properties of rearrangements used in what follows are set forth in [4], [19], [43], [44], [63]. We formulate two of them.

Lemma 2.2. *If a sequence* $\{f_k\} \subset S_0$ *converges in measure to a function* $f \in S_0$*, then* $f_k^*(t) \to f^*(t)$ *at every point of continuity of* f*.*

For the proof see [43, Chap. 2, § 2].

Lemma 2.3. *Let* $f, g \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$*. Then*

$$\int_0^\infty |f^*(t) - g^*(t)|^p \, dt \le \int_{\mathbb{R}^n} |f(x) - g(x)|^p \, dx.$$

The proof can be found in [44, p. 83].

It follows from (2.1) that, for any $0 < p < \infty$,

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx = \int_0^\infty f^*(t)^p \, dt. \tag{2.5}$$

In 1950 G. Lorentz introduced a scale of spaces, defined by two parameters, and including the spaces L^p. Let $0 < p, r < \infty$. A function $f \in S_0(\mathbb{R}^n)$ belongs to the Lorentz space $L^{p,r}(\mathbb{R}^n)$ if

$$\|f\|_{p,r} \equiv \left(\int_0^\infty \left(t^{1/p} f^*(t) \right)^r \frac{dt}{t} \right)^{1/r} < \infty.$$

For $0 < p < \infty$, the space $L^{p,\infty}(\mathbb{R}^n)$ is defined as the class of all $f \in S_0(\mathbb{R}^n)$ such that

$$\|f\|_{p,\infty} \equiv \sup_{t>0} t^{1/p} f^*(t) < \infty.$$

By (2.5) we have that $\|f\|_{p,p} = \|f\|_p$. Further, for a fixed p, the Lorentz spaces $L^{p,r}$ increase as the secondary index r increases. That is, we have the strict embedding $L^{p,r} \subset L^{p,s}$ for $r < s$; in particular,

$$L^{p,r} \subset L^{p,p} \equiv L^p \quad (0 < r < p).$$

More exactly, the following inequality holds (see [63, p. 192]):

$$\|f\|_{p,s} \le \left(\frac{p}{s} \right)^{1/s} \left(\frac{r}{p} \right)^{1/r} \|f\|_{p,r} \quad (0 < r < s \le \infty). \tag{2.6}$$

The difference

$$w(t) \equiv f^{**}(t) - f^*t) = \frac{1}{t} \int_0^t [f^*(u) - f^*(t)] \, du$$

measures the mean oscillations of the rearrangement f^*. We have the equality

$$f^{**}(t) = \int_t^\infty \frac{f^{**}(u) - f^*(u)}{u}\, du. \tag{2.7}$$

The space $W(\mathbb{R}^n) \equiv$ weak-$L^\infty(\mathbb{R}^n)$ consists of all functions $f \in S_0(\mathbb{R}^n)$ such that

$$\|f\|_W \equiv \sup_{t>0} [f^{**}(t) - f^*(t)] < \infty.$$

This space was introduced in [3] by Bennett, DeVore and Sharpley. They proved that BMO $\subset W$.

For a function $f \in S_0(\mathbb{R}^n)$, we consider also the quantity

$$\widetilde{w}(t) \equiv f^*(t) - f^*(2t).$$

Observe that, for any $t > 0$,

$$\frac{1}{2}\widetilde{w}\Big(\frac{t}{2}\Big) \le f^{**}(t) - f^*(t) \le \frac{2}{t}\int_0^t \widetilde{w}(u)\, du. \tag{2.8}$$

The left-hand inequality is immediate since

$$f^{**}(t) - f^*(t) \ge \frac{1}{t}\int_0^{t/2} \big[f^*(u) - f^*(t)\big]\, du \ge \frac{1}{2}\widetilde{w}\Big(\frac{t}{2}\Big).$$

Next, for any $t > 0$ and any $0 < \varepsilon < t$,

$$\int_\varepsilon^t \widetilde{w}(u)\, du = \int_\varepsilon^t f^*(u)\, du - \frac{1}{2}\int_{2\varepsilon}^{2t} f^*(u)\, du \ge \frac{1}{2}\left[\int_\varepsilon^t f^*(u)\, du - tf^*(t)\right].$$

This implies the right-hand inequality in (2.8).

Similarly, we have that, for any $f \in S_0(\mathbb{R}^n)$ and any $t > 0$,

$$f^*(2t) \le \frac{1}{\ln 2}\int_t^\infty \frac{f^*(u) - f^*(2u)}{u}\, du. \tag{2.9}$$

Let $1 \le p, r < \infty$. For a function $f \in S_0(\mathbb{R}^n)$, set

$$\|f\|_{p,r}^* = \left(\int_0^\infty \big[t^{1/p}(f^*(t) - f^*(2t))\big]^r \frac{dt}{t}\right)^{1/r} \tag{2.10}$$

and

$$\|f\|^*_{p,\infty} = \sup_{t>0} t^{1/p}\big[f^*(t) - f^*(2t)\big].$$

It follows from (2.9) and Hardy's inequality (2.4) that

$$\|f\|_{p,r} \le \frac{2^{1/p}p}{\ln 2}\|f\|^*_{p,r} \quad (1 \le p < \infty,\ 1 \le r \le \infty). \tag{2.11}$$

Thus, $\|\cdot\|_{p,r}$ and $\|\cdot\|^*_{p,r}$ for $1 \le p < \infty$, $1 \le r \le \infty$ are equivalent. However, the latter quasi-norm can be finite in the case $p = \infty$, too.

For any $1 \le r < \infty$, the space $L^{\infty,r}(\mathbb{R}^n)$ is defined as the class of all functions $f \in S_0(\mathbb{R}^n)$ such that

$$\|f\|_{\infty,r} \equiv \left(\int_0^\infty [f^{**}(t) - f^*(t)]^r \, \frac{dt}{t}\right)^{1/r} < \infty$$

(see [2], [46]). Set also

$$\|f\|^*_{\infty,r} \equiv \left(\int_0^\infty \big[f^*(t) - f^*(2t)\big]^r \, \frac{dt}{t}\right)^{1/r}. \tag{2.12}$$

It follows from (2.8) that

$$\frac{1}{2}\|f\|^*_{\infty,r} \le \|f\|_{\infty,r} \le 2\|f\|^*_{\infty,r}.$$

3. Estimates of rearrangements

In this section we present the simplest versions of rearrangement estimates and derive from them the basic Sobolev-type inequality. We include the complete proofs to describe the main ideas of the method of rearrangements.

3.1. Estimates in terms of derivatives. The following lemma was obtained in [35, Lemma 5.1].*

Lemma 3.1. *Let $f \in S_0(\mathbb{R}^n)$ be a locally integrable function which has all weak derivatives $\partial f/\partial x_k \in L^1_{\mathrm{loc}}$, $k = 1, \dots, n$. Then*

$$f^{**}(t) - f^*(t) \le \sqrt{n}\; t^{1/n}(|\nabla f|)^{**}(t). \tag{3.1}$$

*In [35] the left-hand side of the corresponding estimate contained the difference $\widetilde{w}(t)$ instead of $f^{**}(t) - f^*(t)$; by (2.8), this is equivalent to (3.1).

Proof. Let $x \in \mathbb{R}^n$ and $t > 0$. Denote by $Q_x(t)$ the cube centered at x with the side length $(2t)^{1/n}$. Fix x and set

$$A_{t,x} = \{y \in Q_x(t) : |f(y)| \le f^*(t)\}.$$

Then $|A_{t,x}| \ge t$. For any $y \in A_{t,x}$,

$$|f(x)| - f^*(t) \le |f(x)| - |f(y)| \le |f(x) - f(y)|.$$

Integrating over $A_{t,x}$ we obtain

$$\begin{aligned} |f(x)| - f^*(t) &\le \frac{1}{t} \int_{A_{t,x}} |f(x) - f(y)|\, dy \\ &\le \frac{1}{t} \int_{Q_0(t)} |f(x) - f(x+h)|\, dh \\ &\le \sqrt{n}\; t^{1/n-1} \int_0^1 d\tau \int_{Q_0(t)} |\nabla f(x+\tau h)|\, dh \end{aligned}$$

for almost all $x \in \mathbb{R}^n$. Let $E \subset \mathbb{R}^n, |E| = t$. Then for all $\tau \in [0,1]$ and $h \in Q_0(t)$,

$$\int_E |\nabla f(x+\tau h)|\, dx \le \int_0^t (|\nabla f|)^*(u)\, du.$$

Applying (2.2), we get (3.1). □

It follows from (3.1) and (2.7) that

$$f^{**}(t) \le \sqrt{n} \int_t^\infty u^{1/n-1} (|\nabla f|)^{**}(u)\, du.$$

Using induction, we immediately obtain

Corollary 3.2. *For any $f \in C_0^\infty$ and any $r \in \mathbb{N}$,*

$$f^{**}(t) \le c \int_t^\infty u^{r/n-1} (D_r f)^{**}(u)\, du,$$

where $D_r f(x) = \sum_{|\alpha|=r} |D^\alpha f(x)|$.

By (2.3), the average rearrangement operator $\varphi \mapsto \varphi^{**}$ is bounded in L^p for $p > 1$. Therefore the above estimates can be applied to the study of Sobolev spaces W_p^r in the case $p > 1$. However, this way leads to a “bad” constant and fails in the case $p = 1$. The following lemma may be more useful.

Lemma 3.3. *Let $f \in S_0(\mathbb{R}^n)$ be a locally integrable function which has all weak derivatives $\partial f/\partial x_k$ that belong to L^p on any set of finite measure $(1 \le p < \infty)$. Let*

$$E_t = \{x : |f(x)| > f^*(t)\}.$$

Then for all $0 < h \le t$,

$$f^*(t) - f^*(t+h) \le 2\sqrt{n}\, t^{1/n-1} \int_{E_{t+h}\setminus E_t} |\nabla f(x)|\, dx. \tag{3.2}$$

Moreover, f^* is absolutely continuous on each interval $[\alpha, \beta]$, $0 < \alpha < \beta < \infty$, and

$$\left|\frac{d}{dt} f^*(t)\right| \le 2\sqrt{n}\, t^{1/n-1} \left(\frac{d}{dt}\int_{E_t} |\nabla f(x)|^p\, dx\right)^{1/p} \tag{3.3}$$

for almost all $t > 0$.

Proof. We can assume that $f \ge 0$. Set

$$g(x) = \begin{cases} \min\{f(x), f^*(t)\} - f^*(t+h), & \text{if } x \in E_{t+h}, \\ 0, & \text{if } x \notin E_{t+h}. \end{cases}$$

It is easy to see that the function g has all weak derivatives $\partial g/\partial x_k$ and, for almost all x,

$$\nabla g(x) = \begin{cases} \nabla f(x), & \text{if } x \in E_{t+h} \setminus E_t, \\ 0, & \text{if } x \notin E_{t+h} \setminus E_t. \end{cases}$$

We have $g^*(t+h) = 0$ and

$$g^{**}(t+h) \ge \frac{t}{t+h}\big[f^*(t) - f^*(t+h)\big].$$

Applying Lemma 3.1, we obtain (3.2). In turn, (3.2) yields that f^* is absolutely continuous on each interval $[\alpha, \beta]$, $0 < \alpha <$ ¡∞. If $(f^*)'(t)$ exists and is different from 0, then $|E_{t+h} \setminus E_t| \le h$, and we get from (3.2)

$$f^*(t) - f^*(t+h) \le 2\sqrt{n}\, t^{1/n-1} h^{1-1/p} \left(\int_{E_{t+h}\setminus E_t} |\nabla f(x)|^p\, dx\right)^{1/p}.$$

This implies (3.3). □

Remark 3.4. Inequalities (3.2) and (3.3) were proved in [35] (the proof was based on the Loomis–Whitney theorem [45]). The local absolute continuity of the rearrangement was first proved in [29] (see also [35]).

Let $W_p^1(\mathbb{R}^n)$ be the Sobolev space of all $f \in L^p(\mathbb{R}^n)$ for which every first-order weak derivative exists and belongs to $L^p(\mathbb{R}^n)$. The simplest version of the classical Sobolev inequality is the following.

Theorem 3.5. *Let $n \ge 2$, $1 \le p < n$, and $q^* = np/(n-p)$. Then for any $f \in W_p^1(\mathbb{R}^n)$,*

$$\|f\|_{q^*} \le c \sum_{k=1}^{n} \|D_k f\|_p. \tag{3.4}$$

Sobolev proved this inequality in 1938 for $p > 1$; his method, based on integral representations, did not work in the case $p = 1$. Only at the end of fifties Gagliardo and Nirenberg gave simple proofs of the inequality (3.4) for all $1 \le p < n$. We will discuss Gagliardo's approach below.

It is well known that the left-hand side in (3.4) can be replaced by the stronger $L^{q^*,p}$-Lorentz norm. Namely, the inequality

$$\|f\|_{q^*,p} \le c \sum_{k=1}^{n} \|D_k f\|_p \quad \left(1 \le p < n,\ q^* = \frac{np}{n-p}\right) \tag{3.5}$$

holds (see [50], [57], [60]). For $p > 1$ this result can be obtained by interpolation (although the direct proof is simpler). There are numerous proofs of (3.5) in the case $p = 1$; most of them are related to rearrangements, properties of level sets, and geometric inequalities. Here we observe that for all $1 \le p < n$ the inequality (3.5) can be immediately derived from (3.3). Indeed, by Hardy's inequality (2.4) and (3.3) we have

$$\begin{aligned}
\|f\|_{q^*,p} &= \left(\int_0^\infty t^{p/q^*-1} f^*(t)^p \, dt\right)^{1/p} \\
&= \left(\int_0^\infty t^{p/q^*-1} \left(\int_t^\infty |(f^*)'(u)| \, du\right)^p dt\right)^{1/p} \\
&\le q^* \left(\int_0^\infty t^{-p/n+p} |(f^*)'(t)|^p \, dt\right)^{1/p} \\
&\le 2\sqrt{n}\, q^* \left(\int_0^\infty \frac{d}{dt} \int_{E_t} |\nabla f(x)|^p \, dx dt\right)^{1/p} \\
&= 2\sqrt{n}\, q^* \|\nabla f\|_p.
\end{aligned}$$

In the limiting case $p = n$ the estimate (3.1) and Hardy's inequality immediately imply that, for any function $f \in W_n^1(\mathbb{R}^n)$ $(n \geq 2)$,

$$\left(\int_0^\infty \left[f^{**}(t) - f^*(t)\right]^n \frac{dt}{t}\right)^{1/n} \leq c_n \|\nabla f\|_n, \tag{3.6}$$

i.e., $W_n^1(\mathbb{R}^n) \subset L^{\infty,n}(\mathbb{R}^n)$ (see [2], [46]). Observe also that, in view of (3.10), the inequality (3.6) can be considered as a special case of (3.13) below.

3.2. Estimates in terms of moduli of continuity. For any function $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$, its modulus of continuity is defined by

$$\omega(f;\delta)_p = \sup_{|h|\leq\delta}\left(\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p\, dx\right)^{1/p} \quad (0 < \delta < \infty).$$

Observe that $\omega(f;\cdot)_p$ is non-decreasing and subadditive function. In particular,

$$\omega(f;2\delta)_p \leq 2\omega(f;\delta)_p \quad \text{for any } \delta \geq 0. \tag{3.7}$$

It follows that $\omega(f;2^n\delta)_p \leq 2^n\omega(f;\delta)_p$ for any $n \in \mathbb{N}$ and any $\delta > 0$. Hence, if $\omega(f;\delta)_p \not\equiv 0$, then $\omega(f;\delta)_p > 0$ for all $\delta > 0$ and

$$\omega(f;\delta)_p \geq c_\omega \delta \quad (c_\omega = \omega(1/2) > 0)$$

for all $\delta \in [0,1]$. It follows from the Lebesgue differentiation theorem that $\omega(f;\delta)_p \equiv 0$ if and only if f is equivalent to 0. It can be easily seen that, for any $f \in W_p^1(\mathbb{R}^n)$ $(1 \leq p < \infty)$,

$$\omega(f;\delta)_p \leq \|\nabla f\|_p\, \delta. \tag{3.8}$$

Moreover, by the Hardy-Littlewood theorem [53, §4.8], for any $1 < p < \infty$ and $\delta > 0$,

$$\omega(f;\delta)_p = O(\delta) \quad \text{if and only if} \quad f \in W_p^1(\mathbb{R}^n). \tag{3.9}$$

Further, for any $1 \leq p < \infty$ and any $f \in W_p^1(\mathbb{R}^n)$,

$$\lim_{\delta\to 0+} \frac{\omega(f;\delta)_p}{\delta} = \sup_{\delta>0} \frac{\omega(f;\delta)_p}{\delta} = \|\nabla f\|_p \tag{3.10}$$

(for the proof, see [42]).

Let $0 < \alpha < 1$, $1 \le p < \infty$ and $1 \le \theta \le \infty$. The Besov space $B^\alpha_{p,\theta}(\mathbb{R}^n)$ consists of all functions $f \in L^p(\mathbb{R}^n)$ such that

$$\|f\|_{b^\alpha_{p,\theta}} \equiv \left(\int_0^\infty \left(t^{-\alpha}\omega(f;t)_p\right)^\theta \frac{dt}{t}\right)^{1/\theta} < \infty$$

if $\theta < \infty$, and

$$\|f\|_{b^\alpha_{p,\infty}} \equiv \sup_{t>0} t^{-\alpha}\omega(f;t)_p < \infty$$

if $\theta = \infty$. Set also $B^\alpha_p = B^\alpha_{p,p}$. The space $B^\alpha_{p,\theta}$ is a Banach space with respect to the norm

$$\|f\|_{B^\alpha_{p,\theta}} = \|f\|_p + \|f\|_{b^\alpha_{p,\theta}}.$$

Observe that this space is not complete with respect to the norm $\|\cdot\|_{b^\alpha_{p,\theta}}$.

We shall consider estimates of rearrangements in terms of moduli of continuity. First of all, our interest in these estimates is motivated by the following problem due to Ul'yanov.

Let $f \in L^p(\mathbb{R}^n)$. Assume that a function φ is defined on $\mathbb{R}_+$ and $\varphi(t)t^{-p}$ increases. Find sharp estimates of the integral

$$\int_{\mathbb{R}^n} \varphi(|f(x)|)\,dx$$

in terms of $\omega(f;\delta)_p$.

Ul'yanov [64] studied this problem in the one-dimensional case for some special functions φ, in particular, for $\varphi(t) = t^q$. His approach was based on the following lemma.

Lemma 3.6. *Let $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Then for any $t > 0$,*

$$f^{**}(t) - f^*(t) \le 2^{1/p} t^{-1/p} \omega(f;t^{1/n})_p\,. \tag{3.11}$$

This lemma was first proved by Ul'yanov [64] in the one-dimensional case (see [38, p. 148] for an alternative proof). Further, the stronger version of (3.11),

$$\int_0^t [f^*(s) - f^*(t)]^p\,ds \le c\omega(f;t^{1/n})_p \quad (1 \le p < \infty,\ n \in \mathbb{N}) \tag{3.12}$$

was proved in [31]. A simpler proof in the general case is contained in [33, Theorem 1]; this proof is similar to the one given in the Lemma 3.1. The estimate (3.12) is efficient for $n = 1$. However, if $n \ge 2$ and $1 \le p < n$, then (3.12) is not sufficiently strong. A sharp estimate is contained in the following theorem proved in [33].

Theorem 3.7. *Let $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, $n \in \mathbb{N}$. Then for any $\delta > 0$,*

$$\int_{\delta^n}^{\infty} t^{-p/n} \int_0^t \big(f^*(u) - f^*(t)\big)^p \, du \frac{dt}{t} \le c_{p,n} \left(\frac{\omega(f,\delta)_p}{\delta} \right)^p. \tag{3.13}$$

In particular, Theorem 3.7 enabled us to obtain sharp estimates of the integral $\int_{\mathbb{R}^n} \varphi(|f(x)|)\, dx$ for functions φ satisfying the condition $\varphi(2t) \le c\varphi(t)$ (see [33]).

An interesting *open problem* is to obtain inequalities similar to (3.13) in terms of partial moduli of continuity (see Sections 4 and 7 below).

4. Smoothness of rearrangements

4.1. The Pólya-Szegö principle. Let $f \in S_0(\mathbb{R}^n)$. The spherically symmetric rearrangement of f is defined by

$$f_s^*(x) = f^*(v_n|x|^n), \quad x \in \mathbb{R}^n,$$

where $v_n = \pi^{n/2}/\Gamma(n/2+1)$ is the measure of the n-dimensional unit ball. The function f_s^* is equimeasurable with f, it possesses the spherical symmetry and decreases as $|x|$ increases.

The classical Pólya-Szegö principle states that, for any $f \in C_0^\infty(\mathbb{R}^n)$ the rearrangement f_s^* is differentiable almost everywhere and for any $1 \le p \le \infty$,

$$\|\nabla f_s^*\|_p \le \|\nabla f\|_p.$$

A stronger version of this principle is represented by the inequality

$$(|\nabla f_s^*|)^{**}(t) \le (|\nabla f|)^{**}(t) \quad (0 < t < \infty) \tag{4.1}$$

(see [1], [36]). By virtue of the Hardy-Littlewood lemma [4, p. 88], (4.1) implies that for any nonnegative and convex function φ on $[0, +\infty)$ with $\varphi(0+) = 0$,

$$\int_{\mathbb{R}^n} \varphi(|\nabla f_s^*(x)|)\, dx \le \int_{\mathbb{R}^n} \varphi(|\nabla f(x)|)\, dx.$$

An extension of the Pólya-Szegö principle to arbitrary rearrangement-invariant spaces was obtained by Cianchi and Pick [18].

Observe that

$$g(x) \equiv |\nabla f_s^*(x)| = v_n n |x|^{n-1} |(f^*)'(v_n|x|^n)|$$

and $g^*(t) = \varkappa_n^{-1} h^*(t)$, where $h(z) = z^{1-1/n}|(f^*)'(z)|$, $z > 0$, and $\varkappa_n = v_n^{-1/n} n^{-1}$ is the isoperimetric constant. Thus, (4.1) is equivalent to the inequality

$$\int_0^t h^*(u)\,du \le \varkappa_n \int_0^t (|\nabla f|)^*(u)\,du. \tag{4.2}$$

Such inequality with a worse constant follows also from Lemma 3.3.

Note that for $n = 1$ we have the following pointwise inequality

$$((f^*)')^*(t) \le (f')^*(t) \quad (t > 0)$$

(see [35]). However, for $n \ge 2$ the inequality $(|\nabla f_s^*|)^*(t) \le c(|\nabla f|)^*(t)$ fails to hold.

Cianchi [17] proved a second-order version of the Pólya-Szegö principle in a form patterned on (4.2) (see also [15]).

The Pólya-Szegö principle expresses variation reducing properties of the rearrangements. Similar properties are also contained in the estimates of the moduli of continuity of rearrangements.

4.2. Moduli of continuity (functions of one variable). In 1968 Ul'yanov [64] posed the following problem: estimate the L^p-modulus of continuity of the rearrangement f^* in terms of the modulus of continuity of a given function f.

The first sharp results were obtained in the one-dimensional case by Oswald [54] and Wik [66].

Theorem 4.1. *For any $f \in L^p[0,1]$, $1 \le p < \infty$,*

$$\int_0^\delta \omega(f^*;t)_p^p\,dt \le \int_0^\delta \omega(f;t)_p^p\,dt \quad \Big(0 \le \delta \le \frac{1}{2}\Big).$$

It follows that

$$\omega(f^*;\delta)_p \le 2\omega(f;\delta)_p \quad \Big(0 \le \delta \le \frac{1}{2}\Big). \tag{4.3}$$

The sharp constant in this inequality is still unknown.

Brudnyi [13] obtained a simpler proof of the inequality (4.3), however, with the constant $2^{1+1/p}+1$ instead of 2. He used the Pólya-Szegö principle and approximations by the Steklov averages.

Theorem 4.1 was derived from the inequality

$$\iint_{|t-s|\le\delta} \varphi(f^*(t) - f^*(s))\,dtds \le \iint_{|x-y|\le\delta} \varphi(f(x) - f(y))\,dxdy, \tag{4.4}$$

where φ is an even nonnegative function, increasing on $[0,+\infty)$ (see [24], [54], [66]).

Even for a function $f \in C_0^\infty$ its rearrangement may be non-differentiable in some points. However, Cianchi [16] obtained some sharp results concerning the second order modulus of smoothness of the rearrangement.

Denote

$$\Delta^r(h)f(x) = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}f(x+jh).$$

If $f \in L^p[0,1]$, $1 \le p < \infty$, then its modulus of continuity of order r is defined by

$$\omega^r(f;\delta)_p = \sup_{0\le h\le\delta}\left(\int_0^{1-rh}|\Delta^r(h)f(x)|^p\,dx\right)^{1/p} \quad (0 \le \delta \le 1/r).$$

Let $1 \le p < \infty$, $1 \le \theta \le \infty$ and $\alpha > 0$. Denote by r the least integer such that $r > \alpha$. The Besov space $B^\alpha_{p,\theta}[0,1]$ is defined as the class of all $f \in L^p[0,1]$ such that

$$\|f\|_{B^\alpha_{p,\theta}} \equiv \|f\|_p + \left(\int_0^{1/r}\left[t^{-\alpha}\omega^r(f;t)_p\right]^\theta \frac{dt}{t}\right)^{1/\theta} < \infty.$$

It was proved by Oswald [56] and, independently, by Bourdaud and Meyer [8], that the operator $f \mapsto |f|$ is bounded in $B^\alpha_{p,\theta}$ if and only if $0 < \alpha < 1 + 1/p$.

Cianchi [16] obtained a similar result for the operator $f \mapsto f^*$.

Theorem 4.2. *Let $1 \le p < \infty$, $1 \le \theta \le \infty$ and $0 < \alpha < 1 + 1/p$. Assume that $f \in B^\alpha_{p,\theta}[0,1]$. Then $f^* \in B^\alpha_{p,\theta}[0,1]$ and*

$$\|f^*\|_{B^\alpha_{p,\theta}} \le c\|f\|_{B^\alpha_{p,\theta}}. \tag{4.5}$$

It was also shown in [16] that (4.5) does not hold if $1 \le \theta < \infty$ and $\alpha \ge 1 + 1/p$, or if $\theta = \infty$ and $\alpha > 1 + 1/p$. The case $\theta = \infty$, $\alpha = 1 + 1/p$ is open.

It would be interesting to obtain a general estimate of $\omega^2(f^*;t)_p$ in terms of $\omega^2(f;t)_p$ which would include (4.5) as a special case.

4.3. Moduli of continuity (multidimensional case). Garsia [23] and Milne [49] obtained the following multidimensional analogue of the inequality (4.4).

Theorem 4.3. *Let φ be an even, nonnegative and nondecreasing function on $[0,+\infty)$. Then for any measurable, any almost everywhere finite function f on $[0,1]^n$ and any $\delta \in [0,1]$,*

$$\iint_{|t-s|\le c_n\delta^n} \varphi\big(f^*(t)-f^*(s)\big)\,dtds \le \iint_{|x-y|\le\delta} \varphi\big(f(x)-f(y)\big)\,dxdy, \tag{4.6}$$

where c_n is a constant depending only on n.

Budagov [14] proved a stronger inequality which takes into account the difference in the behaviour of f in directions of different axes.

Theorem 4.4. *Let φ be an even, nonnegative and nondecreasing function on $[0,+\infty)$. Let $\delta_j \in (0,1]$ $(j=1,\dots,n)$ and $\delta = \delta_1\cdots\delta_n$. Then for any measurable and almost everywhere finite function f on $[0,1]^n$,*

$$\begin{aligned}\iint_{|t-s|\le 4^{-n}\delta^n} &\varphi\big(f^*(t)-f^*(s)\big)\,dtds\\ &\le \iint_{\substack{|x_j-y_j|\le\delta_j\\ j=1,\dots,n}} \varphi\big(f(x)-f(y)\big)\,dxdy.\end{aligned} \tag{4.7}$$

However, in contrast to the one-dimensional case, the inequalities (4.6) and (4.7) for $n\ge 2$ are not sharp. If we take $\varphi(t)=t^p$, then (4.6) and (4.7) can be interpreted as estimates of the L^p-modulus of continuity of f^* in terms of the L^p-moduli of continuity of f. The inequality (4.6) implies that

$$\omega(f^*;\delta)_p \le c\omega(f;\delta^{1/n})_p. \tag{4.8}$$

Nevertheless, this estimate is not sharp. The following sharpening of (4.8) was proved in [35].

Theorem 4.5. *Let $f\in L^p(\mathbb{R}^n)$, $1\le p<\infty$, $n\ge 2$. Then for any $\delta>0$,*

$$\int_{\delta^n}^\infty t^{-p/n}\omega(f^*;t)_p^p\,\frac{dt}{t} \le c_{p,n}\left(\frac{\omega(f;\delta)_p}{\delta}\right)^p. \tag{4.9}$$

We emphasize that (4.9) fails to hold for $n=1$. The proof of (4.9) was obtained by the use of Lemma 3.3 and of approximations by the Steklov averages. Inequalities (4.9) and (3.8) immediately imply the following:

Corollary 4.6. *Let $f \in W_p^1(\mathbb{R}^n)$, $1 \le p < \infty$, $n \ge 2$. Then*

$$\left(\int_0^\infty t^{-p/n} \omega(f^*; t)_p \, \frac{dt}{t}\right)^{1/p} \le c\|\nabla f\|_p.$$

That is, if $f \in W_p^1(\mathbb{R}^n)$, then f^* belongs to the Besov space $B_p^{1/n}(\mathbb{R}_+)$.

It would be interesting to obtain a sharpening of (4.6) and (4.7) for arbitrary functions φ.

Notice that for symmetric rearrangements Wik [67] proved the following analogue of Theorem 4.1.

Theorem 4.7. *Let $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Then for any $\delta > 0$,*

$$\int_0^\delta \omega(f_s^*; t)_p^p \, dt \le \int_0^\delta \omega(f; t)_p^p \, dt. \tag{4.10}$$

In principle, (4.10) implies (4.8). However, the stronger inequality (4.9) cannot be derived directly from (4.10).

Now we consider partial moduli of continuity. Let $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, and $k \in \{1, \dots, n\}$. The partial modulus of continuity of f in L^p with respect to the kth variable x_k is defined by

$$\omega_k(f; \delta)_p = \sup_{0 \le h \le \delta} \left(\int_{\mathbb{R}^n} |f(x + he_k) - f(x)|^p \, dx\right)^{1/p}$$

(e_k is the kth unit coordinate vector). It is easy to see that

$$\max_k \omega_k(f; \delta)_p \le \omega(f; \delta)_p \le \sum_{k=1}^n \omega_k(f; \delta)_p. \tag{4.11}$$

The function

$$\overline{\omega}(f; \delta)_p = \inf_{\substack{\delta_1 \cdots \delta_n = \delta \\ \delta_j \ge 0}} \max_{1 \le j \le n} \omega_j(f; \delta_j)_p \tag{4.12}$$

is called *the average modulus of continuity* (see [30], [31]). It follows from the definition that

$$\overline{\omega}(f; \delta)_p \le \omega(f; \delta^{1/n})_p.$$

P. Oswald [55] proved the following theorem.

Theorem 4.8. *Let $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Then for any $\delta \ge 0$,*

$$\omega(f^*;\delta)_p \le c_n \overline{\omega}(f;\delta)_p. \tag{4.13}$$

Oswald based his proof on the use of some combinatorial methods. A simpler proof was given in [35].

The inequality (4.13) looks like the one-dimensional inequality (4.3). However, in view of Theorem 4.5, it is clear that (4.13) is not sharp. Sharp estimates in terms of *partial* moduli of continuity are known only in the setting of Lipschitz classes.

Let $1 \le p < \infty$ and $0 < \alpha_k \le 1$ $(k = 1, \dots, n)$. Denote by $\Lambda_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$ the class of all $f \in L^p(\mathbb{R}^n)$ such that

$$\|f\|_{\lambda_p^{\alpha_1,\dots,\alpha_n}} \equiv \max_{k=1,\dots,n} \sup_{\delta>0} \delta^{-\alpha_k}\omega_k(f;\delta)_p < \infty.$$

Set

$$\overline{\alpha} = \left(\sum_{k=1}^n \frac{1}{\alpha_k}\right)^{-1}.$$

Then for any $f \in \Lambda_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$, we have $\overline{\omega}(f;\delta)_p = O(\delta^{\overline{\alpha}})$ and, by (4.13),

$$\omega(f^*;\delta)_p \le c\delta^{\overline{\alpha}}.$$

This estimate is sharp if $0 < \alpha_k < 1$, $k = 1, \dots, n$. However, it can be strengthened if at least one α_k is equal to 1. Namely, we proved the following theorem in [36]:

Theorem 4.9. *Let $\alpha_1, \dots, \alpha_n \in (0,1]$ $(n \ge 2)$ and let ν be the number of those α_k that are equal to 1. Let*

$$\overline{\alpha} = \left(\sum_{k=1}^n \frac{1}{\alpha_k}\right)^{-1} \quad \text{and} \quad s = \frac{p}{\overline{\alpha}\nu}.$$

Then for any $f \in \Lambda_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$,

$$\left(\int_0^\infty \left[t^{-\overline{\alpha}}\omega(f^*;t)_p\right]^s \frac{dt}{t}\right)^{1/s} \le c\|f\|_{\lambda_p^{\alpha_1,\dots,\alpha_n}}.$$

That is, if $f \in \Lambda_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$, then f^* belongs to the Besov space $B^{\overline{\alpha}}_{p,s}(\mathbb{R}_+)$. Note that Corollary 4.6 is a special case of Theorem 4.9. However, we emphasize that sharp estimates similar to (4.9) in terms of partial moduli of continuous are *unknown*.

5. Iterative rearrangements

Let $x = (x_1, \dots, x_n)$. As usual, we denote by $\widehat{x}_k$ the $(n-1)$-dimensional vector obtained from the n-tuple x by removal of its kth coordinate. We shall write $x = (x_k, \widehat{x}_k)$ (let us emphasize that in this notation x_k is the kth coordinate of the vector x).

Let $f \in S_0(\mathbb{R}^n)$ and $1 \le k \le n$. We fix $\widehat{x}_k \in \mathbb{R}^{n-1}$ and consider the $\widehat{x}_k$-section of the function f

$$f_{\widehat{x}_k}(x_k) = f(x_k, \widehat{x}_k), \quad x_k \in \mathbb{R}.$$

We have $f_{\widehat{x}_k} \in S_0(\mathbb{R})$ for almost all $\widehat{x}_k \in \mathbb{R}^{n-1}$. Set

$$\mathcal{R}_k f(u, \widehat{x}_k) = f^*_{\widehat{x}_k}(u), \quad u \in \mathbb{R}_+.$$

We emphasize that the kth argument of the function $\mathcal{R}_k f$ is equal to u. The function $\mathcal{R}_k f$ is defined almost everywhere on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$; we call it the rearrangement of f with respect to the kth variable. Using approximation by step functions, Lemma 2.2 and Fubini's theorem, one can easily show that $\mathcal{R}_k f$ is a measurable function equimeasurable with $|f|$. For each ν-tuple $\{k_1, \dots, k_\nu\}$ of pairwise different indices $1 \le k_j \le n$ we set $\mathcal{R}_{k_1,\dots,k_\nu} f = \mathcal{R}_{k_\nu} \cdots \mathcal{R}_{k_1} f$. Next, let $\mathcal{P}_n$ be the collection of all permutations $\sigma = \{k_1, \dots, k_n\}$ of the set $\{1, \dots, n\}$. For each $\sigma \in \mathcal{P}_n$ we call the function

$$\mathcal{R}_\sigma f(t) \equiv \mathcal{R}_{k_1,\dots,k_n} f(t), \quad t \in \mathbb{R}^n_+,$$

the $\mathcal{R}_\sigma$-rearrangement of f. Thus, we obtain $\mathcal{R}_\sigma f$ by "rearranging" f in a non-increasing order successively with respect to the variables $x_{k_1}, \dots, x_{k_n}$. Doing so, we replace successively the arguments $x_{k_1}, \dots, x_{k_n}$ with the arguments $t_{k_1}, \dots, t_{k_n}$. It is easy to see that $\mathcal{R}_\sigma f$ is decreasing with respect to each variable. In view of the above observation, $\mathcal{R}_\sigma f$ is equimeasurable with $|f|$.

In analogy with Lemma 2.2, we have the following:

Lemma 5.1. *Let $f_k \in S_0(\mathbb{R}^n)$ $(k \in \mathbb{N})$ and assume that the sequence $\{f_k\}$ converges in measure to a function $f \in S_0(\mathbb{R}^n)$. Then for each permutation $\sigma \in \mathcal{P}_n$,*

$$\lim_{k\to\infty} \mathcal{R}_\sigma f_k(t) = \mathcal{R}_\sigma f(t) \quad \textit{for almost all } \; t \in \mathbb{R}^n_+.$$

Further, the following theorem holds for the partial moduli of continuity of the iterative rearrangements.

Theorem 5.2. *Let $f \in L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$. Then for each permutation $\sigma \in \mathcal{P}_n$ and each $k = 1, \dots, n$,*

$$\omega_k(\mathcal{R}_\sigma f; \delta)_p \le c\omega_k(f; \delta)_p \quad \textit{for all} \;\; \delta \ge 0,$$

where c is an absolute constant.

This theorem can be easily derived from (4.3) by induction, with the use of Lemma 2.3.

In what follows we set

$$\pi(t) = \prod_{k=1}^{n} t_k, \quad t = (t_1, \dots, t_n) \in \mathbb{R}^n_+.$$

Let $0 < p, r < \infty$ and let $\sigma \in \mathcal{P}_n$ $(n \ge 2)$. We denote by $\mathcal{L}^{p,r}_\sigma(\mathbb{R}^n)$ the class of all functions $f \in S_0(\mathbb{R}^n)$ such that

$$\|f\|_{p,r;\sigma} \equiv \left(\int_{\mathbb{R}^n_+} \left[\pi(t)^{1/p}\mathcal{R}_\sigma f(t)\right]^r \frac{dt}{\pi(t)}\right)^{1/r} < \infty$$

(see [6]). The choice of a permutation σ is essential. We also set

$$\mathcal{L}^{p,r}(\mathbb{R}^n) = \bigcap_{\sigma \in \mathcal{P}_n} \mathcal{L}^{p,r}_\sigma(\mathbb{R}^n), \quad \|f\|_{\mathcal{L}^{p,r}} = \sum_{\sigma \in \mathcal{P}_n} \|f\|_{p,r;\sigma}.$$

The following result was obtained in [68].

Theorem 5.3. *Let $f \in S_0(\mathbb{R}^n)$. Then for any $\sigma \in \mathcal{P}_n$,*

$$\|f\|_{p,r} \le 2^{1/r-1/p}\|f\|_{p,r;\sigma} \quad \textit{if} \;\; 0 < r \le p < \infty \tag{5.1}$$

and

$$\|f\|_{p,r;\sigma} \le 2^{1/p-1/r}\|f\|_{p,r} \quad \textit{if} \;\; 0 < p < r < \infty. \tag{5.2}$$

Proof. Denote $F(t) = \mathcal{R}_\sigma f(t)$. We may suppose that

$$|\{t \in \mathbb{R}^n_+ : F(t) = y\}| = 0 \quad \text{for any} \;\; y > 0.$$

Set

$$A_\nu = \{t \in \mathbb{R}^n_+ : f^*(2^{-\nu+1}) \le F(t) < f^*(2^{-\nu})\}, \quad \nu \in \mathbb{Z}.$$

If $t = (t_1, \dots, t_n) \in A_\nu$ and $s = (s_1, \dots, s_n) \in \mathbb{R}^n_+$ with $0 < s_k \le t_k$, $k = 1, \dots, n$, then $F(s) \ge f^*(2^{-\nu+1})$. Hence, $\pi(t) \le 2^{-\nu+1}$ for all $t \in A_\nu$. Let $0 < r < p$. We have

$$\begin{aligned}
\|f\|^r_{p,r;\sigma} &= \int_{\mathbb{R}^n_+} \pi(t)^{r/p-1} F(t)^r \, dt \\
&= \sum_{\nu\in\mathbb{Z}} \int_{A_\nu} \pi(t)^{r/p-1} F(t)^r \, dt \\
&\ge \sum_{\nu\in\mathbb{Z}} 2^{(r/p-1)(1-\nu)} \int_{A_\nu} F(t)^r \, dt \\
&= \sum_{\nu\in\mathbb{Z}} 2^{(r/p-1)(1-\nu)} \int_{2^{-\nu}}^{2^{-\nu+1}} f^*(u)^r \, du \\
&\ge 2^{r/p-1} \sum_{\nu\in\mathbb{Z}} \int_{2^{-\nu}}^{2^{-\nu+1}} u^{r/p-1} f^*(u)^r \, du \\
&= 2^{r/p-1} \|f\|^r_{p,r}.
\end{aligned}$$

Hence, we obtain (5.1). The proof of (5.2) is similar. □

Thus, for any $\sigma \in \mathcal{P}_n$,

$$\mathcal{L}^{p,r}_\sigma \subset L^{p,r} \quad (r \le p), \qquad L^{p,r} \subset \mathcal{L}^{p,r}_\sigma \quad (p \le r).$$

If $p \ne r$, then these embeddings are strict (see [68]). Moreover, we have the following statement.

Proposition 5.4. *Let $0 < r < p < \infty$. There exists a measurable set $E \subset \mathbb{R}^2$ with $|E| < \infty$ such that $\chi_E \notin \mathcal{L}^{p,r}_{1,2}(\mathbb{R}^2) \cup \mathcal{L}^{p,r}_{2,1}(\mathbb{R}^2)$.*

Proof. Set

$$\varphi(x) = \frac{1}{x(\ln(2/x))^{p/r}}, \quad 0 < x \le 1,$$

and

$$E = \{(x, y) : 0 < y \le \varphi(x),\ 0 < x \le 1\}.$$

Then $\mathcal{R}_{1,2}\chi_E = \mathcal{R}_{2,1}\chi_E = \chi_E$ and we have

$$\begin{aligned}
\iint_{\mathbb{R}^2_+} (ts)^{r/p-1} \chi_E(t,s)^r \, dtds &= \int_0^1 t^{r/p-1} \, dt \int_0^{\varphi(t)} s^{r/p-1} \, ds \\
&= \frac{p}{r} \int_0^1 t^{r/p-1} \varphi(t)^{r/p} \, dt = \frac{p}{r} \int_0^1 \frac{1}{t\ln(2/t)} \, dt = \infty.
\end{aligned}$$

□

It was shown in our work [39] that in the Sobolev-type inequalities the usual Lorentz norm at the left-hand side can be replaced by a stronger $\mathcal{L}^{q^*,p}$-norm. We shall consider the simplest special case of this result which gives a refinement of the inequality (3.5).

Theorem 5.5. *Let $n \geq 2$, $1 \leq p < n$, and $q^* = np/(n-p)$. Then for any $f \in W_p^1(\mathbb{R}^n)$,*

$$\|f\|_{\mathcal{L}^{q^*,p}} \leq c \sum_{k=1}^{n} \|D_k f\|_p. \tag{5.3}$$

Proof. Let $\sigma = \{1, \ldots, n\}$. We estimate $\|f\|_{\mathcal{L}_\sigma^{q^*,p}}$. First, we consider the case $p = 1$. We have

$$|f(x)| \leq \frac{1}{2} \int_{\mathbb{R}} |D_j f(u, \widehat{x}_j)|\, du \equiv \frac{1}{2} \psi_j(\widehat{x}_j), \quad j = 1, \ldots, n.$$

This implies that

$$\mathcal{R}_\sigma f(t) \leq \frac{1}{2} \min_{1 \leq j \leq n} \mathcal{R}_{\widehat{\sigma}_j} \psi_j(\widehat{t}_j), \quad t \in \mathbb{R}_+^n, \tag{5.4}$$

where $\widehat{\sigma}_j$ is the $(n-1)$-tuple obtained from σ by removal of the jth coordinate. Set

$$A_j = \{t \in \mathbb{R}_+^n : t_j \leq \pi(t)^{1/n}\}, \quad j = 1, \ldots, n.$$

Then $\mathbb{R}_+^n = \cup_{j=1}^n A_j$. Further, by (5.4), for any $j = 1, \ldots, n$, we have

$$\begin{aligned}
&\int_{A_j} \pi(t)^{-1/n} \mathcal{R}_\sigma f(t)\, dt \\
&\qquad \leq \frac{1}{2} \int_{R_+^{n-1}} \pi(\widehat{t}_j)^{-1/n} \mathcal{R}_{\widehat{\sigma}_j} \psi_j(\widehat{t}_j) \int_0^{\pi(\widehat{t}_j)^{1/(n-1)}} t_j^{-1/n}\, dt_j d\widehat{t}_j \\
&\qquad \leq \int_{R_+^{n-1}} \mathcal{R}_{\widehat{\sigma}_j} \psi_j(\widehat{t}_j)\, d\widehat{t}_j = \|D_j f\|_1,
\end{aligned}$$

where $\pi(\widehat{t}_j) = \prod_{k \neq j} t_k$. Thus,

$$\|f\|_{\mathcal{L}_\sigma^{n',1}} \leq \sum_{j=1}^{n} \|D_j f\|_1.$$

Similar estimates hold for any $\sigma \in \mathcal{P}_n$, which proves (5.3) for $p = 1$.

Let now $p > 1$. By virtue of Lemma 5.1, we may assume that $\|f\|_{\mathcal{L}^{q^*,p}} < \infty$. Set $K = (2n!)^{q^*}$. For any $j = 1, \ldots, n$ we have

$$|f(x)| \le \mathcal{R}_j f(Kt_j, \widehat{x}_j) + \int_{x_j}^{x_j+Kt_j} |D_j f(u, \widehat{x}_j)|\, du$$
$$\le \mathcal{R}_j f(Kt_j, \widehat{x}_j) + (Kt_j)^{1-1/p}\psi_j(\widehat{x}_j),$$

where

$$\psi_j(\widehat{x}_j) = \left(\int_{\mathbb{R}} |D_j f(u, \widehat{x}_j)|^p\, du\right)^{1/p}.$$

It follows that for any $j = 1, \ldots, n$

$$\mathcal{R}_\sigma f(t) \le \mathcal{R}_{\sigma'_j} f(Kt_j, \widehat{t}_j) + (Kt_j)^{1-1/p}\mathcal{R}_{\widehat{\sigma}_j}\psi_j(\widehat{t}_j),$$

where σ'_j is obtained from σ by moving the jth coordinate to the first place. As above, from these estimates we easily obtain that

$$\int_{A_j} \pi(t)^{p/q^*-1}\mathcal{R}_\sigma f(t)^p\, dt \le K^{-p/q^*}\int_{\mathbb{R}^n_+} \pi(t)^{p/q^*-1}\mathcal{R}_{\sigma'_j} f(t)^p\, dt$$
$$+ c\int_{R^{n-1}_+} \mathcal{R}_{\widehat{\sigma}_j}\psi_j(\widehat{t}_j)\, d\widehat{t}_j$$
$$= K^{-p/q^*}\|f\|^p_{\mathcal{L}^{q^*,p}_{\sigma'_j}} + c\|D_j f\|_p^p.$$

This implies

$$\int_{\mathbb{R}^n_+} \pi(t)^{p/q^*-1}\mathcal{R}_\sigma f(t)^p\, dt \le \left(\frac{1}{2n!}\right)^p \|f\|^p_{\mathcal{L}^{q^*,p}} + c\sum_{j=i}^n \|D_j f\|_p^p$$

and therefore

$$\|f\|_{\mathcal{L}^{q^*,p}} \le c'\sum_{j=i}^n \|D_j f\|_p.$$

□

Now we prove some estimates in terms of moduli of continuity which will be applied below.

Lemma 5.6. *Let $\varphi \in L^p(\mathbb{R}_+)$, $1 \le p < \infty$. Then*

$$\int_0^\infty \frac{1}{x}\int_{x/2}^{2x} |\varphi(y)|^p \, dydx \le 3\|\varphi\|_p^p.$$

Proof. Using Fubini's theorem, we obtain

$$\begin{aligned}\int_0^\infty \frac{1}{x}\int_{x/2}^{2x} |\varphi(y)|^p \, dydx &= \sum_{k\in\mathbb{Z}} \int_{2^{-k}}^{2^{-k+1}} \frac{1}{x}\int_{x/2}^{2x} |\varphi(y)|^p \, dydx \\ &< 3\sum_{k\in\mathbb{Z}} \int_{2^{-k}}^{2^{-k+1}} |\varphi(y)|^p \, dy = 3\|\varphi\|_p^p.\end{aligned}$$

□

Lemma 5.7. *Let $1 \le p < \infty$ and $n \ge 2$. Assume that $f \in L^p(\mathbb{R}^n_+)$ is a nonnegative function nonincreasing in each variable. Then for any $1 \le k \le n$ and any $h > 0$,*

$$\left(\int_{R_+^{n-1}} \int_h^\infty u^{-p}\big[f(u, \widehat{t}_k) - f(2u, \widehat{t}_k)\big]^p \, du d\widehat{t}_k\right)^{1/p} \le 12\frac{\omega_k(f,h)_p}{h}. \quad (5.5)$$

Proof. Fix $\widehat{t}_k \in R_+^{n-1}$ and denote $g(u) = f(u, \widehat{t}_k)$, $u \in \mathbb{R}_+$. Let $0 < h \le u$. Then

$$g(u) - g(2u) \le \frac{2}{h}\int_{u/2}^{2u} [g(z) - g(z+h)] \, dz. \quad (5.6)$$

Indeed,

$$\begin{aligned}\int_{u/2}^{2u} [g(z) - g(z+h)] \, dz &= \int_{u/2}^{2u} g(z)\, dz - \int_{u/2+h}^{2u+h} g(z)\, dz \\ &\ge \int_{u/2}^{u/2+h} g(z)\, dz - hg(2u) \\ &= \int_{u/2}^{u/2+h} [g(z) - g(2u)]\, dz \\ &\ge \frac{h}{2}[g(u) - g(2u)].\end{aligned}$$

Further, (5.6) implies

$$g(u) - g(2u) \le \frac{4u^{1-1/p}}{h} \left(\int_{u/2}^{2u} [g(z) - g(z+h)]^p \, dz \right)^{1/p}.$$

Thus, the left-hand side of (5.5) does not exceed

$$\frac{4}{h} \left(\int_{R_+^{n-1}} \int_0^\infty \frac{1}{u} \int_{u/2}^{2u} \left[f(z, \widehat{t}_k) - f(z+h, \widehat{t}_k) \right]^p dz du d\widehat{t}_k \right)^{1/p}.$$

Applying Lemma 5.6, we obtain (5.5). □

6. Spaces of fractional smoothness

In this section we consider definitions of some anisotropic spaces of fractional smoothness.

Let $r \in \mathbb{N}$, $1 \le p < \infty$, and $1 \le j \le n$. Denote by $W^r_{p;j}(\mathbb{R}^n)$ the Sobolev space of all functions $f \in L^p(\mathbb{R}^n)$ for which there exists the weak partial derivative $D^r_j f \in L^p(\mathbb{R}^n)$. Set also

$$W_p^{r_1,\dots,r_n}(\mathbb{R}^n) = \bigcap_{j=1}^n W^{r_j}_{p;j}(\mathbb{R}^n) \quad (r_j \in \mathbb{N},\ 1 \le p < \infty).$$

Let a function f be given on $\mathbb{R}^n$. For $r \in \mathbb{N}$, $1 \le j \le n$ and $h \in \mathbb{R}$ we set

$$\Delta^r_j(h) f(x) = \sum_{i=0}^r (-1)^{r-i} \binom{r}{i} f(x + ihe_j),$$

where e_j is the unit coordinate vector in $\mathbb{R}^n$. If $f \in L^p(\mathbb{R}^n)$, then the function

$$\omega^r_j(f;\delta)_p = \sup_{0 \le h \le \delta} \|\Delta^r_j(h) f\|_p$$

is called the *partial modulus of continuity* of order r of the function f with respect to the variable x_j in L^p. If $r = 1$, then we omit the superscript in this notation.

If f has the weak derivative $D^r_j f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then

$$\Delta^r_j(h) f(x) = \int_0^h \cdots \int_0^h D^r_j f(x + (u_1 + \cdots + u_r)e_j) \, du_1 \dots du_r \tag{6.1}$$

for almost all x (see [5, §16, (8)]).

Let $f \in L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$, $\alpha > 0$, and $1 \le j \le n$. Let r be the least integer such that $r > \alpha$. The function f belongs to the class $H^\alpha_{p;j}(\mathbb{R}^n)$ if

$$\|f\|_{h^\alpha_{p;j}} \equiv \sup_{\delta>0} \frac{\omega_j^r(f;\delta)_p}{\delta^\alpha} < \infty. \tag{6.2}$$

We emphasize that if $\alpha \in \mathbb{N}$, then in (6.2) we take the modulus of continuity of the order $r = \alpha + 1$.

If $\alpha_j > 0$ $(j = 1, \dots, n)$ and $1 \le p \le \infty$, the Nikol'skii space $H_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$ is defined by

$$H_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n) = \bigcap_{j=1}^n H^{\alpha_j}_{p;j}(\mathbb{R}^n).$$

Assume now that $\alpha > 0$, $1 \le p, \theta < \infty$ and $1 \le j \le n$. As above, let r be the least integer such that $r > \alpha$. A function $f \in L^p(\mathbb{R}^n)$ belongs to the class $B^\alpha_{p,\theta;j}(\mathbb{R}^n)$ if

$$\|f\|_{b^\alpha_{p,\theta;j}} \equiv \left(\int_0^\infty \left[t^{-\alpha} \omega_k^r(f;t)_p \right]^\theta \frac{dt}{t} \right)^{1/\theta} < \infty.$$

Denote also $B^\alpha_{p,p;j} \equiv B^\alpha_{p;j}$.

Let $\alpha_j > 0$ $(j = 1, \dots, n)$ and $1 \le p, \theta < \infty$. Then we set

$$B_{p,\theta}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n) = \bigcap_{j=1}^n B^{\alpha_j}_{p,\theta;j}(\mathbb{R}^n) \quad (B_p^{\alpha_1,\dots,\alpha_n} \equiv B_{p,p}^{\alpha_1,\dots,\alpha_n}).$$

It is easy to see that

$$\|f\|_{h^\alpha_{p;j}} = \lim_{\theta \to +\infty} \|f\|_{b^\alpha_{p,\theta;j}}.$$

This is why we set $B^\alpha_{p,\infty;j}(\mathbb{R}^n) = H^\alpha_{p;j}(\mathbb{R}^n)$ by definition.

It is also well known that

$$B^\alpha_{p,\theta;j} \subset B^\alpha_{p,\eta;j} \quad \text{if} \quad 1 \le \theta < \eta \le \infty$$

(see, e.g., [53]). Moreover, the following estimate holds [41].

Lemma 6.1. *Let* $1 \le p < \infty$, $1 \le \theta < \eta \le \infty$, $0 < \alpha < 1$ *and* $1 \le j \le n$. *Then for any function* $f \in L^p(\mathbb{R}^n)$,

$$\|f\|_{b^\alpha_{p,\eta;j}} \le 8[\alpha(1-\alpha)]^{1/\theta - 1/\eta} \|f\|_{b^\alpha_{p,\theta;j}}.$$

Bourgain, Brezis and Mironescu [9] (see also [11]) found a limiting relation between Sobolev and Besov norms. They proved that a function $f \in L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ belongs to $W^1_p(\mathbb{R}^n)$ if and only if there exists a finite limit

$$\lim_{\alpha \to 1-} (1-\alpha) \|f\|^p_{b^\alpha_p}.$$

Moreover, for any $1 \le p < \infty$ and any $f \in W^1_p(\mathbb{R}^n)$,

$$\lim_{\alpha \to 1-} (1-\alpha) \|f\|^p_{b^\alpha_p} = \frac{1}{p} \|\nabla f\|^p_p. \tag{6.3}$$

For the partial Besov norms we have the following statement.

Lemma 6.2.

$$\lim_{\alpha \to 1-} (1-\alpha)^{1/\theta} \|f\|_{b^\alpha_{p,\theta;k}} = \left(\frac{1}{\theta}\right)^{1/\theta} \sup_{\delta > 0} \frac{\omega_k(f,\delta)_p}{\delta}. \tag{6.4}$$

The proof can be given in the same way as in [42, Proposition 2.5]. Observe that if $f \in W^1_{p;j}(\mathbb{R}^n)$ $(1 \le p < \infty)$, then

$$\sup_{\delta > 0} \frac{\omega_k(f,\delta)_p}{\delta} = \|D_k f\|_p$$

(see [42, Proposition 2.4]).

Let us emphasize again that in the definition of the Nikol'skii space $H^\alpha_{p;j}(\mathbb{R}^n)$ the order r of the modulus of continuity is *strictly greater* than the smoothness exponent α. If $\alpha \in \mathbb{N}$, it is also natural to admit the value $r = \alpha$. However, it leads to completely different spaces – Lipschitz-type spaces.

Assume that $\alpha > 0$ and denote by α^* the least integer $r \ge \alpha$. Let $1 \le p < \infty$ and $1 \le j \le n$. Denote by $\Lambda^\alpha_{p;j}(\mathbb{R}^n)$ the class of all functions $f \in L^p(\mathbb{R}^n)$ such that

$$\|f\|_{l^\alpha_{p;j}} \equiv \sup_{\delta > 0} \frac{\omega_j^{\alpha^*}(f;\delta)_p}{\delta^\alpha} < \infty.$$

Clearly, $\|f\|_{l^\alpha_{p;j}} = \|f\|_{h^\alpha_{p;j}}$ if $\alpha \notin \mathbb{N}$. If $\alpha \in \mathbb{N}$, then we have the strict embedding $\Lambda^\alpha_{p;j} \subset H^\alpha_{p;j}$. Moreover, by the Hardy–Littlewood theorem [53, § 4.8], if $\alpha \in \mathbb{N}$, then

$$\Lambda^\alpha_{p;j}(\mathbb{R}^n) = W^\alpha_{p;j}(\mathbb{R}^n) \quad \text{for } 1 < p \le \infty. \tag{6.5}$$

If $\alpha_j > 0$ $(j = 1, \dots, n)$ and $1 \le p < \infty$, we set

$$\Lambda^{\alpha_1,\dots,\alpha_n}_p(\mathbb{R}^n) = \bigcap_{j=1}^n \Lambda^\alpha_{p;j}(\mathbb{R}^n).$$

We shall also consider the fractional Sobolev spaces.

The Bessel kernel G_α of order $\alpha > 0$ on $\mathbb{R}$ is defined as the function with Fourier transform

$$\widehat{G}_\alpha(\xi) = (1 + 4\pi^2\xi^2)^{-\alpha/2}, \quad \xi \in \mathbb{R}$$

(see [62, p. 130]).

Let $1 \le p \le \infty$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, and $1 \le j \le n$. Let f be a measurable function on $\mathbb{R}^n$. We say that f belongs to the space $L^\alpha_{p;j}(\mathbb{R}^n)$ if there exists a function $f_j \in L^p(\mathbb{R}^n)$ such that for almost all $x \in \mathbb{R}^n$

$$f(x) = \int_{\mathbb{R}} G_\alpha(x_j - t) f_j(t, \widehat{x}_j)\, dt. \tag{6.6}$$

The use of Fubini's theorem together with the arguments given in [62, p. 135] show that the equality (6.6) determines the function f_j uniquely, up to its values on a set of n-dimensional Lebesgue measure zero. We also have

$$\|f\|_p \le \|f_j\|_p.$$

We call f_j the *Bessel derivative* of the function f of order α with respect to x_j and we denote it by $D^\alpha_j f$.

If $\alpha \in \mathbb{N}$, then we set $L^\alpha_{p;j}(\mathbb{R}^n) = W^\alpha_{p;j}(\mathbb{R}^n)$.

The strict embedding

$$L^\alpha_{p;j}(\mathbb{R}^n) \subset \Lambda^\alpha_{p;j}(\mathbb{R}^n) = H^\alpha_{p;j}(\mathbb{R}^n), \quad \alpha \notin \mathbb{N},$$

holds for $1 \le p \le \infty$ (see [53, Chap. 9.3]). Further, if $\alpha \in \mathbb{N}$, then

$$L^\alpha_{1;j}(\mathbb{R}^n) \equiv W^\alpha_{1;j}(\mathbb{R}^n) \subset \Lambda^\alpha_{1;j}(\mathbb{R}^n)$$

and

$$L^{\alpha}_{p;j}(\mathbb{R}^n) \equiv W^{\alpha}_{p;j}(\mathbb{R}^n) = \Lambda^{\alpha}_{p;j}(\mathbb{R}^n) \quad (1 < p < \infty)$$

(see (6.5)).

Let $\alpha_j > 0$ $(j = 1, \ldots, n)$ and $1 \le p \le \infty$. Set

$$L^{\alpha_1,\ldots,\alpha_n}_p(\mathbb{R}^n) = \bigcap_{j=1}^{n} L^{\alpha_j}_{p;j}(\mathbb{R}^n).$$

We shall call $L^{\alpha_1,\ldots,\alpha_n}_p(\mathbb{R}^n)$ the fractional Sobolev space or the Sobolev-Liouville space. Note that this definition is different from the one in the monograph [53] only in the case when $p = 1$ and at least one of the α_j is an odd integer (see [38], [53]).

7. Embeddings

In this section we will study Sobolev-type inequalities in terms of fractional smoothness. Our main objective is to discuss different statements of problems as well as to study relations between different results in this area. Therefore, we consider only the simplest versions of the known theorems.

The origins of the embedding theory are contained in the following basic results due to Hardy and Littlewood [26], [27].

Theorem 7.1. *Let $1 < p < \infty$, $0 < \alpha < 1/p$, and $q^* = p/(1 - \alpha p)$. Assume that $f \in L^p[0, 2\pi]$, $\int_0^{2\pi} f(x)\,dx = 0$, and let f_α be the fractional Weyl integral of f of order α. Then*

$$\|f_\alpha\|_{q^*} \le c\|f\|_p.$$

Note that this theorem is not true for $p = 1$. In 1938 Sobolev extended Theorem 7.1 to the Riesz potentials for functions of several variables.

Theorem 7.2. *Let $1 \le p < \infty$, $0 < \alpha \le 1$, $p < q < \infty$ and $1/p - 1/q < \alpha$. Assume that $f \in L^p[0, 1]$ and $\omega(f; \delta)_p = O(\delta^\alpha)$. Then $f \in L^q[0, 1]$ and $\omega(f; \delta)_q = O(\delta^{\alpha - 1/p + 1/q})$.*

Simple examples show that for $0 < \alpha < 1/p$ the function f may fail to belong to the space L^{q^*} with the limiting exponent $q^* = p/(1 - \alpha p)$.

For the fractional Sobolev-Liouville spaces $L^{\alpha_1,\ldots,\alpha_n}_p(\mathbb{R}^n)$ the embedding into L^q with the limiting exponent was proved by Lizorkin (see [53]). The following result was proved in [38], [39].

Theorem 7.3. *Assume that* $1 < p < \infty$, $n \geq 1$ *or* $p = 1$, $n \geq 2$. *Let* $\alpha_j > 0$ $(j = 1, \dots, n)$ *and let*

$$a \equiv n\left(\sum_{j=1}^{n} \frac{1}{\alpha_j}\right)^{-1} < \frac{n}{p}.$$

Let $q^* = np/(n - \alpha p)$. *Then for every function* $f \in L_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$

$$\|f\|_{q^*,p} \leq c \sum_{j=1}^{n} \|D_j^{\alpha_j} f\|_p. \tag{7.1}$$

Let us emphasize that, in contrast to the case $n = 1$ (see Theorem 7.1), if $n \geq 2$ Theorem 7.3 is true for $p = 1$, too. Observe also that the left-hand side in (7.1) can be replaced by the stronger norm $\|f\|_{\mathcal{L}^{q^*,p}}$ (cf. [39]).

Next, we have the following limiting embedding theorem for Besov spaces (see [5, § 18], [25], [38], [57]).

Theorem 7.4. *Let* $n \in \mathbb{N}$ *and* $\alpha_j > 0$ $(j = 1, \dots, n)$. *Set*

$$\alpha = n\left(\sum_{j=1}^{n} \frac{1}{\alpha_j}\right)^{-1}.$$

Assume that $1 \leq p < n/\alpha$ *and* $1 \leq \theta \leq \infty$. *Let* $q^* = np/(n - \alpha p)$. *Then*

$$B_{p,\theta}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n) \subset L^{q^*,\theta}(\mathbb{R}^n)$$

and, for every function $f \in B_{p,\theta}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$,

$$\|f\|_{q^*,\theta} \leq c\|f\|_{b_{p,\theta}^{\alpha_1,\dots,\alpha_n}}. \tag{7.2}$$

In particular, if $1 \leq p < q < \infty$ and $\alpha = n(1/p - 1/q)$, then for any $f \in B_{p,q}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$

$$\|f\|_q \leq c\|f\|_{b_{p,q}^{\alpha_1,\dots,\alpha_n}}. \tag{7.3}$$

The inequality (7.3) gives a sharp estimate of the L^q-norm of the function f in terms of its $B_{p,q}^{\alpha_1,\dots,\alpha_n}$-norm. However, the problem can be formulated in a different way, posed by Ul'yanov [64]: given a function $f \in L^p(\mathbb{R}^n)$, find sharp estimates of $\|f\|_q$ in terms of partial moduli of continuity of f.

It is more general and may lead to essentially sharper results. The only exception is the case $n = 1$ in which the sharp estimate of $\|f\|_q$ via modulus of continuity coincides with the estimate via Besov norm.

Let $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, and let $p < q < \infty$. Applying Ul'yanov's inequality (3.11), equality (2.7) and Hardy's inequality (2.4), we obtain the estimate

$$\|f\|_q \le c \left(\int_0^\infty t^{-q/p} \omega(f;t)_p^q \, dt \right)^{1/q} \tag{7.4}$$

(cf. [64]). This inequality is sharp in the following sense. We shall call the *modulus of continuity* any non-decreasing, continuous and bounded function $\omega(\delta)$ on $[0, +\infty)$ which satisfies the conditions

$$\omega(\delta + \eta) \le \omega(\delta) + \omega(\eta), \quad \omega(0) = 0.$$

If $\omega(\delta)$ is a given modulus of continuity and $1 \le p < \infty$, denote by $\mathcal{H}_p^\omega(\mathbb{R})$ the class of all functions $f \in L^p(\mathbb{R})$ for which $\omega(f;\delta)_p = O(\omega(\delta))$. Ul'yanov [64] proved that the embedding

$$\mathcal{H}_p^\omega(\mathbb{R}) \subset L^q(\mathbb{R}) \quad (1 \le p < q < \infty)$$

holds if and only if

$$\int_0^\infty t^{-q/p} \omega(t)^q \, dt < \infty.$$

Thus, (7.4) is sharp for *any order* of the modulus of continuity. At the same time, (7.4) coincides with (7.3) (for $n = 1$ and $\alpha < 1$) and can be written as an embedding of a Besov space,

$$B_{p,q}^\alpha(\mathbb{R}) \hookrightarrow L^q(\mathbb{R}), \quad \alpha = \frac{1}{p} - \frac{1}{q}.$$

For $n \ge 2$ the situation is completely different. We start from the *isotropic* case. Let $1 \le p < q < \infty$ and let $\alpha = n(1/p - 1/q)$. Then

$$B_{p,q}^\alpha(\mathbb{R}^n) \subset L^q(\mathbb{R}^n).$$

This embedding was proved in many papers (for the references, see [5, § 18], [32], [38]). The values of parameters are sharp. Observe that for $\alpha < 1$ the inequality

$$\|f\|_q \le c \left(\int_0^\infty \left[t^{-\alpha} \omega(f;t)_p \right]^q \frac{dt}{t} \right)^{1/q} \tag{7.5}$$

follows immediately from the estimate (3.11) and, for any $\alpha > 0$, it can be derived from a similar rearrangement estimate in terms of higher order moduli of continuity [38, Section 10].

Let us return to the problem stated above: given a function $f \in L^p(\mathbb{R}^n)$, find sharp estimate of $\|f\|_q$ in terms of $\omega(f;t)_p$. By virtue of relations (3.9) and (3.10), this problem includes the Sobolev embedding (3.4). It cannot be completely solved with the use of the estimate (3.11) and it requires the use of the stronger estimate (3.13). The corresponding results were obtained in our work [33] in the general setting of Orlicz classes $\varphi(L)$. In the case $\varphi(t) = t^q$ the result reads as follows.

Let $1 \le p < q < \infty$, $1/p - 1/q \le 1/n$. Then for any $f \in L^p(\mathbb{R}^n)$

$$\|f\|_q \le c\left(\sum_{\nu\in\mathbb{Z}} \left[2^{\nu n/p}\omega_\nu\right]^q 2^{-\nu n}\gamma_\nu\right)^{1/q}, \tag{7.6}$$

where $\omega_\nu = \omega(f;2^{-\nu})_p$ and

$$\gamma_\nu = \frac{1}{\omega_\nu}\min\left\{\omega_\nu - \omega_{\nu+1}, \omega_\nu - \frac{\omega_{\nu-1}}{2}\right\}.$$

In comparison with (7.5), we have additional factors γ_ν ($0 \le \gamma_\nu \le 1$) at the right-hand side. These factors play a crucial role. In particular, if $f \in W_p^1(\mathbb{R}^n)$ and $1/p - 1/q = 1/n$, then, by (7.6), (3.8) and (3.7), we obtain

$$\|f\|_q \le c\|\nabla f\|_p^{1-1/q}\left(\sum_{\nu\in\mathbb{Z}}(2^\nu\omega_\nu - 2^{\nu-1}\omega_{\nu-1})\right)^{1/q} \le c\|\nabla f\|_p.$$

Thus, (7.6) contains both the Sobolev inequality (3.4) and the inequality (7.5). At the same time, (3.4) cannot be derived from (7.5) (or (7.3)).

In a different form, the link between estimates in terms of Sobolev and Besov norms was found by Bourgain, Brezis and Mironescu [10]. They proved the following theorem.

Theorem 7.5. *Let $0 < \alpha < 1$ and $1 \le p < n/\alpha$. Then for any $f \in B_p^\alpha(\mathbb{R}^n)$,*

$$\|f\|_q^p \le c_n \frac{1-\alpha}{(n-\alpha p)^{p-1}}\|f\|_{b_p^\alpha}^p \quad \left(q = \frac{np}{n-\alpha p}\right), \tag{7.7}$$

where the constant c_n depends only on n.

In view of (6.3), the Sobolev inequality (3.4) can be considered as a limiting case of (7.7). Note that the proof of (7.7) in [10] was quite complicated. Afterwards, Maz'ya and Shaposhnikova [47] gave a simpler proof of this result. Moreover, they studied the behaviour of the optimal constant as $\alpha \to 0$. Namely, they proved the following inequality

$$\|f\|_q^p \le c_{p,n} \frac{\alpha(1-\alpha)}{(n-\alpha p)^{p-1}} \|f\|_{b_p^\alpha}^p \quad \Big(q = \frac{np}{n-\alpha p}\Big). \tag{7.8}$$

It was observed in [42] that the inequalities (7.7) and (7.8) can be immediately derived from the rearrangement estimate (3.13). More exactly, the following result was obtained in [42].

Theorem 7.6. *Let $0 < \alpha < 1$, $1 \le p < \frac{n}{\alpha}$ and $q = \frac{np}{n-\alpha p}$. Then for any $f \in B_p^\alpha(\mathbb{R}^n)$,*

$$\|f\|_{q,p}^p \le c_{p,n} \frac{\alpha(1-\alpha)}{(n-\alpha p)^p} \|f\|_{b_p^\alpha}^p. \tag{7.10}$$

We note that, by (2.6),

$$\|f\|_q^p \le \frac{n-\alpha p}{n} \|f\|_{q,p}^p\,,$$

and hence (7.10) immediately implies (7.8).

Now we consider estimates in terms of *partial* moduli of continuity. It is clear that such estimates are sharper than those expressed in terms of the "isotropic" modulus of continuity $\omega(f;\delta)_p$ because they take into account the differences in the behaviour of a function with respect to different variables. That is, "bad" properties in some directions can be compensated by "good" properties in other directions. The main problem is to find a right balance. The first approach to this problem can be obtained with the use of the average modulus of continuity (see (4.12)).

The following refinement of the inequality (3.11) was proved in [30]:

$$f^*(t) - f^*(2t) \le c t^{-1/p} \overline{\omega}(f;t)_p, \quad f \in L^p(\mathbb{R}^n),\ 1 \le p < \infty, \tag{7.11}$$

where $\overline{\omega}(f;t)_p$ is the average modulus of continuity. An alternative proof of a more general inequality involving the moduli of continuity of higher order was given in [38, Lemma 10.3]. Applying (7.11) and Hardy's inequality (2.4), we obtain that, for $1 \le p < q < \infty$,

$$\|f\|_q \le c \Big(\int_0^\infty t^{-q/p} \overline{\omega}(f;t)_p^q \, dt\Big)^{1/q} \tag{7.12}$$

(see [30]). It looks exactly like the inequality (7.4) for functions of one variable. However, there are essential differences between (7.4) and (7.12). For any choice of $0 < \alpha_j < 1$ such that

$$\alpha \equiv n\left(\sum_{j=1}^{n} \frac{1}{\alpha_j}\right)^{-1} = n\left(\frac{1}{p} - \frac{1}{q}\right) < 1,$$

(7.12) implies the embedding

$$B_{p,q}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n).$$

Indeed, taking $\delta_j = \delta^{\alpha/(n\alpha_j)}$ in (4.12), we obtain that

$$\overline{\omega}(f;\delta)_p \le \sum_{j=1}^{n} \omega_j(f;\delta^{\alpha/(n\alpha_j)}),$$

which yields that for any $f \in B_{p,q}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$ the right-hand side in (7.12) is finite. However, the infimum in (4.12) is not necessarily attained for the values δ_j of the form $\delta_j = \delta^{\beta_j}$. Therefore, in contrast to one-dimensional case, inequality (7.12) is not equivalent to embeddings of Besov spaces. The second (and the most important) difference is that (7.12) is sharp only under additional conditions on $\omega_k(f;\delta)_p$. That is, in general, the average modulus of continuity cannot give a completely correct description of the behaviour of a function.

Sharp estimates of L^q-norm of a function $f \in L^p(\mathbb{R}^n)$ in terms of its partial moduli of continuity were obtained in our work [32]. It was a solution of the problem posed by Ul'yanov [64]: find necessary and sufficient conditions for the embedding

$$\mathcal{H}_p^{\omega_1,\dots,\omega_n} \subset L^q \quad (1 \le p < q < \infty).$$

Here $\omega_k(\delta)$ are the given moduli of continuity and $\mathcal{H}_p^{\omega_1,\dots,\omega_n}$ is the class of all functions $f \in L^p(\mathbb{R}^n)$ such that

$$\omega_k(f;\delta)_p \le c\omega_k(\delta) \quad \text{for all } \delta \ge 0 \quad (k = 1,\dots,n).$$

Later on, Netrusov [51], [52] extended these results to the moduli of continuity of higher orders. However, his methods do not work for $p = 1$. In the latter case the problem is solved only for the first order moduli of continuity [32]. Observe that the proofs in [32] and [52] are long and complicated (neither the formulations of the results are simple). Therefore, we think that it is necessary to look for simpler approaches. A natural way would be to find sharp estimates of the rearrangement f^* in terms of partial moduli of continuity (an anisotropic analogue of the inequality (3.13)). However, as we have already mentioned above, such estimates are unknown.

In the next section we consider an important special case of the classes $\mathcal{H}_p^{\omega_1,\dots,\omega_n}$.

8. Lipschitz classes

We shall discuss the problem of embedding with the limiting exponent for Lipschitz classes.

For any $\alpha_j \in (0,1]$ and $1 \le p \le \infty$, we have the following embeddings

$$L_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n) \subset \Lambda_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n) \subset H_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n). \tag{8.1}$$

If $1 \le p \le \infty$, then the right embedding in (8.1) becomes equality if and only if $0 < \alpha_j < 1$ for all $j = 1,\dots,n$. In the left embedding the equality takes place if and only if $1 < p \le \infty$ and $\alpha_j = 1$, $j = 1,\dots,n$.

Let $n \ge 2$. Set

$$\alpha \equiv n\left(\sum_{j=1}^{n} \frac{1}{\alpha_j}\right)^{-1}.$$

Assume that $1 \le p < \infty$ and $\alpha < n/p$. Let $q^* = np/(n-\alpha p)$. Then

$$L_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \text{for all } p < q \le q^*$$

and

$$H_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \text{for all } p < q < q^*,$$

but for $q = q^*$ the latter embedding does not hold. The problem arises: what can be said about the embedding

$$\Lambda_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n) \subset L^{q^*}(\mathbb{R}^n)?$$

The solution of this problem was obtained in [32]:

Theorem 8.1. *Let $1 \le p < \infty$, $0 < \alpha_j \le 1$, and*

$$\alpha \equiv n\left(\sum_{j=1}^{n} \frac{1}{\alpha_j}\right)^{-1} < \frac{n}{p}.$$

Let $q^* = np/(n - \alpha p)$. Let ν be the number of α_j that are equal to 1. *The embedding*

$$\Lambda_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n) \subset L^{q^*}(\mathbb{R}^n) \tag{8.2}$$

holds if and only if

$$\nu \ge \frac{n}{\alpha} - p.$$

Remark 8.2. It follows that, in contrast to the Sobolev-Liouville and Nikol'skii spaces, the embedding $\Lambda_p^{\alpha_1,\dots,\alpha_n} \subset L^q$ is not uniquely determined by the value of the harmonic mean α. Roughly speaking, this means that for the spaces $\Lambda_p^{\alpha_1,\dots,\alpha_n}$ the contribution of the variable x_k is not proportional to $1/\alpha_k$.

Theorem 8.1 has been proved in various ways, but there are no simple proofs. Observe that this theorem cannot be derived from the estimate (7.11). Indeed, if $\omega_k(f;t)_p = O(t^{\alpha_k})$, then $\overline{\omega}(f;t)_p = O(t^{\alpha/n})$ and (7.11) gives only the weak estimate $f^*(t) = O(t^{-1/q^*})$.

Netrusov [52] extended Theorem 8.1 to arbitrary values of $\alpha_k > 0$. Moreover, he proved a theorem on embedding of $\Lambda_p^{\alpha_1,\dots,\alpha_n}$ into Lorentz spaces. He proposed another approach based on a modification of the method of integral representations. However, his proof was rather long and complicated, and it did not work for $p = 1$. Applying rearrangements, we proved these results in [38] in a different way, including the case of $p = 1$.

Theorem 8.3. *Let $n \ge 2$ and $\alpha_j > 0$ $(j = 1, \dots, n)$. Let*

$$\alpha = n\left(\sum_{j=1}^n \frac{1}{\alpha_j}\right)^{-1}, \quad 1 \le p < \frac{n}{\alpha} \quad \textit{and} \quad q^* = \frac{np}{n - \alpha p}.$$

Assume that there is an integer among the numbers α_j. Let

$$\alpha' = \left(\sum_{j:\alpha_j\in\mathbb{N}} \frac{1}{\alpha_j}\right)^{-1} \quad \textit{and} \quad s = \frac{n\alpha' p}{\alpha}.$$

Then for every function $f \in \Lambda_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$,

$$\|f\|_{q^*,s} \le c \sum_{j=1}^n \|f\|_{l_{p;j}^{\alpha_j}}.$$

Netrusov also proved that the index s in this theorem cannot be replaced by a smaller one. Note that for a given value of the mean index α, the bigger is the number of the integers among α_j the smaller is the index s. If there are no integers α_j at all, then $s=\infty$. In the other extreme case, if all α_j are integers, we have $s=p$ and Theorem 8.3 coincides with the embedding theorem with the limiting exponent for anisotropic Sobolev spaces $W_p^{\alpha_1,\dots,\alpha_n}$ (cf. Theorem 7.3).

If $0<\alpha_j\le 1$ $(j=1,\dots,n)$, then $s=np/(\nu\alpha)$, where ν is the number of α_j that are equal to 1 (we note that in this case Theorem 8.3 is closely related to Theorem 4.9. We have $s\le q^*$ if and only if $\nu\ge n/\alpha-p$. This is exactly the necessary and sufficient condition for the embedding (8.2) (see Theorem 8.1).

The question arises: how do these results relate to embeddings of Nikol'-skii-Besov spaces? We consider this question for $0<\alpha_j\le 1$. First, we prove the following new (unpublished yet) theorem.

Theorem 8.4. *Let $1\le p<\infty$, $p\le\theta_j\le\infty$, and $0<\beta_j<1$ $(j=1,\dots,n)$. Set*

$$\beta=n\left(\sum_{j=1}^n\frac{1}{\beta_j}\right)^{-1},\quad \theta=\frac{n}{\beta}\left(\sum_{j=1}^n\frac{1}{\beta_j\theta_j}\right)^{-1}.$$

Assume that $1\le p<n/\beta$. Let $q=np/(n-\beta p)$. Then for any function

$$f\in\bigcap_{j=1}^n B^{\beta_j}_{p,\theta_j;j}(\mathbb{R}^n)$$

the estimate

$$\|f\|_{\mathcal{L}^{q,\theta}}\le c\prod_{j=1}^n\left[(1-\beta_j)^{1/\theta_j}\|f\|_{b^{\beta_j}_{p,\theta_j;j}}\right]^{\beta/(n\beta_j)} \tag{8.3}$$

holds, where $c=c_0(4n)^q$ and c_0 is an absolute constant.

Proof. Let $\sigma=1,\dots,n$. Denote $F(t)=\mathcal{R}_\sigma f(t)$, $t\in\mathbb{R}^n_+$. We may assume that f is a continuous function with compact support and that all $\theta_j<\infty$. Then

$$I\equiv\left(\int_{\mathbb{R}^n_+}\prod_{k=1}^n t_k^{\theta/q-1}F(t)^\theta\,dt\right)^{1/\theta}<\infty.$$

Set $r=[q(2+\log_2 n)]+1$ and denote

$$A_\nu=\{t\in\mathbb{R}^n_+ : F(t)\le 2F(2^r t_\nu,\widehat{t_\nu})\}\quad(\nu=1,\dots,n).$$

Set also

$$E = \mathbb{R}_+^n \setminus \Big(\bigcup_{\nu=1}^n A_\nu\Big).$$

Then

$$F(t) \le 2\prod_{k=1}^n \varphi_k(t)^{\beta/(n\beta_k)} \quad \text{for all } t\in E, \tag{8.4}$$

where

$$\varphi_k(t) = F(t_k,\widehat{t}_k) - F(2^r t_k, \widehat{t}_k).$$

On the other hand,

$$\int_{A_\nu} \prod_{k=1}^n t_k^{\theta/q-1} F(t)^\theta \, dt \le 2^{\theta(1-r/q)} I^\theta.$$

This implies that

$$\int_E \prod_{k=1}^n t_k^{\theta/q-1} F(t)^\theta \, dt \ge (1 - 2^{\theta(1-r/q)} n) I^\theta \ge \frac{1}{2} I^\theta.$$

Using this estimate and (8.4), we obtain

$$I \le \left(2\int_{\mathbb{R}_+^n} \prod_{k=1}^n \Big[t_k^{\theta/q-1}(t)\varphi_k(t)^{\beta\theta/(n\beta_k)}\Big]\, dt\right)^{1/\theta} < \infty. \tag{8.5}$$

Set

$$\nu_k = \left(\frac{\theta_k}{p} - \theta_k\beta_k - 1\right)\frac{\theta\beta}{n\theta_k\beta_k}, \quad \mu_k = \left(\frac{\theta_k}{p} - 1\right)\frac{\theta\beta}{n\theta_k\beta_k}.$$

Then

$$\nu_k + \sum_{j\ne k}\mu_j = \frac{\theta}{q} - 1 \quad (k = 1,\dots,n).$$

Applying Hölder's inequality with the exponents $n\theta_k\beta_k/(\theta\beta)$ in (8.5), we obtain

$$I \le 2\prod_{k=1}^n I_k, \tag{8.6}$$

where

$$I_k = \left(\int_{\mathbb{R}_+^n} t_k^{-\theta_k\beta_k}\left(\prod_{j=1}^n t_j\right)^{\theta_k/p-1} \varphi_k(t)^{\theta_k}\, dt\right)^{\beta/(n\theta_k\beta_k)}.$$

We have

$$\varphi_k(t) = \sum_{i=0}^{r-1} \big[F(2^i t_k, \widehat{t_k}) - F(2^{i+1} t_k, \widehat{t_k})\big].$$

Thus,

$$I_k \le 2^{r\beta/(n\beta_k)} J_k, \tag{8.7}$$

where

$$J_k = \left(\int_{\mathbb{R}^n_+} t_k^{-\theta_k \beta_k} \left(\prod_{j=1}^n t_j \right)^{\theta_k/p - 1} \psi_k(t)^{\theta_k}\, dt \right)^{\beta/(n\theta_k\beta_k)}$$

and

$$\psi_k(t) = F(t_k, \widehat{t_k}) - F(2t_k, \widehat{t_k}).$$

By Fubini's theorem,

$$\int_{\mathbb{R}_+} t_k^{\theta_k/p - \theta_k\beta_k - 1} \psi_k(t)^{\theta_k}\, dt_k$$
$$= (1-\beta_k)\theta_k \int_{\mathbb{R}_+} h^{\theta_k(1-\beta_k)-1} \int_h^\infty t_k^{\theta_k/p-1} \left(\frac{\psi_k(t)}{t_k} \right)^{\theta_k} dt_k dh.$$

Hence,

$$J_k^{n\theta_k\beta_k/\beta} = (1-\beta_k)\theta_k \int_{\mathbb{R}_+} h^{\theta_k(1-\beta_k)-1} Q_k(h)\, dh, \tag{8.8}$$

where

$$Q_k(h) = \int_{\mathbb{R}^{n-1}_+} \int_h^\infty t_k^{\theta_k/p-1} \left(\frac{\psi_k(t)}{t_k} \right)^{\theta_k} dt_k d\widehat{t_k}.$$

Observe that

$$\psi_k(t) \le \left(\prod_{j=1}^n t_j \right)^{-1} \int_0^{t_1} \cdots \int_0^{t_n} [F(v) - F(v + 2t_k e_k)]\, dv$$
$$\le 2 \left(\prod_{j=1}^n t_j \right)^{-1/p} \omega_k(F; 2t_k)_p.$$

Thus, by (3.7),

$$Q_k(h) \le 4^{\theta_k - p} \left(\frac{\omega_k(F;h)_p}{h} \right)^{\theta_k - p} \int_{\mathbb{R}^{n-1}_+} \int_h^\infty \left(\frac{\psi_k(t)}{t_k} \right)^p dt_k d\widehat{t_k}.$$

Applying Lemma 5.7 and Theorem 5.2, we obtain

$$Q_k(h) \le 4^{\theta_k+p}\left(\frac{\omega_k(F;h)_p}{h}\right)^{\theta_k} \le c^{\theta_k}\left(\frac{\omega_k(f;h)_p}{h}\right)^{\theta_k},$$

where c is an absolute constant. This estimate and (8.8) imply that

$$J_k \le 2c\left((1-\beta_k)\int_0^\infty \left[h^{-\beta_k}\omega_k(f;h)_p\right]^{\theta_k}\frac{dh}{h}\right)^{\beta/(n\theta_k\beta_k)}.$$

From here, using (8.7) and (8.6), we obtain (8.3). □

Observe that the dependence of the constant c in (8.3) on q, $c = c_0(4n)^q$, certainly is not optimal.

The left-hand side in (8.3) contains the special Lorentz norm $\|f\|_{\mathcal{L}^{q,\theta}}$. We have also a similar theorem in terms of the usual Lorentz norm.

Theorem 8.5. *Let $1 \le p < \infty$, $p \le \theta_j \le \infty$, and $0 < \beta_j < 1$ $(j = 1, \dots, n)$. Set*

$$\beta = n\left(\sum_{j=1}^n \frac{1}{\beta_j}\right)^{-1}, \quad \theta = \frac{n}{\beta}\left(\sum_{j=1}^n \frac{1}{\beta_j\theta_j}\right)^{-1}.$$

Assume that $1 \le p < n/\beta$. Let $q = np/(n-\beta p)$. Then any function

$$f \in \bigcap_{j=1}^n B^{\beta_j}_{p,\theta_j;j}(\mathbb{R}^n)$$

satisfies

$$\|f\|_{q,\theta} \le c\prod_{j=1}^n\left[(1-\beta_j)^{1/\theta_j}\|f\|_{b^{\beta_j}_{p,\theta_j;j}}\right]^{\beta/(n\beta_j)}, \tag{8.9}$$

where $c = c_0(4n)^q$ and c_0 is an absolute constant.

We shall not give here a complete proof of this theorem. If $\theta \le q$, then, by (5.1), $\|f\|_{q,\theta} \le c\|f\|_{\mathcal{L}^{q,\theta}}$ and Theorem 8.5 follows from Theorem 8.4. In the case $\theta > q$ the relation between these norms is opposite (see Theorem 5.3). In this case we apply a different approach.

Observe that the inequality (8.9) without factors $(1-\beta_j)^{1/\theta_j}$ can be readily derived from the estimate (7.11).

Assume that $0 < \alpha_j \le 1$ $(j = 1, \dots, n)$. We shall show that in this case Theorem 8.3 can be obtained as a limiting case of Theorem 8.5. More exactly, we apply Theorem 8.5 to derive the inequality

$$\|f\|_{q^*,s} \le c \prod_{k=1}^{n} \|f\|_{l^{\alpha_k}_{p;k}}^{\alpha/(n\alpha_k)}, \tag{8.10}$$

where $\alpha < n/p$, $q^* = np/(n - \alpha p)$, ν is the number of α_j that are equal to 1, and $s = np/(\nu\alpha)$.

If $\alpha_k < 1$, we take $\beta_k = \alpha_k$, $\theta_k = \infty$. Set $\sigma = \{k : \alpha_k = 1\}$. If $k \in \sigma$, take $\beta_k < 1$, $\theta_k = p$. Then

$$\theta = \frac{np}{\beta}\left(\sum_{k\in\sigma} \frac{1}{\beta_k}\right)^{-1}.$$

Assume now that $\beta_k \to 1$ for each $k \in \sigma$. Then $\beta \to \alpha$, $q \to q^*$, and $\theta \to s = np/(\nu\alpha)$. Hence, the left-hand side in (8.9) tends to $\|f\|_{q^*,s}$. For the corresponding terms in the right-hand side of (8.9), we have, by Lemma 6.2,

$$(1 - \beta_k)^{1/p} \|f\|_{b^{\beta_k}_{p;k}} \to \left(\frac{1}{p}\right)^{1/p} \|f\|_{l^1_{p;k}}.$$

Thus, we obtain (8.10).

We noted already in [38] that the spaces $\Lambda_p^{\alpha_1,\dots,\alpha_n}$ are not yet explored in a satisfactory manner. The study of these spaces requires specific methods. Besides the works cited above, we mention also the paper by Pérez [59] in which an interesting unified approach to the spaces $\Lambda_p^{\alpha_1,\dots,\alpha_n}$ has been developed.

9. Mixed norms

In this section we consider an approach to the Sobolev-type inequalities based on estimates of certain mixed norms. This approach originates in the works due to Gagliardo [22] and Fournier [21].

We have already mentioned in Section 3 that Sobolev's inequality (3.4) with $p = 1$ was proved in 1958 independently by Gagliardo and Nirenberg. The central part of Gagliardo's proof was the following lemma.

Lemma 9.1. *Let $n \ge 2$. Assume that $g_k \in L^1(\mathbb{R}^{n-1})$ $(k = 1, \dots, n)$ are non-negative functions on $\mathbb{R}^{n-1}$. Then*

$$\int_{\mathbb{R}^n} \left(\prod_{k=1}^{n} g_k(\widehat{x}_k)\right)^{1/(n-1)} dx \le \left(\prod_{k=1}^{n} \int_{\mathbb{R}^{n-1}} g_k(\widehat{x}_k)\, d\widehat{x}_k\right)^{1/(n-1)}. \tag{9.1}$$

Assume now that $f \in W_1^1(\mathbb{R}^n)$. Then for almost all $x \in \mathbb{R}^n$ and every $k = 1, \dots, n$,

$$|f(x)| \le \frac{1}{2} \int_{\mathbb{R}} |D_k f(x)|\, dx_k \equiv \frac{1}{2} g_k(\widehat{x}_k).$$

Thus, applying (9.1), we immediately obtain the inequality

$$\|f\|_{n/(n-1)} \le \frac{1}{2} \left(\prod_{k=1}^{n} \|D_k f\|_1 \right)^{1/n}. \tag{9.2}$$

This yields (3.4) with $p = 1$.

However, a stronger statement can be derived from (9.1). Let

$$V_k \equiv L^1_{\widehat{x}_k}(\mathbb{R}^{n-1})[L^\infty_{x_k}(\mathbb{R})] \quad (1 \le k \le n)$$

be the space with the mixed norm

$$\|f\|_{V_k} \equiv \int_{\mathbb{R}^{n-1}} \varphi_k(\widehat{x}_k)\, d\widehat{x}_k,$$

where

$$\varphi_k(\widehat{x}_k) = \operatorname*{ess\,sup}_{x_k \in \mathbb{R}} |f(x)|.$$

Gagliardo's lemma immediately implies the following theorem.

Theorem 9.2. *Assume that $f \in \bigcap_{k=1}^n V_k$, $n \ge 2$. Then $f \in L^{n/(n-1)}(\mathbb{R}^n)$ and*

$$\|f\|_{n/(n-1)} \le \left(\prod_{k=1}^{n} \|f\|_{V_k} \right)^{1/n}.$$

Since

$$\|f\|_{V_k} \le \frac{1}{2} \|D_k f\|_1 \quad (k = 1, \dots, n) \tag{9.3}$$

for $f \in W_1^1(\mathbb{R}^n)$, then (9.2) follows from Theorem 9.2.

As we know, the left-hand side in (3.4) can be replaced by the stronger Lorentz $L^{q^*,p}$-norm (see (3.5)). To prove (3.5) for $p = 1$, Fournier [21] applied the following refinement of the Theorem 9.2.

Theorem 9.3. *Assume that* $f \in \bigcap_{k=1}^{n} V_k$, $n \geq 2$. *Then* $f \in L^{n/(n-1),1}(\mathbb{R}^n)$ *and*

$$\|f\|_{n/(n-1),1} \leq \left(\prod_{k=1}^{n} \|f\|_{V_k} \right)^{1/n}. \tag{9.4}$$

Taking into account (9.3), we immediately obtain (3.5) with $p = 1$. More exactly,

$$\|f\|_{n/(n-1),1} \leq \frac{1}{2} \left(\prod_{k=1}^{n} \|D_k f\|_1 \right)^{1/n}. \tag{9.5}$$

We see that the inequality (9.5) (as well as (9.2)) can be broken down into two successive steps. The main step is the inequality (9.4). To derive (9.5) from (9.4), one has only to apply the following simple fact: if a function $f \in L^1(\mathbb{R}^n)$ has a weak derivative $D_k f \in L^1(\mathbb{R}^n)$, then $f \in V_k$ (see (9.3)).

Fournier [21, p. 66] observed that it is not clear to what extent the methods of his paper can be applied to obtain the inequality (3.5) in the case $1 < p < n$. We studied this question in [41]. One of the main problems in this work was to find an analogue of Theorem 9.3 for more general mixed norm spaces. To clarify this problem, we can consider the following example. Let $n = 2$ and $1 \leq r < \infty$. Assume that

$$f \in L^1_y(\mathbb{R})[L^r_x(\mathbb{R})] \quad \text{and} \quad f \in L^1_x(\mathbb{R})[L^r_y(\mathbb{R})].$$

Which Lorentz space does the function f belong to?

First of all, we studied mixed norm spaces related to the Sobolev spaces W^1_p and inequality (3.5) for *arbitrary* $1 \leq p < n$. We realized that if $D_k f \in L^1(\mathbb{R}^n)$, then $f \in V_k \equiv L^1_{\widehat{x}_k}[L^\infty_{x_k}]$. Suppose now that $D_k f \in L^p(\mathbb{R}^n)$ for some $p > 1$; what is the corresponding space V_k in this case? A similar question arises if a function f belongs to a Besov space with respect to a separate variable x_k. In turn, this question is related to embeddings of anisotropic Besov spaces.

Studying these problems, we introduce a scale of generalized spaces with mixed norms similar to the spaces V_k. In particular, the spaces

$$L^p(\mathbb{R}^{n-1})[L^{r,\infty}(\mathbb{R})] \quad (1 \leq p, r < \infty)$$

are contained in this scale.* First we define the "weak" spaces Λ^σ.

*$L^{r,\infty}$ is the space of all measurable functions f such that $\sup_{t>0} t^{1/r} f^*(t) < \infty$.

Let $\sigma \in \mathbb{R}$. Denote by $\Lambda^\sigma(\mathbb{R})$ the space of all measurable functions f such that

$$\|f\|_{\Lambda^\sigma} \equiv \sup_{t>0} t^\sigma[f^*(t) - f^*(2t)] < \infty. \tag{9.6}$$

If $0 < \sigma < \infty$ and $r = 1/\sigma$, then $\Lambda^\sigma(\mathbb{R}) = L^{r,\infty}(\mathbb{R})$. If $\sigma = 0$, then Λ^σ coincides with the space weak-L^∞ introduced in [3]. If $\sigma < 0$, then (9.6) is a weak version of Lipschitz condition for the rearrangement f^*.

The main result in [41] is the following theorem.

Theorem 9.4. *Assume that $1 \le p < \infty$, $n \ge 2$ $(n \in \mathbb{N})$ and that α_k $(k = 1, \dots, n)$ are positive numbers such that*

$$\alpha \equiv n\left(\sum_{k=1}^n \frac{1}{\alpha_k}\right)^{-1} \le \frac{n}{p}.$$

Let

$$\sigma_k = \frac{1}{p} - \alpha_k, \qquad V_k \equiv L^p_{\widehat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{\sigma_k}_{x_k}(\mathbb{R})]$$

and

$$q = \begin{cases} \dfrac{np}{n - \alpha p} & \text{if } \alpha < \dfrac{n}{p}, \\ \infty & \text{if } \alpha = \dfrac{n}{p}. \end{cases}$$

Suppose that

$$f \in S_0(\mathbb{R}^n) \quad \text{and} \quad f \in \bigcap_{k=1}^n V_k.$$

Then $f \in L^{q,p}(\mathbb{R}^n)$ and

$$\|f\|^*_{q,p} \le c \prod_{k=1}^n \|f\|_{V_k}^{\alpha/(n\alpha_k)}, \tag{9.7}$$

where

$$c = c_n\left(\prod_{k=1}^n (n\alpha_k - \alpha)^{\alpha/(n\alpha_k)}\right)^{-1/p} \tag{9.8}$$

and c_n is a constant depending only on n.

Recall that the modified Lorentz norm $\|\cdot\|^*_{p,r}$ is defined by (2.10) and (2.12). Note that the case $\alpha = n/p$ also is included.

Remark 9.5. If at least one of the numbers α_k tends to 0, then the constant c in (9.7) tends to infinity. We show that the order of growth of this constant given by (9.8) is optimal.

We can now give the answer to a specific problem stated above.

Example 9.6. Let $n = 2$ and $1 \le r \le \infty$. Let

$$f \in L^1_y[L^r_x] \quad \text{and} \quad f \in L^1_x[L^r_y].$$

Applying Theorem 9.4, we obtain that $f \in L^{q,1}(\mathbb{R}^2)$, where $q = 2r/(r+1)$. In the case $r = \infty$ this result coincides with Fournier's theorem.

Remark 9.7. Consider the case when $\alpha_k = 1$, $k = 1, \dots, n$, in Theorem 9.4.

If $p = 1$, then $\sigma_k = 0$ $(k = 1, \dots, n)$, $q = n/(n-1)$, and

$$V_k = L^1_{\widehat{x}_k}[\text{weak-}L^\infty_{x_k}]$$

From Theorem 9.4 we have

$$\|f\|_{n/(n-1),1} \le c \left(\prod_{k=1}^{n} \|f\|_{V_k} \right)^{1/n}.$$

This inequality is slightly stronger than Fournier's inequality (9.4). Indeed, the right-hand side of (9.4) contains the norms in the spaces $L^1_{\widehat{x}_k}[L^\infty_{x_k}]$. We have proved that the interior $L^\infty_{x_k}$-norms can be replaced by weaker norms of the weak-$L^\infty_{x_k}$.

If $1 < p \le n$, then $\sigma_k = 1/p - 1$ $(k = 1, \dots, n)$ and $V_k = L^p_{\widehat{x}_k}[\Lambda^{1/p-1}_{x_k}]$. In this case Theorem 9.4 asserts that

$$\|f\|_{q,p} \le c \left(\prod_{k=1}^{n} \|f\|_{V_k} \right)^{1/n}, \quad \text{where } q = \frac{np}{n-p}.$$

If $p = n$, then $q = \infty$ and we have the norm in $L^{\infty,n}(\mathbb{R}^n)$ at the left-hand side.

It is easy to see that these results are closely related to Sobolev-type inequalities (3.5) and (3.6). Indeed, applying Lemma 3.1, we obtain the following proposition.

Proposition 9.8. *Let $k \in \{1, \dots, n\}$ and $1 \le p < \infty$. Assume that $f \in L^p(\mathbb{R}^n)$ and that f has the weak partial derivative $D_k f \in L^p(\mathbb{R}^n)$.*

Then $f \in V_k \equiv L^p_{\widehat{x}_k}[\Lambda^{1/p-1}_{x_k}]$ *and*

$$\|f\|_{V_k} \le 4\|D_k f\|_p.$$

Recall that

$$W^1_p(\mathbb{R}^n) \hookrightarrow L^{q^*,p}(\mathbb{R}^n) \quad \left(1 \le p \le n,\ q^* = \frac{np}{n-p}\right) \tag{9.9}$$

(see (3.5) and (3.6)). At the same time, by Theorem 9.4,

$$\bigcap_{k=1}^{n} V_k \hookrightarrow L^{q^*,p}(\mathbb{R}^n) \tag{9.10}$$

and by Proposition 9.8,

$$W^1_p(\mathbb{R}^n) \hookrightarrow \bigcap_{k=1}^{n} V_k. \tag{9.11}$$

Thus, we can split (9.9) into two embeddings (9.10) and (9.11). Clearly, (9.10) is the main part of (9.9).

Theorem 9.4 can be also applied to the study of estimates involving certain Besov norms. Namely, consider inequality (7.2) for $0 < \alpha_j < 1$ $(j = 1, \dots, n)$ and $\theta = p$, i.e.,

$$\|f\|_{q,p} \le c\|f\|_{b^{\alpha_1,\dots,\alpha_n}_p}, \quad q = \frac{np}{n-\alpha p}. \tag{9.12}$$

The sharp asymptotics of the constant c in (9.12) as some of the numbers α_k tend to 1 is contained as a special case in Theorem 8.5. However, we obtain an alternative proof of this result, applying Theorem 9.4 and the following proposition [41].

Proposition 9.9. *Let* $0 < \alpha < 1$, $1 \le p < \infty$ *and* $1 \le k \le n$ $(n \ge 2)$. *Assume that* $f \in B^\alpha_{p;k}(\mathbb{R}^n)$. *Then* $f \in V_k \equiv L^p_{\widehat{x}_k}[\Lambda^{1/p-\alpha}_{x_k}]$ *and*

$$\|f\|_{V_k} \le 100[\alpha(1-\alpha)]^{1/p}\|f\|_{b^\alpha_{p;k}}.$$

Theorem 9.4 and Proposition 9.9 immediately imply the following result [41].

Theorem 9.10. *Let* $1 \le p < \infty$, $n \ge 2$ $(n \in \mathbb{N})$ *and* $1/2 < \alpha_k < 1$ $(k = 1, \dots, n)$. *Assume that*

$$\alpha \equiv n \left(\sum_{k=1}^{n} \frac{1}{\alpha_k} \right)^{-1} \le \frac{n}{p}.$$

Let

$$q = \begin{cases} \dfrac{np}{n - \alpha p} & \text{if } \alpha < \dfrac{n}{p}, \\ \infty & \text{if } \alpha = \dfrac{n}{p}. \end{cases}$$

Then for every function $f \in B_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$ *we have* $f \in L^{q,p}(\mathbb{R}^n)$ *and*

$$\|f\|^*_{q,p} \le c \prod_{k=1}^{n} \left[(1-\alpha_k)^{1/p} \|f\|_{b^{\alpha_k}_{p;k}} \right]^{\alpha/(n\alpha_k)}, \tag{9.13}$$

where $c \equiv c_n$ *is a constant depending only on* n.

If $\alpha < n/p$, then, by (2.11) and (9.13), we obtain the inequality

$$\|f\|_{q,p} \le q c_n \prod_{k=1}^{n} \left[(1-\alpha_k)^{1/p} \|f\|_{b^{\alpha_k}_{p;k}} \right]^{\alpha/(n\alpha_k)}, \tag{9.14}$$

where c_n is a constant depending only on n. By (2.6), it follows from (9.14) that

$$\|f\|_{q} \le q^{1-1/p} c_n \prod_{k=1}^{n} \left[(1-\alpha_k)^{1/p} \|f\|_{b^{\alpha_k}_{p;k}} \right]^{\alpha/(n\alpha_k)}. \tag{9.15}$$

Thus, (9.15) implies (7.7).

Assume that for some k there exists a weak derivative $D_k f \in L^p(\mathbb{R}^N)$. Then, by (6.4), for the corresponding term in (9.13) we have

$$(1-\alpha_k)^{1/p} \|f\|_{b^{\alpha_k}_{p;k}} \to \left(\frac{1}{p} \right)^{1/p} \|D_k f\|_p, \quad \text{as } \alpha_k \to 1.$$

Theorem 8.5 shows that (similarly to (9.9)) the embedding

$$B_p^{\alpha_1,\dots,\alpha_k}(\mathbb{R}^n) \hookrightarrow L^{q,p}(\mathbb{R}^n)$$

can be split into two parts. The main part is contained in Theorem 9.4. The factors $(1-\alpha_k)^{\alpha/(pn\alpha_k)}$ in (9.13) appear when we apply Proposition 9.9 (i.e., in the "easy" part of (9.13)). Observe that this approach gives us an alternative explanation of the phenomenon related to these factors.

10. Estimates of moduli of continuity

The problem of estimating the moduli of continuity of a function in L^q in terms of its moduli of continuity in L^p $(1 \le p < q \le \infty)$ has a long history. It emerged with the study of embeddings of Lipschitz classes (E. Titchmarsh, G. H. Hardy and J. E. Littlewood, and S. M. Nikol'skii). Many authors have devoted papers to subsequent investigations of the problem (see [5, § 16], [34], [65]). The following result was obtained in [34].

Theorem 10.1. *Let either* $1 < p < \infty$ *and* $n \ge 1$ *or* $p = 1$ *and* $n \ge 2$. *Suppose that* $f \in L^p(\mathbb{R}^n)$, $p < q < \infty$, *and* $\gamma \equiv n(1/p - 1/q) < 1$. *Then for every* $\delta > 0$,

$$\left(\int_\delta^\infty \left[t^{\gamma-1}\omega(f;t)_q\right]^p \frac{dt}{t}\right)^{1/p} \le c\delta^{\gamma-1}\left(\int_0^\delta \left[t^{-\gamma}\omega(f;t)_p\right]^q \frac{dt}{t}\right)^{1/q}. \tag{10.1}$$

It was also proved in [34] that this theorem is sharp. Namely, let $1 \le p < \infty$, $n \ge 1$, and let $\omega(\delta)$ be a modulus of continuity. Then there exists a function $f \in L^p(\mathbb{R}^n)$ such that $\omega(f;\delta)_p \le \omega(\delta)$ and for every $q \in (p, \infty)$ with $\gamma \equiv n(1/p - 1/q) < 1$ and *every* $\delta > 0$,

$$\left(\int_\delta^\infty \left[t^{\gamma-1}\omega(f;t)_q\right]^p \frac{dt}{t}\right)^{1/p} \ge c\delta^{\gamma-1}\left(\int_0^\delta \left[t^{-\gamma}\omega(t)\right]^q \frac{dt}{t}\right)^{1/q},$$

where $c = c(p,q,n) > 0$.

Furthermore, it was shown in [42] that Theorem 10.1 yields the optimal constant in the different norm inequality for Besov spaces (in the spirit of Theorem 7.5).

Theorem 10.2. *Let* $0 < \alpha < 1$ *and* $p < q < \infty$. *Assume that*

$$\gamma \equiv n\left(\frac{1}{p} - \frac{1}{q}\right) < \alpha$$

and $1 \le \theta < \infty$. *If either* $p > 1$, $n \ge 1$ *or* $p \ge 1$, $n \ge 2$, *then for any* $f \in B^\alpha_{p,\theta}(\mathbb{R}^n)$,

$$\|f\|_{b^{\alpha-\gamma}_{q,\theta}} \le A\frac{(1-s)^{1/\theta^*}}{(\alpha-\gamma)^{1/\theta}}\|f\|_{b^\alpha_{p,\theta}},$$

where $\theta^* = \max\{p,\theta\}$ *and the constant* A *does not depend on* α *and* f.

This assertion does not hold for $p = n = 1$. It was also shown that the exponent $1/\theta^*$ is sharp in a sense.

We return to Theorem 10.1. If f has all first-order generalized derivatives $D_j f \in L^p(\mathbb{R}^n)$ $(j = 1, \dots, n)$, then, by (10.1) and (3.10),

$$\left(\int_0^\infty \left[t^{\gamma-1}\omega(f;t)_q\right]^p \frac{dt}{t}\right)^{1/p} \le c\sum_{k=1}^n \|D_k f\|_p \tag{10.2}$$

(for $p > 1$ and $n \ge 1$ this inequality was proved by Il'in [5, §18]; if $p = n = 1$, then (10.2) fails to hold).

Suppose now that f has a partial derivative $D_j f \in L^p(\mathbb{R}^n)$ with respect to only a single variable x_j. The problem is to estimate the partial moduli of continuity $\omega_j(f;\delta)_q$ with respect to the same variable in L^q, $q > p$. It is clear that for $n \ge 2$ this problem cannot be solved without additional conditions on f. However, it is not necessary to assume smoothness with respect to the other variables. In many cases it suffices to assume, in addition, that f belongs to some space L^r. These conditions lead naturally to multiplicative inequalities of Gagliardo–Nirenberg type. A more general problem of estimating of $\omega_j(f;\delta)_q$ in terms of $\omega_j(f;\delta)_p$ and the norm of f in some L^r also leads to similar inequalities.

Multiplicative inequalities of Gagliardo–Nirenberg type ([5, §15]) are closely related to the Sobolev inequality. As we have seen, the exact integrability exponents for functions in Sobolev spaces are expressed in terms of the Lorentz spaces $L^{q,p}$. Therefore, we study multiplicative inequalities for moduli of continuity in the scale of these spaces.

If $f \in L^{p,s}(\mathbb{R}^n)$, then the function

$$\omega_j^r(f;\delta)_{p,s} = \sup_{0\le h\le\delta} \|\Delta_j^r(h)f\|_{p,s}$$

is called the *partial modulus of continuity* of order r of the function f with respect to the variable x_j in $L^{p,s}$. If $r = 1$, then we omit the superscript in this notation.

Let $1 \le p, s \le \infty$. An ordered pair (p, s) is said to be *admissible* if one of the following conditions holds: (i) $1 < p < \infty$, $1 \le s \le \infty$; (ii) $p = s = 1$; (iii) $p = s = \infty$. We set $L^{\infty,\infty} = L^\infty$.

First, we have the following theorem [40].

Theorem 10.3. *Let (p_0, s_0) and (p_1, s_1) be admissible pairs and let $p_1 > 1$. Let $0 < \theta < 1$ and let numbers p and s be defined by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{s} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}. \tag{10.3}$$

Suppose that a function $f \in L^{p_0,s_0}(\mathbb{R}^n) \cap S_0(\mathbb{R}^n)$ *has the weak derivative* $D_j^r f \in L^{p_1,s_1}(\mathbb{R}^n)$ *with respect to the variable* x_j, $1 \le j \le n$ $(n, r \in \mathbb{N})$. *Then*

$$\left(\int_0^\infty \left[h^{-\theta r}\omega_j^r(f;h)_{p,s}\right]^s \frac{dh}{h}\right)^{1/s} \le c\|f\|_{p_0,s_0}^{1-\theta}\|D_j^r f\|_{p_1,s_1}^\theta, \tag{10.4}$$

where $c = c_r(p')^{1/s'}(p_1')^\theta[\theta(1-\theta)]^{-1/s}$ *and the constant* c_r *depends only on* r.

The proof easily follows by the estimate

$$(\Delta_k^r(h)f)^*(t) \le \min\{2^r f^*(2^{-r}t), h^r(D_k^r f)^{**}(t)\} \tag{10.5}$$

(see (6.1)). Of course, this estimate does not work in the case $p_1 = s_1 = 1$. However, we prove that in this case the inequality (10.4) is still true if $p_0, s_0 < \infty$. This shows that the constant in (10.4) is not optimal and an alternative general approach should be found. Nevertheless, Theorem 10.3 fails if $p_1 = s_1 = 1$ and $p_0 = s_0 = \infty$.

We have the following corollaries of Theorem 10.3.

Corollary 10.4. *Suppose that a function* $f \in L^1(\mathbb{R}^n)$ *has all first-order weak derivatives and* $|\nabla f| \in L^\infty(\mathbb{R}^n)$. *Let* $0 < \theta < 1$ *and* $p = 1/(1-\theta)$. *Then*

$$\|f\|_{b_p^\theta} \le cn[(1-\theta)\theta]^{-1/p}\|f\|_1^{1-\theta}\|\nabla f\|_\infty^\theta, \tag{10.6}$$

where c *is an absolute constant.*

In particular, for $\theta = 1/2$ and $n = 1$ we have

$$\int_0^\infty \int_{\mathbb{R}} [f(x+h) - f(x)]^2\, dx \frac{dh}{h^2} \le c\|f\|_1\|f'\|_\infty. \tag{10.7}$$

This inequality was obtained by Kashin [28] (the proof presented in [28] is due to Besov). A special discrete version of the inequality (10.7) was proved earlier by Bochkarev [7].

Corollary 10.5. *Let* $0 < \theta < 1$, $1 < \nu < \infty$ *and* $p = \nu/\theta$. *Suppose that a function* $f \in L^\infty(\mathbb{R}^n)$ *has the weak derivative* $D_j^r f \in L^\nu(\mathbb{R}^n)$ $(r \in \mathbb{N})$. *Then*

$$\left(\int_0^\infty \left[t^{-\theta r}\omega_j^r(f;t)_p\right]^p \frac{dt}{t}\right)^{1/p} \le c_r(\nu')^\theta(1-\theta)^{-1/p}\|f\|_\infty^{1-\theta}\|D_j^r f\|_\nu^\theta. \tag{10.8}$$

Let us consider the case $r = 1$. By (10.8) and (4.11) we get that

$$\|f\|_{b^{\theta}_{\nu/\theta}} \le K\|f\|_{\infty}^{1-\theta}\|\nabla f\|_{\nu}^{\theta} \quad (1 < \nu < \infty), \tag{10.9}$$

where $K = cn(\nu')^{\theta}(1-\theta)^{-\theta/\nu}$ and c is an absolute constant. We note that inequality (10.9) follows from a more general result of RUNST [61] (see also the paper [12] by BREZIS and MIRONESCU). The authors admit in [12] that they do not know any elementary proof of (10.9) (without using the Littlewood-Paley theory). Such a proof was later obtained by MAZ'YA and SHAPOSHNIKOVA [48]. We see that inequalities (10.6) and (10.9) represent limit cases of Theorem 10.3 (for $r = 1$). We also note that the method of proving Theorem 10.3 differs from the methods used in [48].

Applying Theorem 10.3 and approximation by the generalized Steklov means, we obtain the following result [40].

Theorem 10.6. *Let (p_0, s_0) and (p_1, s_1) be admissible pairs and let $p_1 > 1$. Let $0 < \theta < 1$ and let numbers p and s be defined by (10.3). Suppose that $f \in L^{p_0,s_0}(\mathbb{R}^n) \cap L^{p_1,s_1}(\mathbb{R}^n)$ $(n \in \mathbb{N})$. Let $r \in \mathbb{N}$ and $1 \le j \le n$. Then*

$$\left(\int_{\delta}^{\infty} \left[t^{-\theta r}\omega_j^r(f;t)_{p,s}\right]^s \frac{dt}{t}\right)^{1/s} \le K\|f\|_{p_0,s_0}^{1-\theta}\left[\delta^{-r}\omega_j^r(f;\delta)_{p_1,s_1}\right]^{\theta}$$

for any $\delta > 0$, and

$$\left(\int_{0}^{\infty} \left[t^{-\theta\alpha}\omega_j^r(f;t)_{p,s}\right]^s \frac{dt}{t}\right)^{1/s} \le K'(r-\alpha)^{1/s}\|f\|_{p_0,s_0}^{1-\theta}\left(\int_0^{\infty} \left[t^{-\alpha}\omega_j^r(f;t)_{p_1,s_1}\right]^{s\theta} \frac{dt}{t}\right)^{1/s}$$

for any $0 < \alpha < r$, where $K = c_r(p')^{1/s'}(p_1')^{\theta}[(1-\theta)\theta]^{-1/s}$ and $K' = 2K\theta^{1/s}$.

Corollary 10.7. *Let $r \in \mathbb{N}$, $0 < \alpha < r$, $1 < \nu < \infty$, $0 < \theta < 1$ and $p = \nu/\theta$. Suppose that $f \in L^{\nu}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ $(n \in \mathbb{N})$. Then for any $1 \le j \le n$*

$$\left(\int_{0}^{\infty} \left[t^{-\theta\alpha}\omega_j^r(f;t)_{p}\right]^p \frac{dt}{t}\right)^{1/p} \le K(r-\alpha)^{1/p}\|f\|_{\infty}^{1-\theta}\left(\int_0^{\infty} \left[t^{-\alpha}\omega_j^r(f;t)_{\nu}\right]^{\nu} \frac{dt}{t}\right)^{\theta/\nu}, \tag{10.10}$$

where $K = c_r(\nu')^{\theta}(1-\theta)^{-1/p}$.

Let $r=1$ and $0<\alpha<1$. Applying (10.10) and (4.11), we obtain the inequality

$$\|f\|_{b^{\alpha\theta}_{\nu/\theta}} \le K(1-\alpha)^{\theta/\nu}\|f\|_\infty^{1-\theta}\|f\|_{b^\alpha_\nu}^\theta, \quad 1<\nu<\infty, \tag{10.11}$$

where $K=cn(\nu')^\theta(1-\theta)^{-\theta/\nu}$ and c is an absolute constant.

The inequality (10.11) was proved by MAZ'YA and SHAPOSHNIKOVA [48]. They also showed that the dependence of the constant K on the parameters is exact. We note that the relationship between the norms in (10.11) was obtained earlier (without establishing the exact order of the constant) by RUNST [61]. The problem of the exact constant was posed by BREZIS and MIRONESCU [12].

Theorems 10.3 and 10.6 do not hold for $p_1=1$ and $p_0=\infty$. In generall, the problem becomes much more complicated when $p_1=1$. In [40] this case was considered only for $r=1$ (although similar results hold for arbitrary order of derivatives and moduli of continuity); namely, the following theorem was proved.

Theorem 10.8. *Let $1<p_0<\infty$, $1\le s_0<\infty$ and $0<\theta<1$. Let*

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\theta, \quad \frac{1}{s}=\frac{1-\theta}{s_0}+\theta.$$

Assume that a function $f\in L^{p_0,s_0}(\mathbb{R}^n)$ $(n\in\mathbb{N})$ has a weak derivative $D_kf\in L^1(\mathbb{R}^n)$. Then

$$\left(\int_0^\infty \left[t^{-\theta r}\omega_k(f;t)_{p,s}\right]^s \frac{dt}{t}\right)^{1/s} \le c\|f\|_{p_0,s_0}^{1-\theta}\|D_kf\|_1^\theta\,,$$

where $c=c(p_0,s_0)[(1-\theta)\theta]^{-1/s}$.

Applying Theorem 10.8, we obtain that

$$\int_0^\infty h^{-\alpha}\omega_j(f;h)_{q,1}\frac{dh}{h} \le c\|f\|_{n/(n-1),1}^{1-\alpha}\|D_jf\|_1^\alpha, \tag{10.12}$$

where $\alpha=1-n(1-1/q)$. Let us compare inequalities (10.12) and (10.2) (for $p=1$). By (10.2),

$$\int_0^\infty h^{-\alpha}\omega_j(f;h)_q\frac{dh}{h} \le c\sum_{k=1}^n\|D_kf\|_1. \tag{10.13}$$

In this relation the partial modulus of continuity with respect to the variable x_j is estimated in terms of the norms of the derivatives with respect to *all* the variables. The inequality (10.12) gives a more exact result. Indeed, suppose that a function $f \in S_0(\mathbb{R}^n)$ $(n \geq 2)$ has all first-order weak derivatives $D_k f \in L^1(\mathbb{R}^n)$ $(k = 1, \dots, n)$. In this case, by (3.5),

$$\|f\|_{n/(n-1),1} \leq c \sum_{k=1}^{n} \|D_k f\|_1. \tag{10.14}$$

Thus, inequality (10.13) can be obtained by successive application of inequalities (10.12) and (10.14).

Of course, a similar situation occurs for inequality (10.2) when $1 \leq p < n$.

Theorem 10.8 yields also the following result.

Theorem 10.9. *Let* $1 < p_0 < \infty$, $1 \leq s_0 < \infty$, *and* $0 < \theta < 1$. *Let*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \theta, \qquad \frac{1}{s} = \frac{1-\theta}{s_0} + \theta.$$

Suppose that $f \in L^{p_0,s_0}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ $(n \in \mathbb{N})$ *and let* $1 \leq j \leq n$. *Then*

$$\left(\int_\delta^\infty \left[t^{-\theta} \omega_j(f;t)_{p,s} \right]^s \frac{dt}{t} \right)^{1/s} \leq K \|f\|_{p_0,s_0}^{1-\theta} \left[\frac{\omega_j(f;\delta)_1}{\delta} \right]^\theta \tag{10.15}$$

for any $\delta > 0$, *and*

$$\left(\int_0^\infty \left[t^{-\theta\alpha} \omega_j(f;t)_{p,s} \right]^s \frac{dt}{t} \right)^{1/s} \\ \leq K_1 (1-\alpha)^{1/s} \|f\|_{p_0,s_0}^{1-\theta} \left(\int_0^\infty \left[t^{-\alpha} \omega_j(f;t)_1 \right]^{s\theta} \frac{dt}{t} \right)^{1/s}$$

for any $0 < \alpha < 1$, *where* $K = c(p_0, s_0)[(1-\theta)\theta]^{-1/s}$ *and* $K_1 = 2K\theta^{1/s}$.

Let BV be the space of functions of bounded variation on $\mathbb{R}^n$ (see [69]). It is well known that $\|f\|_{BV}$ is equivalent to $\sup_{\delta>0} \omega(f;\delta)_1/\delta$. Using (10.15) and (4.11), we get the following result.

Corollary 10.10. *Let* $0 < \theta < 1$, $1 < p < \infty$ *and* $1/q = (1-\theta)/p + \theta$. *Suppose that* $f \in L^p(\mathbb{R}^n) \cap \mathrm{BV}(\mathbb{R}^n)$. *Then*

$$\|f\|_{b_q^\theta} \leq c \|f\|_p^{1-\theta} \|f\|_{\mathrm{BV}}^\theta.$$

This result gives a sharpening of the inequality

$$\|f\|_{b^\theta_q} \le c\|f\|^{1-\theta}_{b^0_p}\|f\|^\theta_{\mathrm{BV}},$$

which was proved in [20, Theorem 1.5].

Our proof of Theorem 10.8 is based on the use of rearrangements. We employ also the method of molecular decompositions (due to Pełczyński and Wojciechowski [58]). Let us briefly describe the idea of this method.

We can assume that $f \ge 0$. Denote $\mu_k = f^*(2^{-k})$ and

$$g_k(x) = \max\{f(x), \mu_k\} - \mu_k \quad (k \in \mathbb{Z}).$$

Further, let $u_k(x) = g_k(x) - g_{k+1}(x)$. Then

$$f(x) = \sum_{k\in\mathbb{Z}} u_k(x) \tag{10.16}$$

almost everywhere on $\mathbb{R}^n$. We note that

$$0 \le u_k(x) \le \mu_{k+1} - \mu_k$$

and

$$\big|\{x : u_k(x) > 0\}\big| \le 2^{-k}.$$

If $\mu_{k+1} = \mu_k$, then $u_k(x) \equiv 0$. Let $\sigma = \{k \in \mathbb{Z} : \mu_{k+1} > \mu_k\}$ and

$$G_k = \{x : \mu_k < f(x) < \mu_{k+1}\}, \quad k \in \sigma.$$

Assume that f has the weak partial derivative $D_1 f \in L^1(\mathbb{R}^n)$. Denote by D the set of all $\widehat{x}_1 \in \mathbb{R}^{n-1}$ such that the function $f(x_1, \widehat{x}_1)$ is locally absolutely continuous with respect to the variable x_1 on $\mathbb{R}$. Then

$$\mathrm{meas}_{n-1}(\mathbb{R}^{n-1} \setminus D) = 0.$$

Let $\widehat{x}_1 \in D$ and $k \in \sigma$. The section

$$G_k(\widehat{x}_1) = \{x_1 \in \mathbb{R} : (x_1, \widehat{x}_1) \in G_k\}$$

is an open set in $\mathbb{R}$. The function $u_k(x_1, \widehat{x}_1)$ is also locally absolutely continuous with respect to the variable x_1. This readily implies that

$$D_1 u_k(x) = D_1 f(x)\chi_{G_k}(x) \tag{10.17}$$

almost everywhere on $\mathbb{R}^n$.

For $h > 0$, we set

$$f_h(x) = f(x + he_1) - f(x),$$
$$u_{k,h}(x) = u_k(x + he_1) - u_k(x).$$

By (10.16),

$$f_h(x) = \sum_{k\in\mathbb{Z}} u_{k,h}(x).$$

It follows from (10.17) that

$$|u_{k,h}(x)| \le h \int_{\mathbb{R}} |D_1 u_k(x)|\,dx_1 = h \int_{G_k(\widehat{x}_1)} |D_1 f(x)|\,dx_1. \tag{10.18}$$

Thus,

$$\int_{\mathbb{R}^n} |u_{k,h}(x)|\,dx \le h \int_{G_k} |D_1 f(x)|\,dx \equiv hJ_k$$

for any $k \in \sigma$. Since the sets G_k are pairwise disjoint, we obtain that

$$\sum_{k\in\sigma} J_k \le \|D_1 f\|_1.$$

The latter inequality plays a crucial role in the subsequent proof. The main advantage of the molecular decomposition (10.17) is that the supports of the derivatives $D_1 u_k(x)$ are *pairwise disjoint*. Due to this fact the weak estimate

$$|g(t) - g(t+h)| \le h \int_{\mathbb{R}} |g'(u)|\,du \quad (t, h \in \mathbb{R},\ h > 0)$$

applied to $g = u_{k,h}$ (see (10.18)) leads to sharp results.

We emphasize again that the inequality (10.5) (with $r = 1$) cannot be applied in the case $p_1 = s_1 = 1$ since the operator $\varphi \mapsto \varphi^{**}$ is unbounded in L^1.

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