Some results in the theory of Orlicz spaces and applications to variational problems

Andrea Cianchi

1 Introduction

The purpose of these notes is to report some contributions to the theory of interpolation and of Sobolev inequalities in Orlicz spaces and to present a few applications to nonlinear problems of the calculus of variations and partial differential equations whose nonlinearities are not necessarily of power type.

After recalling the necessary background (Section 2), in Section 3 we discuss inequalities of Sobolev type. In particular, a sharp embedding theorem for Orlicz-Sobolev spaces is exhibited. Section 4 deals with an interpolation theorem for quasilinear operators, a variant of which turns out to be a tool for proving the Sobolev inequalities mentioned above. Applications of the interpolation theorem to various problems, such as fractional integration, a priori bounds for solutions to elliptic equations and Hardy type inequalities are also presented. Sections 5 and 6 are devoted to the regularity of solutions to variational problems. Section 5 deals with global regularity for boundary value problems, and, in particular, with the boundedness of the solutions. In Section 6, problems of a local nature are taken into account and higher integrability properties for the gradient of local minimizers of integral functionals are established.

2 Preliminaries

In this section we recall some definitions and basic facts about Orlicz spaces, rearrangements of functions and quasilinear operators which will be used in the sequel. For an exhaustive treatment of the theory of Orlicz and Orlicz-Sobolev spaces we refer to [A1], [KR] and [RR]. A detailed exposition of properties of decreasing rearrangements and rearrangement invariant spaces is contained in [BS]; in particular, see [BZ], [M] [Ta1] and [Ta5] for Pólya-Szegő type principles involving the spherically symmetric rearrangement of weakly differentiable functions.

2.1 Young functions

A function $A : [0,\infty) \to [0,\infty]$ is called a Young function if it has the form

$$A(s) = \int_0^s a(r)\, dr \qquad \text{for} \quad s \geq 0, \tag{2.1}$$

where $a : [0,\infty) \to [0,\infty]$ is an increasing, left-continuous function which is neither identically zero nor identically infinite on $(0,\infty)$. In particular, if A is finite-valued, vanishes only at 0 and $\lim_{s\to 0+} A(s)/s = \lim_{s\to\infty} s/A(s) = 0$, then A is called an N-function.

The right-continuous generalized inverse of a Young function A is defined on $[0,\infty]$ by

$$A^{-1}(r) = \inf\{s : A(s) > r\} \quad (\inf \emptyset = \infty), \tag{2.2}$$

so that

$$A(A^{-1}(r)) \leq r \leq A^{-1}(A(r)) \qquad \text{for } r \geq 0. \tag{2.3}$$

The Young conjugate of a function A will be denoted either by $\widetilde{A}$ or by $A^{\sim}$and defined as

$$\widetilde{A}(s) = \sup\{sr - A(r) : r > 0\}. \tag{2.4}$$

Notice that, when A is a Young function, then $\widetilde{A}$ is also a Young function and $\widetilde{\widetilde{A}} = A$.

The following relations hold for any Young function A:

$$r \leq A^{-1}(r)\, \widetilde{A}^{-1}(r) \leq 2r \qquad \text{for } r \geq 0. \tag{2.5}$$

Every Young function A satisfies

$$A(s) \leq s\, a(s) \leq A(2s) \qquad \text{for } s \geq 0. \tag{2.6}$$

Hence, in particular, if $A : [0,\infty) \to [0,\infty]$ is any left-continuous function such that $A(s)/s$ is increasing, then

$$A(s/2) \leq \int_0^s \frac{A(r)}{r}\, dr \leq A(s) \qquad \text{for } s \geq 0. \tag{2.7}$$

A function A is said to belong to the class Δ_2, or, equivalently, to satisfy the Δ_2-condition, if there exists a constant $c > 0$ such that

$$A(2s) \leq cA(s) \qquad \text{for } s \geq 0. \tag{2.8}$$

Similarly, A is said to belong to Δ_2 near infinity if it is finite-valued and (2.8) holds for large s.

A function B is said to dominate a function A globally (resp. near infinity) if a positive constant c exists such that

$$A(s) \leq B(cs) \tag{2.9}$$

for $s \geq 0$ (resp. for s greater than some positive number). The functions A and B are called equivalent globally (near infinity) if each dominates the other globally (near infinity). If for every $c > 0$ a number $s_c \geq 0$ exists such that inequality (2.9) holds for $s \geq s_c$, then A is said to increase essentially more slowly than B.

2.2 Orlicz spaces

Let (M, ν) be a positive measure space and let A be a Young function. The Orlicz space $L^A(M, \nu)$ is defined as

$$\begin{aligned} L^A(M,\nu) = \Big\{ f :\ & f \text{ is a } \nu\text{-measurable real-valued function on } M \\ & \text{such that } \int_M A\Big(\frac{|f(x)|}{\lambda}\Big)\, d\nu < \infty \text{ for some } \lambda > 0 \Big\}. \end{aligned} \tag{2.10}$$

The Luxemburg norm $\|f\|_{L^A(M,\nu)}$ is defined as

$$\|f\|_{L^A(M,\nu)} = \inf\Big\{ \lambda > 0 : \int_M A\Big(\frac{|f(x)|}{\lambda}\Big)\, d\nu \leq 1 \Big\}. \tag{2.11}$$

The space $L^A(M,\nu)$, equipped with the norm $\|\cdot\|_{L^A(M,\nu)}$, is a Banach space. Note that, if $A(s) = s^p$ and $p \geq 1$, then $L^A(M,\nu) = L^p(M,\nu)$, the customary Lebesgue space, and $\|\cdot\|_{L^A(M,\nu)} = \|\cdot\|_{L^p(M,\nu)}$; if $A(s) \equiv 0$ for $0 \leq s \leq 1$ and $A(s) \equiv \infty$ otherwise, then $L^A(M,\nu) = L^\infty(M,\nu)$ and $\|\cdot\|_{L^A(M,\nu)} \equiv \|\cdot\|_{L^\infty(M,\nu)}$. In the case when M is a subset of $\mathbb{R}^n$ and ν is the Lebesgue measure, we shall denote ν by m_n and $L^A(M,\nu)$ simply by $L^A(M)$.

The following generalized version of Hölder's inequality holds:

$$\int_M f(x)g(x)d\nu \leq 2\|f\|_{L^A(M,\nu)}\|g\|_{L^{\tilde{A}}(M,\nu)}. \tag{2.12}$$

Furthermore,

$$\|f\|_{L^A(M,\nu)} \leq \sup\left\{\int_M f(x)g(x)\,d\nu/\|g\|_{L^{\tilde{A}}(M,\nu)} : g \in L^{\tilde{A}}(M,\nu)\right\}. \tag{2.13}$$

Embeddings between Orlicz spaces defined by different Young functions are characterized in terms of the notion of domination between the defining Young functions. If (M,ν) is a positive non-atomic σ-finite measure space and A and B are Young functions, then

$$L^B(M,\nu) \to L^A(M,\nu)$$

if and only if B dominates A globally. Here, and in what follows, the arrow "$\to$" denotes a continuous embedding. When $\nu(M) < \infty$, the same embedding holds if and only if B dominates A near infinity.

2.3 Orlicz-Sobolev spaces

Let G be an open subset of $\mathbb{R}^n$. Given a Young function A, the (first order) Orlicz-Sobolev space $W^{1,A}(G)$ is defined as

$$\begin{aligned} W^{1,A}(G) = \{u \in L^A(G) : u \text{ is weakly differentiable} \\ \text{and } |Du| \in L^A(G)\}. \end{aligned} \tag{2.14}$$

Here, D stands for gradient. The space $W^{1,A}(G)$, equipped with the norm $\|u\|_{W^{1,A}(G)} = \|u\|_{L^A(G)} + \|Du\|_{L^A(G)}$, is a Banach space. Clearly, $W^{1,A}(G) = W^{1,p}(G)$, the standard Sobolev space, if $A(s) = s^p$ with $p \geq 1$.

$W_0^{1,A}(G)$ will denote the subspace of $W^{1,A}(G)$ of those functions whose continuation by 0 outside G belongs to $W^{1,A}(\mathbb{R}^n)$.

2.4 Rearrangements

Given a real-valued measurable function f on a positive measure space (M,ν), its distribution function $\mu_f : [0,\infty] \to [0,\infty)$ is defined as

$$\mu_f(t) = \nu(\{x \in M : |f(x)| > t\}) \qquad \text{for } t \geq 0. \tag{2.15}$$

The decreasing rearrangement f^* of f is the right-continuous non-increasing function from $[0,\nu(M))$ into $[0,\infty]$ which is equimeasurable with f. Namely,

$$f^*(s) = \sup\{t \geq 0 : \mu_f(t) > s\} \qquad \text{for } 0 \leq s < \nu(M). \tag{2.16}$$

The equimeasurability of f and f^* implies that

$$\int_M A(|f(x)|)\,d\nu = \int_0^\infty A(|f^*(s)|)\,ds \tag{2.17}$$

for every Young function A. Hence

$$\|f\|_{L^A(M,\nu)} = \|f^*\|_{L^A(0,\nu(M))} \tag{2.18}$$

and, in particular,

$$\operatorname{ess\,sup}|f| = f^*(0). \tag{2.19}$$

A variant of f^* is the signed rearrangement f° defined by

$$f^\circ(s) = \sup\{t \geq 0 : \nu(\{f > t\}) > s\} \qquad \text{for } 0 \leq s < \nu(M). \tag{2.20}$$

Clearly, f° enjoys properties analogous to those of f^*.

Let u be any real-valued weakly differentiable function on $\mathbb{R}^n$ decaying to 0 at infinity, i.e. satisfying $m_n(\{|u| > t\}) < \infty$ for every $t > 0$. Let A be a Young function. If $\int_{\mathbb{R}^n} A(|Du|)\,dx < \infty$, then u^* is locally absolutely continuous on $(0,\infty)$ and the following Pólya-Szegő type inequality holds:

$$\int_{\mathbb{R}^n} A(|Du|)\,dx \geq \int_0^\infty A\Big(n\; C_n^{1/n} s^{1/n'}\Big(-\frac{du^*}{ds}\Big)\Big)\,ds. \tag{2.21}$$

Here, $C_n = \pi^{n/2}/\Gamma(1+n/2)$, the measure of the unit ball in $\mathbb{R}^n$ (see [BZ] and [Ta5]). Notice that the right-hand side of (2.21) agrees with $\int_{\mathbb{R}^n} A(|Du^\bigstar|)\,dx$, where $u^\bigstar(x) = u^*(C_n|x|^n)$, the spherically symmetric rearrangement of u. Clearly, (2.21) implies a corresponding inequality for Luxemburg norms.

Versions of inequality (2.21) for functions $u \in W^{1,A}(G)$, not necessarily vanishing on ∂G, can be proved when G is a sufficiently regular subset of $\mathbb{R}^n$. A suitable regularity assumption is an isoperimetric inequality between the measure of any subset E of G and $P(E;G)$, the perimeter of E relative to G (see e.g. [M]). Recall that $P(E;G)$ agrees with the $(n-1)$-dimensional Hausdorff measure of $\partial E \cap G$, if E is smooth; otherwise it is given by the total variation over G of the gradient of the characteristic function of E.

For $n \geq 2$ and $\sigma \geq 1/n'$, we set

$$\begin{aligned}\mathbf{G}(\sigma) = \{G \subseteq \mathbb{R}^n :\ & \\ & G \text{ is open and positive numbers } N \text{ and } C \\ & \text{exist such that } m(E)^\sigma \leq CP(E;G) \\ & \text{for all } E \subset G \text{ satisfying } m_n(E) \leq N\}.\end{aligned} \tag{2.22}$$

In particular, if G is any connected set from $\mathbf{G}(\sigma)$ having finite measure, then, as a consequence of Lemma 3.2.4 of [M], a positive number C exists such that

$$\min^{\sigma}\{m_n(E), m_n(G-E)\} \le C P(E;G) \tag{2.23}$$

for all $E \subseteq G$. The smallest number C which renders (2.23) true will be denoted by $C_\sigma(G)$ and called the relative isoperimetric constant of G associated with the exponent σ (see [Ci1] for explicit evaluations and estimates of $C_\sigma(G)$ in the case where $n = 2$). Notice that $C_{1/n'}(G)$ is dilation invariant.

Any open set $G \subseteq \mathbb{R}^n$ having finite measure and satisfying the cone property belongs to the class $\mathbf{G}(1/n')$ ([M], Corollary 3.2.1/3). If, in addition, G is connected, then it satisfies an inequality of type (2.23) with $\sigma = 1/n'$. Recall that G has the cone property if there exist a cone Σ such that for any $x \in G$, G contains a cone which is congruent to Σ and whose vertex is x.

If G is any set satisfying (2.23) with $\sigma = 1/n'$ and $u \in W^{1,A}(G)$, then u° is locally absolutely continuous and

$$\begin{aligned}\int_G & A(|Du|)\,dx \\ &\ge \int_0^{m_n(G)} A\Big(C_{1/n'}(G)^{-1}\min^{1/n'}\{s, m_n(G)-s\}\Big(-\frac{du^\circ}{ds}\Big)\Big)\,ds\end{aligned} \tag{2.24}$$

(see [Ci2]). Similarly, if $G \in \mathbf{G}(1/n')$ and u is a function from $W^{1,A}(G)$ such that $m_n(\{|u| > 0\}) \le N$, then

$$\int_G A(|Du|)\,dx \ge \int_0^{m_n(G)} A\Big(C^{-1}s^{1/n'}\Big(-\frac{du^*}{ds}\Big)\Big)\,ds. \tag{2.25}$$

2.5 Quasilinear operators

Let (M_1,ν_1) and (M_2,ν_2) be positive measure spaces. We say that T is a quasilinear operator relative to (M_1,ν_1) and (M_2,ν_2) if its domain is a linear subspace of ν_1-measurable a.e. finite functions on M_1, its range is contained in the set of ν_2-measurable functions on M_2 and a constant $c \ge 1$ exists such that

$$\begin{aligned}&|T(f+g)(y)| \le c(|Tf(y)| + |Tg(y)|)\\ \text{and}\quad &|T(\lambda f)(y)| = |\lambda||Tf(y)|\end{aligned} \tag{2.26}$$

for ν_2-a.e. $y \in M_2$, for all f and g in the domain of T and all $\lambda \in \mathbb{R}$.

Given a positive measure space (M, ν) and a number $r \in [1, \infty)$, we denote by $\Lambda^r(M, \nu)$ the Lorentz space of all real-valued ν-measurable functions f on M for which the quantity

$$\|f\|_{\Lambda^r(M,\nu)} = \int_0^\infty \nu(\{|f| > t\})^{1/r} \, dt \tag{2.27}$$

is finite and by $M^r(M, \nu)$ the Marcinkiewicz space of ν-measurable functions f on M for which the quantity

$$\|f\|_{M^r(M,\nu)} = \sup_{t>0} t\nu(\{|f| > t\})^{1/r} \tag{2.28}$$

is finite. In the case when $r = \infty$, we set $\Lambda^\infty(M, \nu) = M^\infty(M, \nu) = L^\infty(M, \nu)$. For every $r \in [1, \infty]$, the following alternative formulas hold:

$$\|f\|_{\Lambda^r(M,\nu)} = \int_0^\infty f^*(s) \, d\psi_r(s)$$

and

$$\|f\|_{M^r(M,\nu)} = \sup_{s>0} \psi_r(s) f^*(s),$$

where $\psi_r(s) = \|\chi_{[0,s]}\|_{L^r(0,\infty)}$ and χ_Ω denotes the characteristic function of a set Ω.

Assume that $1 \leq p, q \leq \infty$. Then a quasilinear operator T relative to (M_1, ν_1) and (M_2, ν_2) is said to be of weak type (p, q) if a constant N exists such that

$$\|Tf\|_{M^q(M_2,\nu_2)} \leq N \|f\|_{\Lambda^p(M_1,\nu_1)} \tag{2.29}$$

for all $f \in \Lambda^p(M_1, \nu_1)$. The smallest constant N which renders (2.29) true is called the weak (p, q) norm of T. The notion of weak type (p, q) that we are using here is due to Calderón. Note that, since $\|f\|_{L^p(M_1,\nu_1)} \leq \|f\|_{\Lambda^p(M_1,\nu_1)}$ for $p \in [1, \infty]$ and $f \in \Lambda^p(M_1, \nu_1)$ (with equality if $p = 1$ or $p = \infty$), such a notion is less restrictive, for $p \in (1, \infty)$, than that originally given by Marcinkiewicz where the Lebesgue norm $L^p(M_1, \nu_1)$ replaced $\Lambda^p(M_1, \nu_1)$ in (2.29).

Analogously, a quasilinear operator T relative to (M_1, ν_1) and (M_2, ν_2) is said to be of strong type (p, q) if a constant N exists such that

$$\|Tf\|_{L^q(M_2,\nu_2)} \leq N \|f\|_{L^p(M_1,\nu_1)} \tag{2.30}$$

for all $f \in L^p(M_1, \nu_1)$.

3 Sobolev inequalities

3.1 Standard results

The classical Sobolev embedding theorem tells us that if G is a sufficiently smooth open subset of $\mathbb{R}^n$, $n \geq 2$, then

$$W^{1,p}(G) \to L^{p^*}(G) \tag{3.1}$$

where $p^* = np/(n-p)$, the Sobolev conjugate of p, if $p < n$, and

$$W^{1,p}(G) \to C^{0,1-n/p}(G) \tag{3.2}$$

if $p > n$ and $m_n(G) < \infty$ (see e.g. [A1], [KJF], [M], [Zi]). Here $C^{0,\alpha}(G)$ denotes the space of bounded Hölder continuous functions on G with exponent α. When $p = n$, as long as Lebesgue spaces are taken into account, one can only say that

$$W^{1,n}(G) \to L^q(G) \tag{3.3}$$

for every $q \in [n, \infty)$, whereas simple counterexamples show that $W^{1,n}(G) \not\subset L^\infty(G)$.

The embedding (3.3) can be improved if Orlicz spaces are taken into play. Indeed, if $m_n(G) < \infty$, then

$$W^{1,n}(G) \to L^B(G), \qquad B(s) = e^{s^{n'}} - 1, \tag{3.4}$$

where $n' = n/(n-1)$, the Hölder conjugate of n ([Tr]; see also [Po], [Y]). Moreover, the embedding (3.4) is sharp in the sense that there is no Orlicz space, strictly contained in $L^B(G)$, into which $W^{1,n}(G)$ is continuously embedded ([HMT]).

In this section we address ourselves to the general problem of associating with any Young function A a Young function B having the property that, for any sufficiently smooth subset G of $\mathbb{R}^n$, $L^B(G)$ is the smallest Orlicz space into which $W^{1,A}(G)$ is continuously embedded.

3.2 Embeddings for $W_0^{1,A}(G)$

Given $n \geq 2$ and a Young function A satisfying

$$\int_0 \left(\frac{t}{A(t)}\right)^{n'-1} dt < \infty, \tag{3.5}$$

we define $H : [0,\infty) \to [0,\infty)$ as

$$H(r) = \left(\int_0^r \left(\frac{t}{A(t)}\right)^{n'-1} dt\right)^{1/n'} \qquad \text{for } r \geq 0 \tag{3.6}$$

and $A_n : [0,\infty) \to [0,\infty]$ by

$$A_n = A \circ H^{-1}, \tag{3.7}$$

where H^{-1} is the left-continuous inverse of H. The function A_n plays the role of an optimal Sobolev conjugate of A. Actually, we have

Theorem 3.1. *Let $n \geq 2$ and let A be a Young function satisfying* (3.5). *Then there exists a constant K, depending only on n, such that*

$$\|u\|_{L^{A_n}(\mathbb{R}^n)} \leq K\|Du\|_{L^A(\mathbb{R}^n)} \tag{3.8}$$

for every real-valued weakly differentiable function u on $\mathbb{R}^n$ decaying to 0 at infinity. Moreover, the result is sharp, in the sense that condition (3.5) *is necessary for an inequality of type* (3.8) *to hold and $L^{A_n}(\mathbb{R}^n)$ is the smallest Orlicz space which renders* (3.8) *true.*

Let us mention that earlier (non-sharp) embeddings for Orlicz-Sobolev spaces are contained in [A2] and [DT].

Remark 3.2. Assumption (3.5) prevents $A(s)$ from vanishing for $s > 0$. Thus, any function from $W_0^{1,A}(G)$ decays to 0 at infinity. Consequently, inequality (3.8) holds, in particular, for every function $u \in W_0^{1,A}(G)$.

Remark 3.3. If assumption (3.5) is dropped, an inequality of type (3.8) still holds for functions supported in a set having finite measure, with K depending also on such a measure and on A: one has just to replace A in the definitions of H and A_n by any Young function equivalent with A near infinity, for which the integral in (3.5) converges. This is a consequence of the fact that Luxemburg norms over sets of finite measure turn into equivalent norms if the defining convex functions are replaced by functions equivalent near infinity.

Remark 3.4. Inequality (3.8) is equivalent to the integral inequality

$$\int_{\mathbb{R}^n} A_n\left(\frac{|u(x)|}{K(\int_{\mathbb{R}^n} A(|Du|)\,dy)^{1/n}}\right) dx \leq \int_{\mathbb{R}^n} A(|Du|)\,dx. \tag{3.9}$$

Indeed, (3.9) implies (3.8) by the very definition of the Luxemburg norm. Conversely, (3.9) follows on replacing $A(s)$ by $\bar{A}(s) = A(s)/M$ in (3.8), with $M = \int_{\mathbb{R}^n} A(|Du|)\,dx$, and observing that, if $\bar{A}_n$ is the function defined as in (3.6)–(3.7) but with A replaced by $\bar{A}$, then $\bar{A}_n(s) = M^{-1}A_n(M^{-1/n})$.

Remark 3.5. In the case when A_n is everywhere finite, i.e. when $\int^\infty \left(\frac{t}{A(t)}\right)^{n'-1} dt = \infty$, inequality (3.9) enables one to show that

$$\int_{\mathbb{R}^n} A_n\left(\frac{|u(x)|}{\lambda}\right) dx < \infty \qquad \text{for every } \lambda > 0, \tag{3.10}$$

whenever $u \in W^{1,A}(\mathbb{R}^n)$. Notice that, in general, (3.10) yields a stronger information than just $u \in L^{A_n}(\mathbb{R}^n)$. Indeed, any function satisfying (3.10) belongs to the closure of $L^\infty(\mathbb{R}^n)$ in $L^{A_n}(\mathbb{R}^n)$, a space which is strictly contained in $L^{A_n}(\mathbb{R}^n)$ if A_n does not satisfy the Δ_2-condition.

It obviously suffices to prove (3.10) under the assumption that $\int_{\mathbb{R}^n} A(|Du|)\,dx < \infty$. Let us choose t so large that

$$K\left(\int_{\{|u|>t\}} A(|Du|)\,dx\right)^{1/n} \le \lambda/2,$$

where K is the constant appearing in (3.9). Then, by the convexity of A_n,

$$\begin{aligned}
\int_{\mathbb{R}^n} A_n\left(\frac{|u|}{\lambda}\right) dx &= \int_{\{|u|\le t\}} A_n\left(\frac{|u|}{\lambda}\right) dx + \int_{\{|u|>t\}} A_n\left(\frac{|u|}{\lambda}\right) dx \\
&\le \int_{\{|u|\le t\}} A_n\left(\frac{|u|}{\lambda}\right) dx \\
&\quad + \frac{1}{2}\int_{\{|u|>t\}} A_n\left(\frac{2(|u|-t)}{\lambda}\right) dx \\
&\quad + \frac{1}{2}\int_{\{|u|>t\}} A_n\left(\frac{2t}{\lambda}\right) dx \\
&\le \int_{\{|u|\le t\}} A_n\left(\frac{|u(x)|}{\lambda}\right) dx \\
&\quad + \frac{1}{2}\int_{\{|u|>t\}} A_n\left(\frac{|u|-t}{K(\int_{\{|u|>t\}} A(|Du|)\,dy)^{1/n}}\right) dx \\
&\quad + \frac{1}{2}A_n\left(\frac{2t}{\lambda}\right)\mu_u(t).
\end{aligned} \tag{3.11}$$

Notice that the first integral on the right-hand side of (3.11) is finite since $\lim_{s\to 0^+} A_n(\lambda s)/A(s) = 0$ for every $\lambda > 0$, the second one because of (3.9) and the third one since we are assuming that A_n is finite-valued. Hence, (3.10) follows.

A tool to prove Theorem 3.1 is the following interpolation result (see [Ci6]).

Theorem 3.6. *Let (M_1, ν_1) and (M_2, ν_2) be positive non-atomic measure spaces and let T be a linear operator whose domain is a linear subspace of the set of ν_1-measurable functions on M_1 and whose range is contained in the set of ν_2-measurable functions on M_2. Let $p \in (1, \infty)$. Assume that T is of strong type $(1, p')$ with norm $\leq N_0$ and of weak type (p, ∞) with norm $\leq N_1$. Let A be a Young function satisfying condition* (3.5) *with n replaced by p, and let A_p be the Young function defined as in* (3.6)–(3.7) *with n replaced by p. Then there exists a constant K, depending only on N_0, N_1 and p, such that*

$$\|Tf\|_{L^{A_p}(M_2,\nu_2)} \leq K\|f\|_{L^A(M_1,\nu_1)}$$

for every ν_1-measurable function f on M_1 such that $\|f\|_{L^A(M_1,\nu_1)} < \infty$.

Proof of Theorem 3.1, outlined. Let u be any weakly differentiable function decaying to 0 at infinity and such that $\|Du\|_{L^A(\mathbb{R}^n)} < \infty$. Since u^* is locally absolutely continuous, we have

$$u^*(s) = \int_s^\infty \left(-\frac{du^*}{dr}\right) dr \qquad \text{for } s \geq 0.$$

Owing to equation (2.18) and to inequality (2.21), inequality (3.8) will follow if we show that there exists a constant C such that

$$\left\| \int_s^\infty -\frac{du^*}{dr}\, dr \right\|_{L^{A_n}(0,\infty)} \leq C \left\| r^{1/n'} \Big(-\frac{du^*}{dr} \Big) \right\|_{L^A(0,\infty)}. \tag{3.12}$$

Inequality (3.12) is a one-dimensional Hardy type inequality, which can be proved via Theorem 3.6, after observing that the operator T, defined on a locally integrable function $\phi : [0, \infty) \to \mathbb{R}$ by

$$T\phi(s) = \int_s^\infty r^{-1/n'} \phi(r)\, dr \qquad \text{for } s \geq 0,$$

is of strong type $(1, n')$ with norm ≤ 1 and of weak type (n, ∞) with norm $\leq n$.

As for the sharpness of the result, assume that inequality (3.8) holds with A_n replaced by some B. Let us consider radially decreasing test functions u having the form

$$u(x) = \frac{1}{nC_n^{1/n}} \int_{C_n|x|^n}^{\infty} r^{-1/n'} \phi(r)\, dr \tag{3.13}$$

for some measurable function $\phi : [0,\infty) \to [0,\infty)$ with $\|\phi\|_{L^A(0,\infty)} < \infty$. Since $|Du(x)| = \phi(C_n|x|^n)$, we have

$$\left\| \int_s^{\infty} r^{-1/n'} \phi(r)\, dr \right\|_{L^B(0,\infty)} \le nC_n^{1/n} K \|\phi\|_{L^A(0,\infty)}. \tag{3.14}$$

If t is any fixed positive number and the support of ϕ is contained in $[t,\infty)$, then

$$\|\phi\|_{L^A(0,\infty)} = \|\phi\|_{L^A(t,\infty)} \tag{3.15}$$

and

$$\begin{aligned} \left\| \int_s^{\infty} r^{-1/n'} \phi(r)\, dr \right\|_{L^B(0,\infty)} &\ge \left\| \int_s^{\infty} r^{-1/n'} \phi(r)\, dr \right\|_{L^B(0,t)} \\ &\ge \int_t^{\infty} r^{-1/n'} \phi(r)\, dr \;\; \|1\|_{L^B(0,t)} \\ &= \int_t^{\infty} r^{-1/n'} \phi(r)\, dr \, \frac{1}{B^{-1}(1/t)}. \end{aligned} \tag{3.16}$$

Combining (3.14)–(3.16) and making use of inequality (2.13) yield

$$\begin{aligned} KB^{-1}(1/t) &\ge \sup_{\phi \in L^A(t,\infty)} \frac{\int_t^{\infty} r^{-1/n'} \phi(r)\, dr}{\|\phi\|_{L^A(t,\infty)}} \\ &\ge \|r^{-1/n'}\|_{L^{\tilde{A}}(t,\infty)} \quad \text{for } t > 0. \end{aligned} \tag{3.17}$$

Hence, the conclusion follows, owing to Lemmas 3.7 and 3.8 below. □

Lemma 3.7. *Let A be a Young function. Then $\|r^{-1/n'}\|_{L^{\tilde{A}}(s,\infty)} < \infty$ for every $s > 0$ if and only if*

$$\int_0 \frac{\widetilde{A}(t)}{t^{1+n'}}\, dt < \infty. \tag{3.18}$$

Moreover, if we set

$$D_n(s) = (sJ^{-1}(s^{n'}))^{n'} \qquad \text{for } s \geq 0, \tag{3.19}$$

where J^{-1} is the left-continuous inverse of the function given by

$$J(r) = n' \int_0^r \frac{\widetilde{A}(t)}{t^{1+n'}}\, dt \qquad \text{for } r \geq 0, \tag{3.20}$$

then

$$\|r^{-1/n'}\|_{L^{\widetilde{A}}(s,\infty)} = D_n^{-1}(1/s) \qquad \text{for } s > 0, \tag{3.21}$$

where D_n^{-1} is the right-continuous inverse of D_n.

Lemma 3.8. *Let A be a Young function. We have*

$$\int_0 \frac{\widetilde{A}(t)}{t^{1+n'}}\, dt < \infty \quad \text{if and only if} \quad \int_0 \left(\frac{t}{A(t)}\right)^{n'-1} dt < \infty \tag{3.22}$$

and

$$\int^\infty \frac{\widetilde{A}(t)}{t^{1+n'}}\, dt < \infty \quad \text{if and only if} \quad \int^\infty \left(\frac{t}{A(t)}\right)^{n'-1} dt < \infty. \tag{3.23}$$

Moreover, there exist constants c_1 and c_2, depending only on n, such that

$$A_n(c_1 s) \leq D_n(s) \leq A_n(c_2 s) \qquad \text{for } s \geq 0, \tag{3.24}$$

where A_n and D_n are the functions defined by (3.7) *and* (3.19)*, respectively.*

A proof of Lemma 3.7 can be found in [Ci2]. Lemma 3.8 is proved in [Ci8].

3.3 Embeddings for $W^{1,A}(G)$

It is well known that the validity of Poincaré-type inequalities and embedding theorems involving spaces of functions defined in an open set G, which do not necessarily vanish on ∂G, depends on the regularity of G. In our embedding such a regularity will be prescribed in terms of isoperimetric inequalities—see Subsection 2.4.

Theorem 3.9. *Let $n \geq 2$. Let A be any Young function and let A_n be the function defined as in* (3.6)–(3.7) *with A modified, if necessary, near zero in such a way that* (3.5) *is fulfilled.*
i) *If $G \in \mathbf{G}(1/n')$ is connected and has finite measure, then there exists a constant K, depending only on A, $m_n(G)$ and $C_{1/n'}(G)$ such that*

$$\|u - u_G\|_{L^{A_n}(G)} \leq K\|Du\|_{L^A(G)} \tag{3.25}$$

for all $u \in W^{1,A}(G)$. Here,

$$u_G = \frac{1}{m_n(G)} \int_G u(x)\,dx,$$

the mean value of u over G.

The constant K in (3.25) *depends only on $C_{1/n'}(G)$ provided that* (3.5) *holds.*
ii) *The continuous embedding*

$$W^{1,A}(G) \to L^{\bar{A}_n}(G) \tag{3.26}$$

holds for every $G \in \mathbf{G}(1/n')$. Here, $\bar{A}_n$ is the Young function defined by

$$\bar{A}_n(s) = \begin{cases} A_n(s) & \text{if } s \geq s_2 \\ A(s) & \text{if } 0 \leq s \leq s_1 \end{cases} \tag{3.27}$$

for (*suitable*) $0 < s_1 < s_2$.

Moreover, $L^{A_n}(G)$ and $L^{\bar{A}_n}(G)$ are the smallest Orlicz spaces which render (3.25) *and* (3.26), *respectively, true.*

A proof of Theorem 3.9 makes use of inequalities (2.24)–(2.25) and of interpolation techniques as in Theorems 3.6—see [Ci3].

Remark 3.10. Notice that formula (3.27) determines $\bar{A}_n$ up to globally equivalent Young functions; thus, the Orlicz space $L^{\bar{A}_n}(G)$ is uniquely defined. Moreover, since the Young functions A_n and $\bar{A}_n$ are equivalent near infinity, $L^{A_n}(G) = L^{\bar{A}_n}(G)$ when $m(G) < \infty$.

Remark 3.11. A similar argument as in Remark 3.5 shows that, under assumption (3.5), inequality (3.25) is equivalent to the integral inequality

$$\int_G A_n\left(\frac{|u(x) - u_G|}{K(\int_G A(|Du|)\,dy)^{1/n}}\right) dx \leq \int_G A(|Du|)\,dx. \tag{3.28}$$

Remark 3.12. Theorem 3.9 can be extended to the case where less smooth domains $G \in \mathbf{G}(\sigma)$ with $\sigma \in (1/n', 1)$ are taken into account: in the statement, $1/n'$ has to be replaced by σ and A_n by the Young function $A_{1/(1-\sigma)}$ given by (3.6)–(3.7) with $1/(1-\sigma)$ in place of n.

Compact embeddings for $W^{1,A}(G)$ are considered in the following

Theorem 3.13. *Let $n \geq 2$ and let G be any open bounded set from the class $\mathbf{G}(1/n')$. Let A be any Young function. If B is any Young function increasing essentially more slowly near infinity than the function A_n, then the embedding*

$$W^{1,A}(G) \to L^B(G) \tag{3.29}$$

is compact.

Theorem 3.13 follows from Theorem 3.9 via the same arguments as in the proof of Theorem 3.7 of [DT].

3.4 Embeddings into spaces of continuous functions

Let us now focus the case when

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{n'-1} dt < \infty. \tag{3.30}$$

Condition (3.30) is equivalent to saying that $A_n(s) = \infty$ for large s. Thus, Theorem 3.1 yields, in particular, the following corollary (see also [M], [Ta3]).

Corollary 3.14. *Let A be a Young function such that*

$$\int_0^{\infty} \left(\frac{t}{A(t)}\right)^{n'-1} dt < \infty.$$

Then

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq K \|Du\|_{L^A(\mathbb{R}^n)}$$

for every weakly differentiable function u decaying to 0 at infinity such that $\|Du\|_{L^A(\mathbb{R}^n)} < \infty$.

An analogous corollary for functions in $W^{1,A}(G)$ follows from Theorem 3.9.

Under assumption (3.30), a stronger result then Corollary 3.14 is true. Namely, any function from $W^{1,A}(G)$ is in fact continuous and an estimate for its modulus of continuity can be obtained. This is the object of the next result. First, let us introduce the function Θ, defined by

$$\Theta(r) = n' \int_r^\infty \frac{\widetilde{A}(t)}{t^{1+n'}}\, dt \qquad \text{for } r \geq 0,$$

and the function ω, given by

$$\omega(s) = \left(s\Theta^{-1}(s^{n'})\right)^{n'} \qquad \text{for } s \geq 0. \tag{3.31}$$

Notice that, under assumption (3.30), the function $\Theta(r)$ is finite for all $r > 0$ by Lemma 3.8.

Theorem 3.15. *Let G be an open subset of $\mathbb{R}^n$ and let A be a Young function satisfying* (3.30). *Then*

i) *Every $u \in W^{1,A}(G)$ equals a.e. a continuous function.*

ii) *For every compact subset G' of G, a constant C exists such that*

$$|u(x) - u(y)| \leq C\|u\|_{W^{1,A}(G)}\omega^{-1}(|x-y|^{-n}) \tag{3.32}$$

for every $u \in W^{1,A}(G)$ and a.e. $x, y \in G'$, where, ω^{-1} is the right-continuous inverse of the function defined by (3.31).

Under the additional assumption that G is a bounded strongly Lipschitz domain inequality (3.32) *holds for a.e. $x, y \in G$.*

Recall that a bounded open set $G \in \mathbb{R}^n$ is called strongly Lipschitz if, for each $x \in \partial G$, there exist a neighbourhood $\mathbf{U}_x$ of x, a coordinate system $(y_1, \ldots, y_n)$ centred at x and a Lipschitz-continuous function $\Xi = \Xi(y_1, \ldots, y_{n-1})$ such that

$$G \cap \mathbf{U}_x = \{(y_1, \ldots, y_n) : y_n > \Xi(y_1, \ldots, y_{n-1})\}.$$

Proof of Theorem 3.15. Let G by any open subset of $\mathbb{R}^n$. Consider any cube Q_δ contained in G and having sides of length δ. Call v the restriction of u

to Q_δ. By inequality (2.12),

$$\begin{aligned} \operatorname{ess\,sup} v - \operatorname{ess\,inf} v &= \int_0^{\delta^n} -\frac{dv^\circ}{dr}\,dr \\ &\le 2\Big\| C_{1/n'}(Q) \min{}^{-1/n'}\{r, \delta^n - r\} \Big\|_{L^{\tilde{A}}(0,\delta^n)} \qquad (3.33) \\ &\quad \times \Big\| C_{1/n'}(Q)^{-1} \min{}^{1/n'}\{r, \delta^n - r\} \Big(-\frac{dv^\circ}{dr} \Big) \Big\|_{L^A(0,\delta^n)}. \end{aligned}$$

Here, $C_{1/n'}(Q)$ denotes the relative isoperimetric constant of cubes in $\mathbb{R}^n$ for the exponent $1/n'$. By Lemma 3.16 below, we have

$$\Big\| \min{}^{-1/n'}\{r, \delta^n - r\} \Big\|_{L^{\tilde{A}}(0,\delta^n)} \le 2\omega^{-1}(\delta^{-n}). \qquad (3.34)$$

Moreover, by (2.24),

$$\begin{aligned} &\Big\| C_{1/n'}(Q)^{-1} \min{}^{1/n'}\{r, \delta^n - r\} \Big(-\frac{dv^\circ}{dr} \Big) \Big\|_{L^A(0,\delta^n)} \qquad (3.35) \\ &\qquad \le \|Dv\|_{L^A(Q_\delta)} \le \|Du\|_{L^A(G)}. \end{aligned}$$

Combining (3.33)–(3.35) yields

$$\operatorname{ess\,sup} v - \operatorname{ess\,sup} v \le 4C_{1/n'}(Q)\,\omega^{-1}(\delta^{-n})\|Du\|_{L^A(G)}. \qquad (3.36)$$

Inequality (3.32) for a.e. $x, y \in G'$ obviously follows from (3.36).

In the case when G is a strongly Lipschitz domain one may assume, without loss of generality, that G is a cube (see e.g. [DT], Theorem 3.6). Thus, (3.36) implies (3.32) for a.e. $x, y \in G$, since, in this case, for every $x, y \in G$ there exists a cube $Q_{|x-y|}$, having sides of length not exceeding $|x-y|$ which are parallel to those of G, such that $x, y \in Q_{|x-y|} \subseteq G$.

Finally, let us show that u is a.e. equal to a continuous function. For $k \in \mathbb{N}$, denote by $u_k(x)$ the mean value of u over the cube $Q_{1/k}(x)$ centred at x and having sides of length $1/k$ which are parallel to the coordinate axes. Observe that u_k is a continuous function. Inequality (3.36) implies that

$$|u_k(x) - u_h(x)| \le 4C_{1/n'}(Q)\omega^{-1}(k^n)\|Du\|_{L^A(G)}$$

whenever $h > k$ and $Q_{1/k}(x) \subseteq G$. Since $\omega^{-1}(\delta^{-n})$ tends to 0 as δ goes to 0^+, $\{u_k\}$ is a Cauchy sequence in the space of continuous functions on any compact subset of G. Therefore, $\{u_k\}$ converges to a continuous function, say $\bar{u}$, in G. By Lebesgue theorem, $\bar{u} = u$ a.e. in G. □

Lemma 3.16 (see [Ci2]). *Let A be a Young function. Then* $\|r^{-1/n'}\|_{L^{\tilde{A}}(0,s)} < \infty$ *for every* $s > 0$ *if and only if* (3.30) *holds. Moreover,*

$$\|r^{-1/n'}\|_{L^{\tilde{A}}(0,s)} = \omega^{-1}(1/s) \qquad \text{for } s > 0,$$

where ω^{-1} *is the right-continuous inverse of the function defined by* (3.31).

3.5 Examples

Example 3.17. Consider Young functions $A(s)$ which are equivalent to $s^p(\log(s))^q$ near infinity, where either $p > 1$ and $q \in \mathbb{R}$ or $p = 1$ and $q \geq 0$. Let G be any open subset of $\mathbb{R}^n$ having finite measure. Then Theorem 3.1 yields

$$W_0^{1,A}(G) \to L^{A_n}(G), \tag{3.37}$$

where $A_n(s)$ is equivalent near infinity to

$$\begin{cases} s^{np/(n-p)}(\log s)^{nq/(n-p)} & \text{if } 1 \leq p < n \\ \exp\left(s^{n/(n-1-q)}\right) & \text{if } p = n,\ q < n-1 \\ \exp\left(\exp\left(s^{n'}\right)\right) & \text{if } p = n,\ q = n-1. \end{cases}$$

If either $p > n$, or $p = n$ and $q > n - 1$, then Corollary 3.14 tells us that

$$W_0^{1,A}(G) \to L^{\infty}(G).$$

By Theorem 3.9, the same embeddings are true (and optimal) with $W_0^{1,A}(G)$ replaced by $W^{1,A}(G)$ provided that $G \in \mathbf{G}(1/n')$. Notice that when $p \neq n$ and $q = 0$, (3.37) agrees with the Sobolev theorem; when $p = n$, the embedding (3.37) yields (3.4) if $q = 0$ and result of [FLS] if $q < 0$ and of [EGO] if $q = n - 1$.

Example 3.18. Consider Young functions $A(s)$ which are equivalent to $s^p(\log\log(s))^q$ near infinity, where either $p > 1$ and $q \in \mathbb{R}$ or $p = 1$ and $q \geq 0$. Let G be any set from $\mathbf{G}(1/n')$ having finite measure. Then Theorem 3.9 implies that

$$W^{1,A}(G) \to L^{A_n}(G),$$

where $A_n(s)$ is equivalent near infinity to

$$\begin{cases} s^{np/(n-p)}(\log\log(s))^{nq/(n-p)} & \text{if } 1 \leq p < n \\ \exp\left(s^{n'}\right)(\log(s))^{q/(n-1)} & \text{if } p = n. \end{cases}$$

When $p > n$, then

$$W_0^{1,A}(G) \to L^\infty(G).$$

4 Interpolation of operators

4.1 Statement of the result

The Marcinkiewicz interpolation theorem states that, if $1 \le p_i \le q_i \le \infty$ for $i = 0, 1$, and $p_0 \ne p_1$, $q_0 \ne q_1$, then every (quasi)linear operator of weak types (p_0, q_0) and (p_1, q_1) is bounded between the Lebesgue spaces L^p and L^q, provided that p and q satisfy

$$\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}, \quad \frac{1}{q} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1} \tag{4.1}$$

for some $\theta \in (0, 1)$ ([Mark], [Z]). This classical result has been the object of various extensions and of developments in more abstract settings. Let us recall, for instance, a Lorentz space version of it, which tells us that every operator as above (with p_i not necessarily $\le q_i$) is bounded between the Lorentz spaces $L^{p,r}$ and $L^{q,r}$ if p and q are related as in (4.1) and r is any positive number ([Ca], [H]).

As far as interpolation in Orlicz spaces is concerned, contributions can be found in several papers, including [GP], [Kr], [Pe], [Pu], [Ra], [Ri], [To1], [Z]. A survey of results is contained in [Ma1].

In this section we present an optimal version of the Marcinkiewicz theorem for Orlicz spaces. Namely, we exhibit a necessary and sufficient condition on N-functions A and B, extending (4.1), for every quasilinear operator of weak type (p_i, q_i), with $1 \le p_i \le q_i \le \infty$, $i = 0, 1$, to be bounded from L^A into L^B.

The extension of condition (4.1) that we find involves certain functions $E_{\Phi,\beta}$, $F_{\Phi,\beta}$, $G_{\Phi,\beta}$, $H_{\Phi,\beta}$ from $[0, \infty]$ into $[0, \infty]$ associated with an N-function Φ and with $\beta \in [1, \infty]$ according to the following formulas. In each of these formulas, the former equation stands for a definition, the latter follows via an easy computation:

$$E_{\Phi,\beta}(s) = \left\| \left(\frac{r}{\Phi(r)}\right)^{1/\beta'} \right\|_{L^\beta(0,s)} = \begin{cases} \left(\int_0^s \left(\frac{r}{\Phi(r)}\right)^{\beta-1} dr\right)^{1/\beta} & \text{if } 1 \le \beta < \infty \\ \infty & \text{if } \beta = \infty; \end{cases} \tag{4.2}$$

$$F_{\Phi,\beta}(s) = \left\| \left(\frac{r}{\Phi(r)}\right)^{1/\beta'} \right\|_{L^\beta(s,\infty)} = \begin{cases} \infty & \text{if } \beta = 1 \\ \left(\int_s^\infty \left(\frac{r}{\Phi(r)}\right)^{\beta-1} dr\right)^{1/\beta} & \text{if } 1 < \beta < \infty \\ s/\Phi(s) & \text{if } \beta = \infty; \end{cases} \tag{4.3}$$

$$G_{\Phi,\beta}(s) = \left\| \frac{\Phi(r)^{1/\beta}}{r^{1+1/\beta}} \right\|_{L^\beta(0,s)} = \begin{cases} \left(\int_0^s \frac{\Phi(r)}{r^{\beta+1}}\, dr\right)^{1/\beta} & \text{if } 1 \le \beta < \infty \\ \infty & \text{if } \beta = \infty; \end{cases} \tag{4.4}$$

$$H_{\Phi,\beta}(s) = \left\| \frac{\Phi(r)^{1/\beta}}{r^{1+1/\beta}} \right\|_{L^\beta(s,\infty)} = \begin{cases} \infty & \text{if } \beta = 1 \\ \left(\int_s^\infty \frac{\Phi(r)}{r^{\beta+1}}\, dr\right)^{1/\beta} & \text{if } 1 < \beta < \infty \\ 1/s & \text{if } \beta = \infty. \end{cases} \tag{4.5}$$

All these functions are strictly monotone for those Φ and β for which they are not identically equal to ∞. Actually, $E_{\Phi,\beta}$ and $G_{\Phi,\beta}$ are increasing, whereas $F_{\Phi,\beta}$ and $H_{\Phi,\beta}$ are decreasing.

Throughout this section, the generalized inverse of a monotone function $\Psi : [0,\infty] \to [0,\infty]$ will be defined by

$$\Psi^{-1}(r) = \sup\{s \ge 0 : \Psi(s) < r\} \quad \text{for } r \ge 0, \tag{4.6}$$

if Ψ is non-decreasing, and by

$$\Psi^{-1}(r) = \inf\{s \ge 0 : \Psi(s) \le r\} \quad \text{for } r \ge 0, \tag{4.7}$$

if Ψ is non-increasing, where $\sup \emptyset = 0$ and $\inf \emptyset = \infty$. Moreover, monotone functions Ψ defined on $(0,\infty)$ will be understood extended to $[0,\infty]$ on setting $\Psi(0) = \lim_{s\to 0+} \Psi(s)$ and $\Psi(\infty) = \lim_{s\to +\infty} \Psi(s)$. Expressions of the forms $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$ are defined as 0.

Our interpolation theorem can be stated as follows.

Theorem 4.1. *Assume that $1 \le p_0 \le p_1 \le \infty$, $1 \le q_0, q_1 \le \infty$, and $p_i \le q_i$ for $i = 0, 1$. Let A and B be N-functions. Let (M_1,ν_1) and (M_2,ν_2) be infinite, non-atomic and σ-finite measure spaces. Assume that $q_0 \le q_1$.*

Then $L^A(M_1,\nu_1) \subset \Lambda^{p_0}(M_1,\nu_1) + \Lambda^{p_1}(M_1,\nu_1)$ *and every quasilinear operator* T *of weak types* (p_0,q_0) *and* (p_1,q_1) *relative to* (M_1,ν_1) *and to* (M_2,ν_2) *is bounded from* $L^A(M_1,\nu_1)$ *into* $L^B(M_2,\nu_2)$ *if and only if the functions* $E_{A,p_1'}$, $F_{A,p_0'}$, G_{B,q_0}, H_{B,q_1}, *defined as in* (4.2)–(4.5), *are finite on* $(0,\infty)$ *and a constant* $\sigma > 0$ *exists such that*

$$F_{A,p_0'}(E^{-1}_{A,p_1'}(\sigma s))G_{B,q_0}(H^{-1}_{B,q_1}(1/s)) \le \sigma \qquad \text{for } s \ge 0. \tag{4.8}$$

Moreover, if T *is of weak types* (p_i,q_i) *with norms* N_i, $i = 0,1$, *and the above conditions are fulfilled, then*

$$\|Tf\|_{L^B(M_2,\nu_2)} \le c\sigma K \max\{N_0,N_1\}\ \|f\|_{L^A(M_1,\nu_1)} \tag{4.9}$$

for all $f \in L^A(M_1,\nu_1)$, *where* c *is the constant appearing in* (2.26) *and* K *is an absolute constant. When* $q_0 \ge q_1$, *an analogous statement holds with condition* (4.8) *replaced by*

$$F_{A,p_0'}(E^{-1}_{A,p_1'}(\sigma s))H_{B,q_0}(G^{-1}_{B,q_1}(1/s)) \le \sigma \qquad \text{for } s \ge 0. \tag{4.10}$$

Remark 4.2. Unlike the classical Marcinkiewicz theorem, here both the case $p_0 = p_1$ and $q_0 = q_1$ are admissible. It is easily verified that in each of these cases, the sole finiteness of $E_{A,p_1'}$, $F_{A,p_0'}$ and of G_{B,q_0}, H_{B,q_1} or G_{B,q_1}, H_{B,q_0} on $(0,\infty)$, implies (4.8) or (4.10), respectively. In particular, observe that (4.8) and (4.10) are equivalent when $q_0 = q_1$.

Remark 4.3. Clearly, inequalities (4.8) and (4.10) depend only on the asymptotic behaviour of A and B at 0 and at infinity. More precisely, it is easy to see that the validity of those inequalities are invariant under replacement of A and B by equivalent functions. This invariance is consistent with the fact that replacement by equivalent N-functions leaves Orlicz spaces unchanged (up to equivalent norms).

Remark 4.4. A version of Theorem 4.1 can be also proved under the assumption that $\nu_1(M_1)$ or $\nu_2(M_2)$ is finite. The conclusions are the following. Assume that $q_0 \le q_1$ and denote by $\Theta_\sigma(s)$ the left-hand side of (4.8). Then every quasilinear operator of weak types (p_i,q_i) (with p_i and q_i as in Theorem 4.1) is bounded from $L^A(M_1,\nu_1)$ into $L^B(M_2,\nu_2)$ if and only if $\limsup_{s\to+\infty} \Theta_\sigma(s) < \infty$ for some $\sigma > 0$ and either $\nu_1(M_1) < \infty$, $\nu_2(M_2) = \infty$ and $F_{A,p_0'}$, G_{B,q_0}, H_{B,q_1} are finite on $(0,\infty)$, or $\nu_1(M_1) = \infty$, $\nu_2(M_2) < \infty$ and $E_{A,p_1'}$, $F_{A,p_0'}$, H_{B,q_1} are finite on $(0,\infty)$, or $\nu_1(M_1) < \infty$, $\nu_2(M_2) < \infty$ and $F_{A,p_0'}$, H_{B,q_1} are finite on $(0,\infty)$. Assume now that $q_0 \ge q_1$ and denote by $\Xi_\sigma(s)$ the left-hand side of (4.10). Then every quasilinear operator as above is bounded from $L^A(M_1,\nu_1)$ into $L^B(M_2,\nu_2)$ if and only if either $\nu_1(M_1) < \infty$, $\nu_2(M_2) = \infty$, $F_{A,p_0'}$, G_{B,q_1}, H_{B,q_0} are finite on $(0,\infty)$ and $\limsup_{s\to+\infty} \Xi_\sigma(s) < \infty$, or $\nu_1(M_1) = \infty$, $\nu_2(M_2) < \infty$, $F_{A,p_1'}$, $F_{A,p_0'}$, H_{B,q_0} are finite on $(0,\infty)$ and $\limsup_{s\to 0+} \Xi_\sigma(s) < \infty$, or $\nu_1(M_1) < \infty$, $\nu_2(M_2) < \infty$ and $F_{A,p_0'}$, H_{B,q_0} are finite on $(0,\infty)$. This statement can be proved via similar arguments as in Subsection 4.4 below, on making use of the fact that replacing the defining N-functions in an Orlicz space over a non-atomic finite measure space by an N-function equivalent near infinity results in the same Orlicz space with an equivalent norm.

4.2 Examples

Throughout this subsection, we shall assume that $1 \le p_0 \le p_1 \le \infty$, $1 \le q_0, q_1 \le \infty$, and $p_i \le q_i$ for $i = 0,1$.

Example 4.5. Assume that A and B are powers, i.e. $A(s) = s^p$, $B(s) = s^q$ for some p, $q > 1$. Then it follows from Theorem 4.1 that every quasilinear operator of weak types (p_i, q_i), $i = 0,1$, is bounded from $L^p(M_1,\nu_1)$ into $L^q(M_2,\nu_2)$ if and only if $p_0 < p < p_1$, $\min\{q_0,q_1\} < q < \max\{q_0,q_1\}$ and $\dfrac{(p_0-p)p_1}{(p_1-p)p_0} = \dfrac{(q_0-p)q_1}{(q_1-p)q_0}$. Clearly, these conditions are equivalent to requiring that (4.1) holds for some $\theta \in (0,1)$. Thus, the original theorem of Marcinkiewicz is reproduced.

Example 4.6. Let us consider two limiting situations of the preceding example (see also [BR], [BS] and [EOP]). Assume that $q_0 \le q_1$ and that the underlying measure spaces (M_1,ν_1) and (M_2,ν_2) have finite measures. Owing to Remark 4.4, we have the following conclusions. If $1 < p_1 < \infty$, then every quasilinear operator of weak types (p_i,q_i), $i = 0,1$, is bounded from $L^{p_1}(M_1,\nu_1)$ into $L^B(M_2,\nu_2)$ if there exists $k > 0$ such that $B(ks) \le s^{q_1}(\log(s))^{-1-q_1/p_1'}$ or $B(ks) \le \exp(s^{p_1'})$ for large s, according to whether $q_1 < \infty$ or $q_1 = \infty$. If $1 < q_0 < \infty$, then every quasilinear operator of weak types (p_i,q_i), $i = 0,1$, is bounded from $L^A(M_1,\nu_1)$ into $L^{q_0}(M_2,\nu_2)$ if there exists $k > 0$ such that $A(ks) \ge s^{p_0}(\log s)^{p_0-1+p_0/q_0}$ for large s.

Example 4.7. Here we take into account the "diagonal" case where $p_0 = q_0$, $p_1 = q_1$ and $A = B$ in Theorem 4.1. Let us denote by $I(A^{-1})$ and $i(A^{-1})$ the upper and lower Matuszewska-Orlicz indices of A^{-1} respectively, defined by

$$I(A^{-1}) = \lim_{\lambda\to+\infty} \frac{\log\left(\sup\limits_{r>0} \dfrac{A^{-1}(\lambda r)}{A^{-1}(r)}\right)}{\log\lambda}$$

and

$$i(A^{-1}) = \lim_{\lambda \to +\infty} \frac{\log\left(\inf\limits_{r>0} \dfrac{A^{-1}(\lambda r)}{A^{-1}(r)}\right)}{\log \lambda}.$$

Lemma 4 of [Ci5] tells us that $I(A^{-1}) < 1/p_0$ if and only if a constant $k > 0$ exists such that

$$F_{A,p_0'}(ks)G_{A,p_0}(s) \leq k \quad \text{for} \quad s \geq 0. \tag{4.11}$$

Similarly, one can show that, if $p_1 < \infty$, then $i(A^{-1}) > 1/p_1$ if and only if a constant $k > 0$ exists such that

$$E_{A,p_1'}(ks)H_{A,p_1}(s) \leq k \quad \text{for} \quad s \geq 0. \tag{4.12}$$

When $p_1 < \infty$, inequalities (4.11)–(4.12) imply (4.8). Thus, Theorem 4.1 ensures that every quasilinear operator of weak types (p_0, p_0) and (p_1, p_1) is bounded from $L^A(M_1, \nu_1)$ into $L^A(M_2, \nu_2)$ if

$$\frac{1}{p_1} < i(A^{-1}) \leq I(A^{-1}) < \frac{1}{p_0}. \tag{4.13}$$

When $p_1 = \infty$, (4.11) agrees with (4.8); therefore the same conclusion is true provided that

$$I(A^{-1}) < \frac{1}{p_0}. \tag{4.14}$$

On the other hand, condition (4.13) or (4.14) is also necessary for every operator of weak types (p_0, p_0) and (p_1, p_1) to be bounded from $L^A(M_1, \nu_1)$ into $L^A(M_2, \nu_2)$, as can be shown by analogous arguments as in the proof of Theorem 4.1. Thus, in particular, the result of Boyd's interpolation theorem (see e.g. Theorem 5.16, Chap. 3, of [BS]) is recovered in the case where the rearrangement invariant space involved in that theorem is an Orlicz space.

4.3 Applications

As a first application of Theorem 4.1, let us consider boundedness properties of the fractional integral operator, also called Riesz potential. Recall that the fractional integral $R_\alpha f$ of order $\alpha \in (0, n)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$R_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \quad \text{for } x \in \mathbb{R}^n.$$

The operator R_α is known to be of weak types $(1, n/(n-\alpha))$ and $(n/\alpha, \infty)$ (see e.g. [BS], [To2]). Therefore, by Theorem 4.1, R_α is bounded from $L^A(\mathbb{R}^n)$ into $L^B(\mathbb{R}^n)$ if the functions $E_{A,n/(n-\alpha)}$ and $G_{B,n/(n-\alpha)}$ are finite on $(0, \infty)$ and a constant $\sigma > 0$ exists such that

$$G_{B,n/(n-\alpha)}(\sigma^{-1} E_{A,n/(n-\alpha)}(s)) \leq \sigma \, \frac{A(s)}{s} \quad \text{for } s \geq 0. \tag{4.15}$$

Moreover, condition (4.15) is also necessary for R_α to be bounded from $L^A(\mathbb{R}^n)$ into $L^B(\mathbb{R}^n)$, as can be shown on making use of an estimate from below for $R_\alpha f$ when f is radial (see e.g. [Sa]) and of similar arguments as in the proof of the necessity part in Theorem 4.1.

In particular, (4.15) is fulfilled if $A(s) = s^p$ and $B(s) = s^{np/(n-\alpha p)}$ with $1 < p < n/\alpha$. Thus, the Hardy-Littlewood-Sobolev theorem stating that R_α is bounded from $L^p(\mathbb{R}^n)$ into $L^{np/(n-\alpha p)}(\mathbb{R}^n)$ is reproduced. In the case when Ω is a subset of $\mathbb{R}^n$ having finite measure, we recover the borderline results that R_α is bounded from $L^{n/\alpha}(\Omega)$ into $L^B(\Omega)$, where $B(s) = \exp(s^{n/(n-\alpha)}) - 1$ ([Stri], [Tr]), and from $L^A(\Omega)$ into $L^{n/(n-\alpha)}(\Omega)$, where $A(s) = s(\log(1+s))^{(n-\alpha)/n}$ ([O]). More general limiting situations can be considered; for instance, if $A(s) = s^{n/\alpha}(\log(1+s))^q$, then R_α is bounded from $L^A(\Omega)$ into $L^B(\Omega)$ where $B(s) = \exp(s^{n/(n-\alpha-\alpha q)}) - 1$ if $q < (n-\alpha)/\alpha$ (cf. [FLS]) and $B(s) = \exp(\exp(s^{n/(n-\alpha)})) - 1$ if $q = (n-\alpha)/\alpha$ (cf. [EGO]); when $q > (n-\alpha)/\alpha$, R_α is bounded from $L^A(\Omega)$ into $L^B(\Omega)$ for every B and the norm of R_α is independent of B, whence R_α is in fact bounded from $L^A(\Omega)$ into $L^\infty(\Omega)$.

Notice that the Hardy-Littlewood maximal operator, which is of weak types (1,1) and (∞, ∞), is another classical operator that can be easily dealt with by Theorem 4.1.

We next take into account a priori estimates for solutions to uniformly elliptic boundary value problems of the type

$$\begin{cases} -\displaystyle\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + c(x)u = f(x) \quad \text{in } G \\ u = 0 \quad \text{on } \partial G. \end{cases} \tag{4.16}$$

Here, G is an open subset of $\mathbb{R}^n$, $n \geq 3$; the coefficients $a_{ij}(x)$ and $c(x)$ are functions from $L^\infty(G)$ satisfying

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^n \text{ and a.e. } x \in G,$$

and

$$c(x) \geq 0 \quad \text{for a.e. } x \in G.$$

Moreover, $f \in L^{2n/(n+2)}(G)$.

Theorem 1 of [Ta2] ensures that, if u is the weak solution from $W_0^{1,2}(G)$ to problem (4.16), then

$$u^*(s) \leq \frac{1}{n(n-2)C_n^{2/n}} \Big(s^{-1+2/n} \int_0^s f^*(r)\, dr + \int_s^{m_n(G)} f^*(r) r^{-1+2/n}\, dr \Big) \tag{4.17}$$

for $s \in (0, m_n(G))$.

By Lemma 4.10, Chap. 4 of [BS], inequality (4.17) implies that the linear operator which associates with the datum f the corresponding solution u to (4.16) is of weak types $(1, n/(n-2))$ and $(n/2, \infty)$. By Theorem 4.1, we have that the a priori estimate

$$\|u\|_{L^B(G)} \leq \text{Const. } \|f\|_{L^A(G)} \tag{4.18}$$

holds whenever $E_{A,n/(n-2)}$ and $G_{B,n/(n-2)}$ are finite on $(0, \infty)$ and inequality (4.15) holds with $\alpha = 2$. This reproduces, for instance, the classical estimate where $A(s) = s^p$ with $1 < p < n/2$ and $B(s) = s^{np/(n-2p)}$ in (4.18). When $m_n(G) < \infty$, the limiting estimate where $A(s) = s^{n/2}$ and $B(s) = \exp(s^{n/(n-2)}) - 1$ in (4.18) follows from Example 4.6. Notice that similar results can be proved for problems of type (4.16) when the datum on the right-hand side of the equation is in divergence form or when the boundary condition is of Neumann type.

An alternative characterization of those Orlicz spaces between which R_α is bounded or for which an estimate of type (4.18) is true is given in [Ci4]. See also [CS] for a maximal function approach to the study of R_α in Orlicz spaces and for capacitary estimates of the Lebesgue points of $R_\alpha f$.

We conclude this section by considering n-dimensional Hardy type inequalities. The standard version of this inequality states that if G is an open bounded subset of Euclidean space $\mathbb{R}^n$ having a smooth boundary ∂G and $d(x)$ denotes the distance of the point $x \in G$ from ∂G, then, for every $p \in [1, \infty]$ and $\alpha > -1 + 1/p$, a positive constant C exist such that

$$\left\| \frac{u}{d^{1+\alpha}} \right\|_{L^p(G)} \leq C \left\| \frac{Du}{d^\alpha} \right\|_{L^p(G)} \tag{4.19}$$

for all sufficiently smooth functions u vanishing on ∂G. A proof of inequality (4.19) can be found e.g. in [K], [OK]. Let us mention that such inequalities and their generalizations have applications, for example, in the theory of degenerate partial differential equations.

We are concerned here with the problem of finding necessary and sufficient conditions on the real number α and on the Young functions A and B for the inequality

$$\left\| \frac{u}{d^{1+\alpha}} \right\|_{L^B(G)} \le C \left\| \frac{Du}{d^{\alpha}} \right\|_{L^A(G)} \tag{4.20}$$

to hold for every smooth and bounded open subset G of $\mathbb{R}^n$ and all functions u from the space

$$\begin{aligned} V_0^{1,A}(G \,|\, d^{-\alpha}) = \{u : u \text{ is a real-valued function on } G \text{ such that}\\ \text{the continuation of } u \text{ by } 0 \text{ outside } G\\ \text{is a weakly differentiable function on } \mathbb{R}^n\\ \text{and } |Du| d^{-\alpha} \in L^A(G)\}. \end{aligned} \tag{4.21}$$

These conditions are provided by the following

Theorem 4.8. *Let A and B be Young functions and let $\alpha > -1$. Then for every open set G having a Lipschitz-continuous boundary there exists a positive constant C such that inequality (4.20) holds for all $u \in V_0^{1,A}(G \,|\, d^{-\alpha})$ if and only if either*

$$\begin{gathered} \alpha > 0 \text{ and there exist numbers } k > 0 \text{ and } \bar{s} \ge 0 \text{ such that}\\ B(s) \le A(ks) \quad \text{for } s \ge \bar{s}, \end{gathered} \tag{4.22}$$

or

$$\begin{gathered} \alpha = 0 \text{ and there exist numbers } k > 0 \text{ and } \bar{s} \ge 0 \text{ such that}\\ s \int_{\bar{s}}^{s} \frac{B(r)}{r^2}\, dr \le A(ks) \quad \text{for } s \ge \bar{s}, \end{gathered} \tag{4.23}$$

or

$$\begin{gathered} -1 < \alpha < 0 \text{ and there exist numbers } k > 0 \text{ and } \bar{s} \ge 0 \text{ such that}\\ \left(\int_{ks}^{\infty} \left(\frac{r}{A(r)} \right)^{-1-1/\alpha} dr \right)^{-\alpha} \left(\int_{\bar{s}}^{s} \frac{B(r)}{r^{1+1/(\alpha+1)}}\, dr \right)^{\alpha+1} \le k\\ \text{for } s \ge \bar{s}. \end{gathered} \tag{4.24}$$

If $\alpha \le -1$, inequality (4.20) cannot be true whatever A and B are.

As an example, consider the limiting case of inequality (4.19) when $\alpha = -1 + 1/p$. Then Theorem 4.8 ensures that inequality (4.20) holds with $B(s) = s^p$ and $A(s) = (s \log(e + s))^p$.

In the one-dimensional situation, inequality (4.20) is equivalent to

$$\left\| s^{-1-\alpha} \int_0^s \phi(r)\, dr \right\|_{L^B(0,V)} \le C \left\| \frac{\phi}{s^\alpha} \right\|_{L^A(0,V)}, \tag{4.25}$$

where V is a positive number and $\phi : [0, V] \to \mathbb{R}$. The operator defined at ϕ by $T\phi(s) = s^{-1-\alpha} \int_0^s \phi(r)\, dr$ is easily verified to be of type (∞, ∞). Moreover, T is of strong type (1,1) if $\alpha > 0$ and of weak type $(1/(1 + \alpha), 1/(1 + \alpha))$ if $-1 < \alpha \le 0$. Thus, a version of Theorem 4.8 for inequality (4.25) follows from an interpolation theorem of Calderón (Theorem 2.2, Chap. 3 of [BS]) when $\alpha > 0$ and from Theorem 4.1 when $-1 < \alpha \le 0$. The n-dimensional case can be treated via a similar approach, after making use of local coordinates. The proof of necessity of conditions (4.22)–(4.24) is based on the choice of a ball as domain G and of radial test functions in inequality (4.20), but requires some technical lemmas. We refer to [Ci5] for a specific treatment of the subject.

4.4 Proof of Theorem 4.1, sketched

We present here the main steps of the proof of Theorem 4.1. The details can be found in [Ci7]. Consider the case when p_i and q_i are finite and $q_0 \le q_1$ (the other cases are analogous). Inequality (4.9) will follow if we show that there exists a constant C (having the form specified in the statement) such that, if f is any ν_1-measurable function on M_1 satisfying

$$\int_{M_1} A(|f(x)|)\, d\nu_1 \le 1, \tag{4.26}$$

then

$$\int_{M_2} B\left(\frac{|Tf(y)|}{C} \right) d\nu_2 \le 1. \tag{4.27}$$

Let a and b be the left-continuous derivatives of A and B, respectively. Thus,

$$\int_{M_1} A(|f(x)|)\, d\nu_1 = \int_0^\infty \nu_1(\{|f| > s\}) a(s)\, ds$$

and

$$\int_{M_2} B\left(\frac{|Tf(y)|}{K}\right) d\nu_2 = \int_0^\infty \nu_2(\{|Tf| > Ks\}) b(s)\, ds.$$

Setting $f = f_{\rho(t)} + f^{\rho(t)}$, where $f_{\rho(t)} = \operatorname{sign}(f)\min\{\rho(t), |f|\}$ and $\rho(t)$ is a function to be specified later, making use of the subadditivity of T and of the endpoint weak type estimates, and recalling (2.6), we reduce the problem to the inequalities

$$\begin{aligned}\Big(\int_0^\infty \frac{B(t)}{t^{1+q_1}}\Big(\int_0^{\rho(t)} \nu_1(\{|f| > s\})^{1/p_1}\, ds\Big)^{q_1} dt\Big)^{1/q_1} \\ \leq \text{Const.}\Big(\int_0^\infty \nu_1(\{|f| > t\})\frac{A(t)}{t}\, dt\Big)^{1/p_1}\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}\Big(\int_0^\infty \frac{B(t)}{t^{1+q_0}}\Big(\int_{\rho(t)}^\infty \nu_1(\{|f| > s\})^{1/p_0}\, ds\Big)^{q_0} dt\Big)^{1/q_0} \\ \leq \text{Const.}\Big(\int_0^\infty \nu_1(\{|f| > t\})\frac{A(t)}{t}\, dt\Big)^{1/p_0}.\end{aligned} \tag{4.29}$$

Inequalities (4.28)–(4.29) are weighted Hardy type inequalities, whose characterization (see [OK]) ensures that they hold, whatever $\nu_1(\{|f| > s\})$ is, provided that condition (4.8) is fulfilled and $\rho(t) = E^{-1}_{A,p_1'}(\sigma/H_{B,q_1}(t))$ for $t \geq 0$. This proves the sufficiency of condition (4.8). As for its necessity, one can use an argument by Calderón ([Ca]) to deduce the boundedness from $L^A(0,\infty)$ into $L^B(0,\infty)$ of the one-dimensional operator S, defined on ϕ by

$$S\phi(s) = \int_0^\infty \phi(r)\, \frac{d}{dr}\min\left\{\frac{r^{1/p_0}}{s^{1/q_0}}, \frac{r^{1/p_1}}{s^{1/q_1}}\right\} dr \quad \text{for } s > 0.$$

Such a boundedness is in turn equivalent to a couple of weighted Hardy type inequalities in Orlicz spaces. Necessary conditions for those inequalities to hold are not difficult to derive (see e.g. the proof of the sharpness of Theorem 3.1). The remaining part of the proof consists in a technical lemma showing that those conditions imply (4.8).

5 Boundedness of solutions to variational problems

In this and in the next section we prove some regularity properties of solutions to variational problems under general growth conditions. In the present section boundary value problems, i.e. global problems, are focused, whereas Section 6 deals with local questions. We stress that the approaches in these two settings are of quite different types.

Let us mention that results concerning differential problems with not necessarily polynomial growth have been considered in a number of papers, including [BL], [D], [FS], [Go1], [Go2], [Go3], [GM], [GIS], [LM], [L], [Mar], [MP1], [MP2], [MT].

5.1 Statement of the problem and results

Consider the problem of the calculus of variations

$$\begin{cases} \min \int_G F(x, u, Du)\, dx \\ u = u_0 \quad \text{on } \partial G, \end{cases} \tag{5.1}$$

where G is an open subset of $\mathbb{R}^n$, $n \geq 2$, having finite measure; F is a Carathéodory function from $G \times \mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}$; u_0 is a prescribed boundary datum.

Our assumptions on the integrand F amount to requiring that there exist A, B and s_0 such that

$$F(x, s, \xi) \geq A(|\xi|) - B(|s|) \tag{5.2}$$

$$F(x, s, 0) \leq B(|s|) \tag{5.3}$$

for $|s| \geq s_0$, $\xi \in \mathbb{R}^n$ and a.e. $x \in G$. Here, s_0 is a non-negative number, A is a finite-valued Young function and B is an increasing function from $[0, \infty)$ into $[0, \infty)$.

The boundary datum u_0 is assumed to be a bounded weakly differentiable function on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} A(|Du_0|)\, dx < \infty$.

The competing functions u in problem (5.1) are taken from the class $K^A_{u_0}$ defined as

$$\begin{aligned} K^A_{u_0} = \Big\{ u : \ & u \text{ is a real-valued weakly differentiable function in } G, \\ & \int_G A(|Du|)\, dx < \infty \\ & \text{and the continuation of } u - u_0 \text{ by } 0 \text{ outside } G \\ & \text{is a weakly differentiable function in } \mathbb{R}^n \Big\}. \end{aligned} \tag{5.4}$$

We are concerned with conditions on A and B ensuring that any minimizer of problem (5.1) is bounded in G. Our result can be stated as follows.

Theorem 5.1. *Assume that a positive constant c exists such that*

$$B(s) \leq A_n(cs) \tag{5.5}$$

for large s, where A_n *is the Sobolev conjugate of A defined by* (3.7). *Then any minimizer u of problem* (5.1) *is bounded.*

Remark 5.2. When

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{n'-1} dt < \infty,$$

$A_n(s) = \infty$ for large s, so that condition (5.5) is certainly satisfied. In fact, every function from the class $K^A_{u_0}$ is automatically bounded in this case, by Corollary 3.14.

Remark 5.3. Let us point out that, unlike most results in the theory of calculus of variations and of partial differential equations, the boundedness result of Theorem 5.1 does not have a corresponding a priori estimate for the maximum.

Example 5.4. In the special case where A and B are powers, Theorem 1 reproduces and slightly improves classical results appearing in [LU] and [S]. Indeed, choose $A(s) = s^p$ for some $p \in [1, n]$ (when $p > n$ every $u \in K^A_{u_0}$ is bounded by the Sobolev embedding theorem). Then Theorem 5.1 ensures that any minimizer of problem (5.1) is bounded provided that either $p < n$ and $B(s) \leq cs^{p^*}$ or $p = n$ and $B(s) \leq e^{cs^{n'}}$ for some $c > 0$ and for large s. The boundedness of minimizers of (5.1) follows from Theorem 3.2, Chap. 5 of [LU] or Theorem 6.2 of [S] under the stronger assumption that $B(s) \leq cs^q$ for some $q < p^*$ in case $p < n$ and for any $q > 0$ in case $p = n$.

Theorem 5.1 can also be shown to improve a result from [Ta4].

Example 5.5. The preceding example can be generalized on taking into account functions $A(s)$ having the form $s^p \log^q(e + s)$, where either $p > 1$ and $q \in \mathbb{R}$, or $p = 1$ and $q \geq 0$. Theorem 5.1 and Example 3.17 tell us that minimizers of (5.1) are bounded if

$$B(s) \leq \begin{cases} cs^{p^*} (\log s)^{nq/(n-p)} & \text{if } 1 \leq p < n \\ \exp\left(cs^{n/(n-1-q)}\right) & \text{if } p = n,\ q < n-1 \\ \exp\left(\exp\left(cs^{n'}\right)\right) & \text{if } p = n,\ q = n-1 \end{cases}$$

for large s. When either $p > n$ or $p = n$ and $q > n - 1$, then every $u \in K^A_{u_0}$ is bounded (see Remark 5.2).

Now let us discuss the boundedness of solutions to boundary value problems of the type

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x,u,Du) + b(x,u,Du) = 0 & \text{in } G \\ u = u_0 & \text{on } \partial G. \end{cases} \tag{5.6}$$

Here, a_i, $i = 1, \ldots, n$, and b are Carathéodory functions from $G \times \mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}$ satisfying growth conditions of the form

$$\sum_{i=1}^{n} a_i(x,u,\xi)\xi_i \geq A(|\xi|) - B(|s|) \tag{5.7}$$

$$\operatorname{sign}(s)\, b(x,s,\xi) \leq C(|s|) + D(|s|)E(|\xi|) \tag{5.8}$$

for $|s| \geq s_0$, $\xi \in \mathbb{R}^n$ and a.e. $x \in G$, where s_0 is a positive number, A is a finite-valued Young function and B, C, D, E are increasing functions from $[0, \infty)$ into $[0, \infty)$.

We consider weak solutions to problem (5.6) from the class $K^A_{u_0}$, where u_0 is a function as above (in particular bounded). A function $u \in K^A_{u_0}$ will be called a weak solution to (5.6) if

$$\int_G \sum_{i=1}^{n} a_i(x,u,Du)\frac{\partial \phi}{\partial x_i} - b(x,u,Du)\phi(x)\, dx = 0 \tag{5.9}$$

for all test functions $\phi \in K^A_0$. Here, K^A_0 is defined as in (5.4) with $u_0 \equiv 0$.

The next theorem gives conditions on the functions A, B, C, D, E ensuring that every weak solution to (5.6) is bounded in G.

Theorem 5.6. *Assume that:*
i) $A \circ E^{-1}$ *is a Young function;*
ii) *There exist constants* $c > 0$ *and* $k > 1$ *such such*

$$B(s) \leq A_n(cs) \tag{5.10}$$

$$C(s) \leq \frac{1}{s} A_n(cs) \tag{5.11}$$

$$D(s) \leq \frac{1}{ks}(((A \circ E^{-1})^{\sim})^{-1} \circ A_n)(cs) \tag{5.12}$$

for large s. *Then any weak solution to problem* (5.6) *is bounded.*

Remark 5.7. When $A, \ldots, E$ are positive powers, $A(s) = s^p$, $B(s) = s^\alpha$, $C(s) = s^\beta$, $D(s) = s^\gamma$, $E(s) = s^q$, say, then Theorem 5.6 states that weak solutions to problem (5.6) are bounded provided that

$$1 \le p < n, \quad q \le p - 1 + \frac{p}{n}, \quad \alpha \le \frac{np}{n-p},$$
$$\beta \le \frac{np}{n-p} - 1, \quad \gamma \le n\frac{p-q}{n-p} - 1.$$

This result should be compared with Theorems 7.1, Chap. 4, and 3.1, Chap. 5 of [LU], where equality is not allowed in the inequalities involving α, β and γ.

5.2 Proof of Theorem 5.1

In this section we outline the proof of Theorem 5.1, that of Theorem 5.6 being similar. In order to avoid technical complications, we drop the term $B(cs)$ in condition (5.2), i.e. we assume that F satisfies (5.3) and

$$F(x, s, \xi) \ge A(|\xi|). \tag{5.13}$$

Moreover, we assume that (5.3) and (5.13) are fulfilled for every $s \ge 0$.

Suppose, by contradiction, that $\operatorname{ess\,sup}|u| = \infty$. Let $t > 0$ and set $v(x) = \operatorname{sign}(u)\min\{t, |u(x)|\}$. Clearly, $v \in K^A_{u_0}$ provided that $t > t_0$, where $t_0 = \sup|u_0|$. The minimum property of u ensures that

$$\int_G F(x, u, Du)\, dx \le \int_G F(x, v, Dv)\, dx. \tag{5.14}$$

Hence, by conditions (5.3) and (5.13), and assumption (5.5), one easily deduces that

$$\int_{\{|u|>t\}} A(|Du|)\, dx \le A_n(ct)\mu(t) \tag{5.15}$$

where $\mu(t) = \mu_u(t)$, the distribution function of u. By Jensen's inequality we have

$$A\left(\frac{1}{\mu(t)}\int_{\{|u|>t\}} |Du|\, dx\right) \le \frac{1}{\mu(t)}\int_{\{|u|>t\}} A(|Du|)\, dx. \tag{5.16}$$

Moreover, on making use of the coarea formula ([M], [Zi]) and of the standard isoperimetric inequality in $\mathbb{R}^n$ one can show that

$$nC_n^{1/n}\int_t^\infty \mu(\tau)^{1/n'}\, d\tau \le \int_{\{|u|>t\}} |Du|\, dx \tag{5.17}$$

if $t > t_0$ (see e.g. [Ta4], proof of Theorem 1]). Combining (5.15)–(5.17) and integrating yields

$$\frac{C_n^{1/n'}}{n}\left(\int_s^\infty \frac{dt}{[A^{-1}(A_n(ct))]^{1/n'}}\right)^n \le \int_s^\infty \mu(t)^{1/n'}\,dt. \tag{5.18}$$

This is already a contradiction in the case when the integral on the left-hand side of (5.18) diverges. On the contrary, if such an integral converges, one can conclude as follows. Remark 3.5 ensures that $\int_G A_n(\lambda|u|)\,dx < \infty$ for every $\lambda > 0$. Thus, the function

$$\omega_\lambda(t) = \int_{\{|u|>t\}} A_n(\lambda|u|)\,dx$$

is finite for every $\lambda > 0$ and $t \ge 0$, and

$$\lim_{t\to+\infty} \omega_\lambda(t) = 0. \tag{5.19}$$

Since $\omega_\lambda(t) \ge A_n(\lambda t)\mu(t)$ for $t \ge 0$, we have

$$\int_s^\infty \frac{\omega_\lambda(t)^{1/n'}}{A_n(\lambda t)^{1/n'}}\,dt \ge \int_s^\infty \mu(t)^{1/n'}\,dt \quad \text{for } s \ge 0. \tag{5.20}$$

On combining (5.18) and (5.20) and making use of (5.19), we get

$$\lim_{s\to+\infty}\left(\int_s^\infty \frac{1}{[A^{-1}(A_n(t))]^{1/n'}}\,dt\right)\left(\int_s^\infty \frac{1}{A_n(kt)^{1/n'}}\,dt\right)^{-1} = 0 \tag{5.21}$$

for every $k > 0$. This is impossible by the following lemma ([Ci6]).

Lemma 5.8. *Let $k > 2$. Then there exists a positive constant C, depending only n and k, such that*

$$C\int_s^\infty \frac{dt}{A_n(kt)^{1/n'}} \le \left(\int_s^\infty \frac{dt}{[A^{-1}(A_n(t))]^{1/n'}}\right)^n \quad \textit{for } s > 0. \tag{5.22}$$

6 Higher integrability of the gradient of minimizers

6.1 Statement of the problem

In the present section we deal with local minimizers of functionals having the form

$$J(u,G) = \int_G F(x,u,Du)\,dx, \tag{6.1}$$

where G is an open subset of $\mathbb{R}^n$, $n \geq 2$; $u : G \to \mathbb{R}^N$, $N \geq 1$, is a weakly differentiable function; F is a Carathéodory function from $G \times \mathbb{R}^n \times \mathbb{R}^{nN}$ into $\mathbb{R}$.

Our assumptions on F read as follows:

$$\begin{aligned} A(|\xi|) - b(x)E(|z|) - a(x) &\leq F(x, z, \xi) \\ &\leq cA(|\xi|) + b(x)E(c|z|) + a(x) \end{aligned} \tag{6.2}$$

for every $(z, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ and a.e. $x \in G$. Here, A is a generalized Young function, i.e. a function from $[0, \infty)$ into $[0, \infty)$ such that

$$\frac{A(s)}{s} \text{ is increasing and } A(s) = 0 \text{ if and only if } s = 0, \tag{6.3}$$

which will be assumed to satisfy the Δ_2-condition. E is an increasing function from $[0, \infty)$ into $[0, \infty)$, a and b are non-negative locally integrable functions on G and c is a constant ≥ 1.

A function u will be called a local minimizer of J if for every open set $G' \subset\subset G$

$$\int_{G'} A(|Du|)\, dx < \infty$$

and

$$J(u, G') \leq J(u + v, G')$$

for every weakly differentiable function $v : G' \to \mathbb{R}^N$, with compact support, such that $\int_{G'} A(|Dv|)\, dx < \infty$.

The local boundedness of local minimizers of J can be proved in the scalar case ($N = 1$) under suitable assumptions on E, a and b ([L]; see also [MP1], [MP2]). Here we are concerned with the regularity of the gradient of minimizers of J and, specifically, with a higher integrability property. Notice that such a property is well-known in the case where the functions A and E are powers ([GG]).

6.2 A model case

Let us begin by considering the model case where

$$A(|\xi|) \leq F(x, z, \xi) \leq c(1 + A(|\xi|)). \tag{6.4}$$

Our result is stated in Theorem 6.1 below. Earlier results in the same direction are contained in [BL], [FS], [GIS].

Theorem 6.1. *Let u be a local minimizer of* (6.1). *Assume that* (6.4) *holds. Under the above assumptions on A, for every subset $G' \subset\subset G$ there exists $\delta > 0$ such that*

$$\int_{G'} A(|Du|) \left(\frac{A(|Du|)}{|Du|}\right)^{\delta} dx < \infty.$$

The proof of Theorem 6.1 consists of the following three steps:
a Caccioppoli inequality, a Sobolev inequality and a Gehring type lemma.

Under assumption (6.4), the Caccioppoli inequality has the form

$$\int_{Q_R} A(|Du|)\, dx \le c \left(\int_{Q_{2R}} A\left(\frac{|u-\lambda|}{R}\right) dx + R^n \right) \tag{6.5}$$

for every $\lambda \in \mathbb{R}^N$, every $Q_{2R} \subset\subset G$ and some positive constant c. Here, and in what follows, Q_R denotes *a* cube having sides of length R and Q_{2R} denotes a cube with the same centre as Q_R and sides (of length $2R$) parallel to those of Q_R. The proof of inequality (6.5) is based on the choice of test functions v given by

$$v = \eta(u - \lambda),$$

where η is a suitable cut-off function, and proceeds along standard lines—see [GG] for the case when A is a power and [Sb] for a general A satisfying (6.3) and the Δ_2-condition.

The Sobolev inequality is needed in order to estimate the integral on the right-hand side of (6.5) by a term of the type

$$\int_{Q_{2R}} B(|Du|)\, dx,$$

where $B : [0, \infty) \to [0, \infty)$ grows at infinity more slowly that A (in a suitable sense). When $A(t) = t^p$ such an estimate holds with $B(t) = t^q$ and $1 \le q < p \le q^*$, by the classical Sobolev inequality. In the general case, a norm inequality like (3.25) or an integral inequality like (3.28) seem to be of no use in the present framework. A new Sobolev inequality in integral form, having a different structure, has to be used.

Theorem 6.2. *Let A be any function satisfying* (6.3). *Set*

$$B(s) = A(s) \left(\frac{A(s)}{s}\right)^{-1/n} \qquad \textit{for } s > 0. \tag{6.6}$$

Then there exists a constant c, depending only on n, such that

$$A^{-1}\left(\frac{1}{m_n(Q_R)}\int_{Q_R} A\left(\frac{|u-m(u)|}{R}\right)dx\right) \leq cB^{-1}\left(\frac{1}{m_n(Q_R)}\int_{Q_R} B(c|Du|)\,dx\right) \tag{6.7}$$

for every cube $Q_R \subset \mathbb{R}^n$ and every weakly differentiable function $u : Q_R \to \mathbb{R}$. Here

$$m(u) = \sup\left\{t \in \mathbb{R} : m_n(\{x \in Q_R : u(x) > t\}) > \frac{m_n(Q_R)}{2}\right\}, \tag{6.8}$$

the median of u in Q_R.

Notice that Theorem 6.2 applies in a very general setting, but may fail to reproduce optimal results in standard situations. For instance, it does not include the classical Sobolev inequality. This fact should not surprise, since, unlike the classical Sobolev inequality, the constant c in (6.7) depends only on the dimension n. However, this lack of optimality does not affect the proof of Theorem 6.1.

The next result contains the Gehring type lemma, which we need in a local version. Such a result yields a higher integrability property for functions satisfying a reverse Jensen inequality.

Lemma 6.3. *Let A be any function satisfying* (6.3) *and the Δ_2-condition. Let Q be any cube in $\mathbb{R}^n$ and let f and g be non-negative integrable functions in Q. Assume that there exists a constant c such that*

$$\frac{1}{m_n(Q_R)}\int_{Q_R} A(f)\,dx \leq c\left(A\left(\frac{1}{m_n(Q_{2R})}\int_{Q_{2R}} f\,dx\right) + \frac{1}{m_n(Q_{2R})}\int_{Q_{2R}} A(g)\,dx\right) \tag{6.9}$$

for every cube $Q_{2R} \subset Q$. Furthermore, assume that

$$\int_Q A(g)\left(\frac{A(g)}{g}\right)^{\sigma} dx < \infty$$

for some $\sigma > 0$. Then for every subcube $Q' \subset\subset Q$ there exists $\varepsilon \in (0,\sigma)$, depending on n, A, c, $\operatorname{dist}(Q', \partial Q)$ and $\int_Q A(f)\,dx$, such that

$$\int_{Q'} A(f)\left(\frac{A(f)}{f}\right)^{\varepsilon} dx < \infty. \tag{6.10}$$

We refer to [CF] for the proofs of Theorem 6.2 and Lemma 6.3, as well as for the proofs of the results of the next section. Let us just say here that, owing to (2.24), inequality (6.7) can be reduced to a one-dimensional Hardy type inequality, which can be established by techniques related to those of [BK]. As for Lemma 6.3, even if the basic idea goes back to the paper by Gehring [Ge], the approach that we use is that of [Stre].

Proof of Theorem 6.1. From inequality (6.5) and Theorem 6.2 (applied to the components of u) we deduce that a positive constant c exists such that

$$\begin{aligned}\frac{1}{m_n(Q_R)}\int_{Q_R} A(|Du|)\,dx \\ \leq c\Big(A\circ B^{-1}\Big(\frac{1}{m_n(Q_{2R})}\int_{Q_{2R}} B(|Du|)\,dx\Big)+1\Big)\end{aligned} \tag{6.11}$$

for every cube $Q_{2R}\subset G$. Notice that in (6.11) we have made use of the Δ_2-condition for B and $A\circ B^{-1}$. The result then follows from Lemma 6.3 with $f=B(|Du|)$ and A replaced by $A\circ B^{-1}$, a function satisfying (6.3). □

6.3 The general case

Let us consider now the general situation when the integrand F satisfies the complete growth condition (6.2). In this context, assumptions on the summability of $a(x)$ and a balance between the degree of summability of $b(x)$ and the growth of E are needed. To this purpose, let us introduce the functions F_q and G_q defined for $q>1$ and $s>0$ by

$$F_q(s)=A(s)\left(\frac{A(s)}{s}\right)^{-1/q}, \quad G_q(s)=A\circ F_q^{-1}(s). \tag{6.12}$$

Notice that both F_q and G_q satisfy (6.3) and that $F_n\equiv B$, the function defined in (6.6).

Our assumption on $a(x)$ is that a number $r_1>1$ exists such that

$$\int_{G'} G_n^{-1}(a(x))\left(\frac{a(x)}{G_n^{-1}(a(x))}\right)^{r_1'} dx<\infty \qquad \text{for every } G'\subset\subset G. \tag{6.13}$$

As far as $b(x)$ and E are concerned, we require that there exist $\alpha>0$ and $r_2>1$ such that

$$\int_{G'} G_n^{-1}(b_\alpha(x))\left(\frac{b_\alpha(x)}{G_n^{-1}(b_\alpha(x))}\right)^{r_2'} dx<\infty \qquad \text{for every } G'\subset\subset G. \tag{6.14}$$

Here, $b_\alpha(x) = \widetilde{\Phi}_\alpha(b(x))$, where $\Phi_\alpha(s) = F_{r_2} \circ H^{-1}(\alpha E^{-1}(s))$ and H is the function defined by (3.6). The inverses H^{-1} and E^{-1} are taken left-continuous.

Notice that, at least when $\lim_{s\to\infty} A(s)/s = \infty$ (the only case of interest in this section), condition (6.14) forces E to grow slower than the Sobolev conjugate A_n of A defined by (3.7), in the sense that

$$\lim_{s\to\infty} \frac{A_n(\alpha s)}{E(s)} = \infty \tag{6.15}$$

for some $\alpha > 0$. Indeed, if (6.14) holds for some $\alpha > 0$ and some function $b(x)$ which does not vanish identically, then $\widetilde{\Phi}_\alpha(s)$ must be finite for some $s > 0$. This is in turn true if and only if $\liminf_{s\to\infty} \Phi_\alpha(s)/s > 0$, i.e. if and only if $\liminf_{s\to\infty} F_{r_2} \circ H^{-1}(\alpha s)/E(s) > 0$. The last inequality obviously implies (6.15).

In particular, if (3.30) is fulfilled, then $\widetilde{\Phi}_\alpha$ is linear at infinity whatever E is. Consequently, E can be any function and condition (6.14) can be equivalently written with b_α replaced by b. This is consistent with the fact that u, an hence $E(|u|)$, is locally bounded whenever (3.30) is in force.

For technical reasons, in this subsection we shall also assume, without loss of generality, that A is linear near 0. Indeed, A can be replaced, if necessary, by the function $\widehat{A}(s)$ which equals $A(s)$ for $s \geq 1$ and agrees with $A(1)s$ for $0 \leq s < 1$. The new function $\widehat{A}$ still satisfies (6.3) and the Δ_2-condition. Moreover, condition (6.2) is fulfilled with A and $a(x)$ replaced by $\widehat{A}$ and $\widehat{a}(x) = a(x) + A(1)$, respectively. After these replacements, conditions (6.13)–(6.14) are easily verified to be equivalent to the original ones. Such a modification of A does not affect the higher integrability result contained in the next theorem.

Theorem 6.4. *Let u be a local minimizer of* (6.1). *Assume that* (6.2) *holds. Under the above assumptions on A, E, a and b, for every subset $G' \subset\subset G$ there exists $\delta > 0$ such that*

$$\int_{G'} A(|Du|) \left(\frac{A(|Du|)}{|Du|} \right)^\delta dx < \infty.$$

Example 6.5. Consider the case where $A(s) = s^p$ for some $p > 1$. If $p < n$, then our assumptions are certainly satisfied if $E(s) = s^q$ with $q < p^*$, $b \in L^\sigma_{\mathrm{loc}}(G)$ for some $\sigma > p^*/(p^* - q)$ and $a \in L^\gamma_{\mathrm{loc}}(G)$ for some q and $\gamma > 1$. Notice that these are exactly the hypotheses of [GG]. When $p > n$, (6.13)–(6.14) amount to saying that $a, b \in L^\sigma_{\mathrm{loc}}(G)$ for some $\sigma > 1$. In the limiting situation where $p = n$, assumptions (6.13)–(6.14) are fulfilled provided that $E(s) = e^{cs^{n'}}$ for some $c > 0$, and $a, b \in L^\sigma_{\mathrm{loc}}(G)$ for some $\sigma > 1$.

Besides Theorem 6.2 and Lemma 6.3, the proof of Theorem 6.4 requires a Caccioppoli type estimate which now takes the following form.

Lemma 6.6. *Let* u *be a local minimizer of* (6.1). *Assume the same hypotheses as in Theorem 6.4. Let* $G' \subset\subset G$. *Then there exist positive constants* c *and* R_0 *such that for every* $\lambda \in \mathbb{R}^N$

$$\int_{Q_R} A(|Du|)\,dx \le c\Big(\int_{Q_{2R}} A\left(\frac{|u-\lambda|}{R}\right)dx + \int_{Q_{2R}} G_\rho^{-1} \circ A_n(c|u|)\,dx + \int_{Q_{2R}} (b_\alpha(x) + a(x) + 1)\,dx\Big) \tag{6.16}$$

if $R \le R_0$ *and* $Q_{2R} \subset G'$. *Here* $\rho = \max\{r_1, r_2, n\}$.

Let us point out that in the derivation of Lemma 6.6 and Theorem 6.4 the Poincaré inequality for Orlicz-Sobolev functions, in the form of (3.28), has also to be used.

References

[A1] R. A. Adams, *Sobolev spaces.* Academic Press, New York 1975.

[A2] R. A. Adams, *On the Orlicz-Sobolev imbedding theorem.* J. Funct. Anal. **24** (1977), 241–257.

[BR] C. Bennett and K. Rudnick, *On Lorentz-Zygmund spaces.* Dissertationes Mathematicae, CLXXV, Warszawa 1980.

[BS] C. Bennett and R. Sharpley, *Interpolation of operators.* Pure and Applied Mathematics, Vol. 129, Academic Press, Boston 1988.

[BL] T. Bhattacharya and F. Leonetti, *A new Poincaré inequality and its applications to the regularity of minimizers of integral functionals with nonstandard growth.* Nonlinear Anal. **17** (1991), 833–839.

[BK] S. Bloom and R. Kerman, *Weighted* L_Φ *integral inequalities for operators of Hardy type.* Studia Math. **110** (1994), 35–52.

[BZ] J. E. Brothers and W. P. Ziemer, *Minimal rearrangements of Sobolev functions.* J. Reine Angew. Math. **384** (1988), 153–179.

[Ca] A.P. Calderón, *Spaces between* L^1 *and* L^∞ *and the theorem of Marcinkiewicz.* Studia Math. **26** (1966), 273–299.

[Ci1] A. Cianchi, *On relative isoperimetric inequalities in the plane.* Boll. Un. Mat. Ital. (7) **3-B** (1989), 289–325.

[Ci2] A. Cianchi, *Continuity properties of functions from Orlicz-Sobolev spaces and embedding theorems.* Ann. Scuola Norm. Sup. Pisa. Cl. Sci. (4) **23** (1996), 575–608.

[Ci3] A. Cianchi, *A sharp embedding theorem for Orlicz–Sobolev spaces.* Indiana Univ. Math. J. **45** (1996), 39–65.

[Ci4] A. Cianchi, *Strong and weak type inequalities for some classical operators in Orlicz spaces.* To appear in J. London Math. Soc.

[Ci5] A. Cianchi, *Hardy inequalities in Orlicz spaces.* To appear in Trans. Amer. Math. Soc.

[Ci6] A. Cianchi, *Boundedness of solutions to variational problems under general growth conditions.* Comm. Partial Differential Equations **22** (1997), 1629–1646.

[Ci7] A. Cianchi, *An optimal interpolation theorem of Marcinkiewicz type in Orlicz spaces.* J. Funct. Anal. **153** (1998), 357–381.

[Ci8] A. Cianchi, *A fully anisotropic Sobolev inequality.* Preprint no. 15, 1998, Dip. Mat. "U. Dini", Univ. di Firenze.

[CF] A. Cianchi and N. Fusco, *Gradient regularity for minimizers under general growth conditions.* J. reine angew. Math. **227** (1998), 166–186.

[CS] A. Cianchi and B. Stroffolini, *An extension of Hedberg's convolution inequality and applications.* To appear in J. Math. Anal. Appl.

[D] T. K. Donaldson, *Nonlinear elliptic boundary value problems in Orlicz-Sobolev spaces.* J. Differential Equations **10** (1971), 507–528.

[DT] T. K. Donaldson and N. S. Trudinger, *Orlicz-Sobolev spaces and imbedding theorems.* J. Funct. Anal. **8** (1971), 52–75.

[EGO] D. E. Edmunds, P. Gurka and B. Opic, *Double exponential integrability of convolution operators in generalized Lorentz-Zygmund spaces.* Indiana Univ. Math. J. **44** (1995), 19–44.

[EOP] W. D. Evans, B. Opic and L. Pick, *Interpolation of operators on scales of generalized Lorentz-Zygmund spaces.* Math. Nachr. **182** (1996), 127–181.

[FLS] N. Fusco, P. L. Lions and C. Sbordone, *Some remarks on Sobolev imbeddings in borderline cases.* Proc. Amer. Math. Soc. **124** (1996), 561–565.

[FS] N. Fusco and C. Sbordone, *Higher integrability of the gradient of minimizers of functionals with nonstandard growth conditions.* Comm. Pure Appl. Math. **43** (1990), 673–683.

[Ge] F. W. Gehring, *The L^p-integrability of the partial derivatives of a quasi conformal mapping.* Acta Math. **130** (1973), 265–277.

[GG] M. Giaquinta and E. Giusti, *Quasi-minima.* Ann. Inst. H. Poincaré, Anal. Non Linéaire **1** (1984), 79–107.

[Go1] J.-P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients.* Trans. Amer. Math. Soc. **190** (1974), 163–205.

[Go2] J.-P. Gossez, *Orlicz-Sobolev spaces and nonlinear elliptic boundary value problems.* In: Nonlinear Analysis, Function Spaces and Applications, Vol. 1. S. Fučík and A. Kufner (eds.), Teubner, Leipzig 1979, 59–94.

[Go3] J.-P. Gossez, *A strong nonlinear elliptic problem in Orlicz-Sobolev spaces.* Proc. Sympos. Pure Math. **45** (1986), 455–462.

[GM] J.-P. Gossez and V. Mustonen, *Variational inequalities in Orlicz-Sobolev spaces.* Nonlinear Anal. **11** (1987), 379–392.

[GIS] L. Greco, T. Iwaniec and C. Sbordone, *Variational integrals of nearly linear growth.* Differential Integral Equations **10** (1997), 687–716.

[GP] J. Gustavsson and J. Peetre, *Interpolation of Orlicz spaces.* Studia Math. **60** (1977), 33–59.

[HMT] J. A. Hempel, G. R. Morris and N. S. Trudinger, *On the sharpness of a limiting case of the Sobolev imbedding theorem.* Bull. Austral. Math. Soc. **3** (1970), 369–373.

[H] R. A. Hunt, *An extension of the Marcinkiewicz interpolation theorem to Lorentz spaces.* Bull. Amer. Math. Soc. **70** (1964), 803–807.

[KR] M. A. Krasnosel'skii and Ya. B. Rutickii, *Convex functions and Orlicz spaces.* P. Noordhoff, Groningen 1961.

[Kr] W. T. Kraynek, *Interpolation of sublinear operators in generalized Orlicz and Hardy-Orlicz spaces.* Studia Math. **43** (1972), 93–123.

[K] A. Kufner, *Weighted Sobolev spaces.* Teubner, Leipzig 1980.

[KJF] A. Kufner, O. John and S. Fučík, *Function Spaces.* Noordhoff Int. Publ., Leyden 1977.

[LU] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and quasilinear elliptic equations.* Academic Press, New York 1968.

[LM] R. Landes and V. Mustonen, *Pseudo monotone mappings in Sobolev-Orlicz spaces and nonlinear boundary value problems on unbounded domains.* J. Math. Anal. Appl. **88** (1982), 25–36.

[L] G. M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations.* Comm. Partial Differential Equations **16** (1991), 311–361.

[Mal] L. Maligranda, *Orlicz spaces and interpolation.* Seminários de Matemática, Univ. Estadual de Campinas 1989.

[Mar] P. Marcellini, *Regularity for elliptic equations with general growth conditions.* J. Differential Equations **105** (1993), 296–333.

[Mark] J. Marcinkiewicz, *Sur l'interpolation d'operateurs.* C. R. Acad. Sci. Paris (1939), 1272–1273.

[MP1] E. Mascolo and G. Papi, *Local boundedness of minimizers of integrals of the calculus of variations.* Ann. Mat. Pura Appl. **167** (1994), 323–339.

[MP2] E. Mascolo and G. Papi, *Harnack inequality for minimizers of integral functionals with general growth conditions.* Nonlinear Differential Equations Appl. **3** (1996), 232–244.

[M] V. G. Maz'ya, *Sobolev spaces.* Springer-Verlag, Berlin 1985.

[MT] V. Mustonen and M. Tienari, *An eigenvalue problem for generalized Laplacian in Orlicz-Sobolev spaces.* To appear in Proc. Royal Soc. Edinburgh A.

[O] R. O'Neil, *Fractional integration in Orlicz spaces.* Trans. Amer. Math. Soc. **115** (1965), 300–328.

[OK] B. Opic and A. Kufner, *Hardy-type inequalities.* Longman Scientific & Technical, Harlow 1990.

[Pe] J. Peetre, *A new approach in interpolation spaces.* Studia Math. **34** (1970), 23–42.

[Po] S.I. Pokhozhaev, *On the imbedding Sobolev theorem for* $pl = n$ (Russian.) Doklady conference, Section Math., Moscow Power Institute (1965), 158–170.

[Pu] E.I. Pustyl'nik, *On optimal interpolation and some interpolation properties of Orlicz spaces.* Soviet Math. Dokl. **27** (1983).

[Ra] M.M. Rao, *Interpolation, ergodicity and martingales.* J. Math. Mech. **16** (1966), 543–567.

[RR] M.M. Rao and Z.D. Ren, *Theory of Orlicz spaces.* Marcel Dekker Inc., New York 1991.

[Ri] W.J. Riordan, *On the interpolation of operators.* PhD Dissertation, Univ. Chicago 1956 (see also Notices Amer. Math. Soc. **5** (1958), 590).

[Sa] E.T. Sawyer, *Boundedness of classical operators on classical Lorentz spaces.* Studia Math. **96** (1990), 145–158.

[Sb] C. Sbordone, *On some integral inequalities and their applications to the Calculus of Variations.* Boll. Un. Mat. Ital., Analisi Funzionale e Applicazioni, Ser VI **5** (1986), 73–94.

[S] G. Stampacchia, *On some regular multiple integral problems in the calculus of variations.* Comm. Pure Appl. Math. **16** (1963), 383–421.

[Stre] E.W. Stredulinsky, *Higher integrability from reverse Hölder inequalities.* Indiana Univ. Math. J. **29** (1980), 407–413.

[Stri] R.S. Strichartz, *A note on Trudinger's extension of Sobolev's inequalities.* Indiana Univ. Math. J. **21** (1972), 841–842.

[Ta1] G. Talenti, *Best constant in Sobolev inequality.* Ann. Mat. Pura Appl. **110** (1976), 353–372.

[Ta2] G. Talenti, *Elliptic equations and rearrangements.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **3** (1976), 697–718.

[Ta3] G. Talenti, *An embedding theorem.* Essays of Math. Analysis in honour of E. De Giorgi, Birkhäuser, Boston 1989.

[Ta4] G. Talenti, *Boundedness of minimizers.* Hokkaido Math. J. **XIX** (1990), 259–279.

[Ta5] G. Talenti, *Inequalities in rearrangement invariant function spaces.* In: Nonlinar Analysis, Function Spaces and Applications, vol. 5. M. Krbec, A. Kufner, B. Opic and J. Rákosník (eds.), Prometheus Publishing House, Prague 1994, 177–230.

[To1] A. Torchinsky, *Interpolation of operators and Orlicz classes.* Studia Math. **59** (1976), 177–207.

[To2] A. Torchinsky, *Real variable methods in harmonic analysis.* Academic Press, San Diego 1986.

[Tr] N.S. Trudinger, *On imbeddings into Orlicz spaces and some applications.* J. Math. Mech. **17** (1967), 473–483.

[Y] V. I. Yudovich, *Certain estimates connected with integral operators and solutions of elliptic equations.* Dokl. Akad. Nauk SSSR **138** (1961), 805–808.

[Zi] W. P. Ziemer, *Weakly differentiable functions.* Springer-Verlag, New York 1989.

[Z] A. Zygmund, *On a theorem of Marcinkiewicz concerning interpolation of operations.* J. Math. Pures Appl. **35** (1956), 223–248.