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WEIGHTED ESTIMATES FOR CLASSICAL INTEGRAL OPERATORS

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This paper deals with the problem of a full characterization of couples of weighted function spaces for which a given integral operator is bounded from one of these spaces into the other. According to coincidence and non-coincidence of the weights involved these problems are called one weight and two weight problems, resp.

Today the weighted inequalities and their applications to various mathematical problems are very rapidly developing part of the harmonic analysis. In last years, solutions to many serious problems in the theory of functions turned out to be closely linked with the weight theory.

Particularly, extensive and intensive study of integral operators in weighted function spaces began in the early 70's after the paper [1] by B. Muckenhoupt who showed that the weighted inequalities for the classical maximal operators are true iff the weight function satisfies the so called A_p condition. Later, solutions were given to one weight problems for Hilbert transform, fractional maximal functions, fractional integrals, and other classical integral operators [2], [3].

The two weight problem in Lebesgue spaces for the classical maximal functions was solved by E. Sawyer [4]. The same author solved in [5] the problem of a full charecterization of pairs of weight functions in the two weight weak type inequality, and another equivalent condition, perhaps more easily verifiable, was given by M. Gabidzashvili [6]. Quite recently, E. Sawyer [7] has found a characterization of pairs of weights, guaranteeing validity of the two weight strong type inequality for fractional integrals. It turned out that, using the results of [6] and [7], the solution to the last problem can be given in a more visible form.

In this lecture we present solutions of some weight (and unweight) problems for classical integral operators, in particular, for maximal functions, potentials, Riesz transforms, and for their generalizations in the spaces of the homogeneous type. These results have been obtained recently by participants of the seminar on weighted function spaces and integral operators in the Tbilisi Institute of Mathematics of the Georgian Academy of Sciences and some of them also in collaboration with our Czech colleagues from the Mathematical Institute of the Czechoslovak Academy of Sciences in Prague.

1. Fractional integrals in $\mathbf{R^n}$

First of all we will discuss some unweighted problems for the potential type integral operator

$$I_\gamma(f d\mu) = \int_{R^n} \frac{f(y)d\mu}{|x-y|^{\alpha-\gamma}}\,, \quad 0<\gamma<\alpha\leq n\,,$$

where μ is a Borel measure.

The question is as follows: What is the full description of such measures μ for which the operator I_γ acts continuously from $L^p(R^n, d\mu)$ into $L^q(R^n, d\mu)$ when $1 < q \leq \frac{\alpha p}{\alpha - p\gamma}$, $1 < p < \frac{\alpha}{\gamma}$?

For the measure μ we introduce the following function

$$\Omega(x) = \sup_{r>0} \frac{\mu B(x,r)}{r^\alpha}\,.$$

PROPOSITION 1.1. *Let us suppose that the function $\Omega(x)$ is finite almost everywhere on a μ-measurable set E. Then for arbitrary f from $L^p(R^n, d\mu)$ with $1 < p < \frac{\alpha}{\gamma}$, the function $I_\gamma(f d\mu)$ is finite almost everywhere (in μ sense) on E.*

THEOREM 1.1 [9]. *Suppose that $0<\gamma<\alpha\leq n$, $1<p<\frac{\alpha}{\gamma}$, $\frac{1}{q}=\frac{1}{p}-\frac{\gamma}{\alpha}$. Then there exists a positive constant c_1 such that for every F from $L^p(R^n, d\mu)$ we have*

$$\left(\int_{R^n} (I_\gamma(f d\mu))^q\,(\Omega(x))^{(\gamma/\alpha-1)q}\,d\mu\right)^{1/q} \leq c_1 \left(\int_{R^n} |f(x)|^p\,d\mu\right)^{1/q}$$

This theorem and some other consideration yield the following

THEOREM 1.2 [9]. *If $\frac{1}{q}=\frac{1}{p}-\frac{\gamma}{\alpha}$, then the following conditions are equivalent:*

i) $I_\gamma(f d\mu)$ acts continuously from $L^p(R^n, d\mu)$ into $L^q(R^n, d\mu)$ when $1<p<\frac{\alpha}{\gamma}$.

ii) I_γ is of the weak type $(1, \frac{\alpha}{\alpha-\gamma})$ (with respect to the measure μ)

iii) There is a positive constant c_2 such that

$$\mu B(x,r) \leq c_2\, r^\alpha$$

for every $x\in R$, $r>0$.

Here $B(x,r)$ denotes a ball in R^n centered at x and with the radius r.

In general we have the following

THEOREM 1.3 [9]. *Suppose that* $1 < q \le \dfrac{\alpha p}{\alpha - p\gamma}$, $1 < p \le \dfrac{\alpha}{\gamma}$. *Then the operator* I_γ *acts continuously from* $L^p(R^n, d\mu)$ *into* $L^q(R^n, d\mu)$ *if and only if there exists a positive constant* c_3 *such that*

$$\mu B(x,r) \;\le\; c_3 r^{((\alpha-\gamma)pq)/(pq-q+p)} .$$

Theorem 1.3 contains the well-known Sobolev embedding theorem [11]. Let E be a Borel subset of R^n with the condition that its Hausdorff α-measure $\mathcal{H}_\alpha E$ is positive.

We have the following

THEOREM 1.4 [9]. *Suppose that* $1 < p < \alpha\gamma^{-1}$, $1 < q \le \dfrac{\alpha p}{\alpha - \gamma p}$. *Then the inequality*

$$\left(\int\limits_E \left(\int\limits_E \frac{f(y)\, d\mathcal{H}_\alpha}{|x-y|^{\alpha-\gamma}}\right)^q d\mathcal{H}_\alpha\right)^{1/q} \le c \left(\int\limits_E |f(x)|^p \, d\mathcal{H}_\alpha\right)^{1/p}$$

holds for every $f \in L^p(E, \mathcal{H}_\alpha)$ *with a constant* $c > 0$ *independent of* f *if and only if*

$$\mathcal{H}_\alpha(E \cap B(x,r)) \;\le\; c_1 r^{((\alpha-\gamma)pq)/(pq+p-q)} ,$$

where c *does not depend on* x *and* r.

Now suppose that Γ is a rectifiable curve in the complex plane (with finite or infinite length). We shall consider the potential type operators

$$\mathcal{K}_\gamma f(t) = \int\limits_\Gamma \frac{f(\tau)}{|t-\tau|^{1-\gamma}}\, d\tau , \quad 0 < \gamma < 1 .$$

THEOREM 1.5 [8]. *If* $1 < p < \gamma^{-1}$, $1 < q \le \dfrac{p}{1-p\gamma}$, *then the following two statements are equivalent:*

i) K_γ *acts continuously from* $L^p(\Gamma)$ *into* $L^q(\Gamma)$.

ii) The curve Γ *satisfies the condition*

$$|\Gamma \cap B(z,r)| \le c\, r^{((1-\gamma)pq)/(pq-q+p)} ,$$

where c *does not depend on* $z \in \Gamma$ *and* $r > 0$.

Note that a generalization of these results for another variant of the fractional order integral in spaces of the homogeneous type was proposed in [12].

Now we consider weight problems for integral operators with positive kernels in weighted Lorentz space. This is a set of measurable functions $f : R^n \to R^1$ for which the following quasinorm

$$\|f\|_{L^{ps}_w} = \begin{cases} \left(p \int\limits_0^\infty (w\{y \in R^n : |f(x)| > \lambda\})^{s/p} \lambda^{s-1}\, d\lambda\right)^{1/s} & \text{for } 1 \le p < \infty,\ 1 \le s < \infty , \\ \sup\limits_\lambda \lambda (w\{x \in R^n : |f(x)| > \lambda\})^{1/p} & \text{for } 1 \le p < \infty,\ s = \infty , \end{cases}$$

is finite.

Further, we shall say that a function $w : R \to R$ is weight function, if w is local integrable and positive almost everywhere. For the weight function w and a Borel set E we denote

$$wE = \int_E w(x)\,dx\,.$$

Let us consider the integral operator

$$\mathcal{K}f(x) = \int_{R^n} K(x,y)f(y)\,dy\,,$$

where $K : R^n \times R^n \to R^1$ is positive and $f \geq 0$.

THEOREM 1.6 [14]. *Let $1 \leq s \leq p < q < \infty$, $w : R^n \to R^1$ be a weight function, let $\psi : R^n \to R^1$ be a nonnegative locally integrable function, and ν a Borel measure. If*

$$\sup_{\substack{x\in R^n\\ r>0}} (\nu B(x,2r))^{1/q}\,\|\chi_{R^n\setminus B(x,r)}w^{-1}\psi K(x,.)\|_{L_w^{p's'}} < \infty\,, \tag{1.1}$$

then there exists a positive constant c_4 such that for any $\lambda > 0$ and an arbitrary nonnegative f from L_w^{ps} we have

$$\nu\{x \in R^n : \mathcal{K}(f\psi)(x) > \lambda\} \;\leq\; c_4\,\lambda^{-q}\,\|f\|^q_{L_w^{ps}}\,.$$

In [17] a condition for pairs of weight functions can be found which is necessary and sufficient for the operator $\mathcal{K}$ to be the weak type mapping from one weighted Lorentz space into another one in the case that the kernel K is an anisotropic radial decreasing function and $1 < p \leq q < \infty$.

Let us consider the generalized potential operator

$$T_\gamma(\phi,t) = \int_{R^n} \frac{\phi(y)}{(|x-y|+t)^{n-\gamma}}\,dy\,, \qquad 0<\gamma<n\,, \quad t>0.$$

We suppose that β is a measure on $R_+^{n+1} = R^n \times [0,+\infty)$, $\widehat{B}(x,r) = B(x,r) \times [0,2r)$.

THEOREM 1.7 [15]. *Let $1 \leq s \leq p < q < \infty$. Then the following statements are equivalent:*

i) There is a positive constant c_5 such that the inequality

$$\beta\{(x,t) \in R_+^{n+1} : T_\gamma(f\psi)(x,t) > \lambda\} \;\leq\; c\lambda^{-q}\|f\|^q_{L_w^{ps}}\,,$$

holds for every $\lambda > 0$ and an arbitrary function f from L_w^{ps}

ii) $\sup\limits_{\substack{x\in R^n\\ r>0,t>0}} (\beta\widehat{B}(x,r))^{1/q}\|\chi_{R^n\setminus B(x,r)}\psi(|x-.|+t)^{\gamma-n}w^{-1}\|_{L_w^{p's'}} < \infty.$

From the above results and solution of the two weight problem for the Riesz potential which was recently given by E. Sawyer [7] we are able to give a necessary and sufficient condition for validity of the two weight strong type inequality which is more convenient from the point of view of the verification.

THEOREM 1.8. *Let $1 < p < q < \infty$. Then the inequality*

$$\Big(\int_{R^n} (I_\gamma(f)(x))^q v(x)\,dx\Big)^{1/q} \le c\Big(\int_{R^n} |f(x)|^p w(x)\,dx\Big)^{1/q}$$

with a constant $c > 0$ independent of $f \in L^p_w(R^n)$ holds if and only if

$$\sup_{\substack{x\in R^n\\ r>0}} (vB(x,r))^{1/q} \|\chi_{R^n\setminus B(x,r)} w^{-1} |x - .|^{\gamma-n}\|_{L^{p'}_w} < \infty$$

and

$$\sup_{\substack{x\in R^n\\ r>0}} (w^{1-p'}(B(x,r)))^{1/p'} \|\chi_{R^n\setminus B(x,r)} |x - .|^{\gamma-n}\|_{L^q_v} < \infty.$$

2. Maximal functions and Riesz transforms

We start this section with an unweighted problem for the Hardy–Littlewood maximal function and the Riesz transforms. For a locally integrable function $f : R^n \to R^1$, the Hardy-Littlewood maximal function is defined by

$$Mf(x) = \sup \frac{1}{|B|}\int_B |f(y)|\,dy\,,$$

where sup is taken over all balls B containing the point x.

For a measurable function $f : R^n \to R^1$ satisfying the condition

$$\int_{R^n} \frac{|f(x)|\,dx}{(1+|x|)^n} < \infty,$$

the Riesz transforms $R_j f$ $(j = 1, 2, \dots, n)$ are defined as follows:

$$R_j f(x) = \lim_{\varepsilon\to 0} c_n \int_{|y|>\varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y)\,dy\,,$$

where $y = (y_1, \dots, y_n)$ and $c_n = \pi^{-(n+1)/2}\Gamma((n+1)/2)$.

We shall say that a function $\phi : R^1 \to R^1$ belongs to the class Φ, if ϕ is an even, nonnegative, nondecreasing function on $(0,\infty)$ such that $\phi(0+) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$. By $\phi(L)$ we denote the set of all measurable functions f for which

$$\int_{R^n} (\phi \circ f)(x)\,dx < \infty\,.$$

Definition. A nonnegative function $\phi : R^1 \to R^1$ is called *quasiconvex* if there exists a convex function ψ and a constant $c > 1$ such that

$$\psi(t) \le \phi(t) \le c\,\psi(ct) \qquad \text{for all } t \in R^1.$$

PROPOSITION 2.1. *If $\phi \in \Phi$, then the inequality*

$$\phi(\lambda)\, m\{\, x \in R^n : Mf(x) > \lambda\} \le c \int\limits_{R}^{n} (\phi \circ (cf))(x)\, dx$$

holds for every $\lambda > 0$ and $f \in \phi(L)$ with a constant c independent of λ and f if and only if the function ϕ is quasiconvex.

PROPOSITION 2.2. *Let $\phi \in \Phi$. Then the following two conditions are equivalent:*

i) There is a positive constant c such that

$$\phi(\lambda)\, m\{\, x \in R^n : R_j f(x) > \lambda\} \le c \int\limits_{R}^{n} (\phi \circ (cf))(x)\, dx\,.$$

ii) ϕ is quasiconvex and satisfies the Δ_2 condition.

THEOREM 2.1 [18]. *The following statements are equivalent:*

i) There exists a positive constant c_1 such that

$$\int\limits_{R}^{n} (\phi \circ Mf)(x)\, dx \le c_1 \int\limits_{R}^{n} (\phi \circ (c_1 f))(x)\, dx$$

for every f such that $c_1 f \in \phi(L)$.

ii) ϕ^α is quasiconvex for some $\alpha \in \phi(L)$.

iii) There exists a positive constant c_2 such that

$$\int\limits_{0}^{\sigma} \frac{\phi(s)}{s^2}\, ds \le c_2 \frac{\phi(c_2\sigma)}{\sigma}\,, \qquad 0 < \sigma < \infty\,.$$

If ϕ is a Young function, then it can be shown that all the above statements are equivalent to the Δ_2 condition for the function complementary to function ϕ.

For the Riesz transforms the following theorem is true.

THEOREM 2.2. *For the validity of the inequalities*

$$\int\limits_{R}^{n} (\phi \circ R_j f)(x)\, dx \le c \int\limits_{R}^{n} (\phi \circ (cf))(x)\, dx \qquad (j = 1, 2, \ldots, n)$$

it is necessary and sufficient that $\phi \in \Delta_2$ and ϕ^α be quasiconvex for some α, $0 < \alpha < 1$.

For the Hilbert transform and the conjugate function the analogous problem was solved in [21], [22], [23] in a different way. Further we discuss the following question: What is the relation between the classes of functions ϕ for which either the inequality for the scalar maximal function (the Riesz transform) holds and the classes of those functions ϕ for which the corresponding inequality for the vector–valued maximal function (the Riesz transform) is true?

THEOREM 2.3 [19]. *Suppose that $1 < \theta < \infty$. Then the following conditions are equivalent:*

i) The inequality

$$\phi(\lambda)\, m\{ x \in R^n : \Big(\sum_{j=1}^{\infty}(Mf_j(x))^\theta\Big)^{1/\theta} > \lambda\} \le c_1 \int_{R^n} (\phi \circ f)(x)\, dx$$

is valid for some positive c_1 independent of $f = (f_1, \cdots, f_n, \cdots)$ and $\lambda > 0$.

ii) The inequality

$$\phi(\lambda)\, m\{x \in R^n : \Big(\sum_{j=1}^{\infty}|R_k f_j(x)|^\theta\Big)^{1/\theta} > \lambda\} \le c_2 \int_{R^n} \phi \circ \Big(\sum_{j=1}^{\infty}\|f_j(x)\|^\theta\Big)^{1/\theta} dx$$

is valid for some positive c_2 independent of $f = (f_1, \ldots, f_n, \ldots)$ and $\lambda > 0$.

iii) ϕ is quasiconvex and $\phi \in \Delta_2$.

THEOREM 2.4 [19]. *Let us suppose that $\phi \in \Phi$ and $1 < \theta < \infty$. Then the following statements are equivalent:*

i) There is a positive constant c_1 such that for each $f = (f_1, \ldots, f_n, \ldots)$ the inequality

$$\int_{R^n} \phi \circ \Big(\sum_{j=1}^{\infty}(Mf_j)^\theta\Big)^{1/\theta}(x)\, dx \le c_1 \int_{R^n} \phi \circ \Big(\sum_{j=1}^{\infty} |f_j|^\theta\Big)^{1/\theta}(x)\, dx$$

holds.

ii) There is a positive constant c_2 such that

$$\int_{R^n} \phi \circ \Big(\sum_{j=1}^{\infty} |R_k f_j|^\theta\Big)^{1/\theta}(x)\, dx \le c_2 \int_{R^n} \phi \circ \Big(\sum_{j=1}^{\infty} |f_j|^\theta\Big)^{1/\theta}(x)\, dx .$$

iii) $\phi \in \Delta_2$ and ϕ^α is quasi-convex for some $\alpha \in (0, 1)$.

In the sequel, we present some general inequalities with Carleson measures for the maximal operator

$$\widetilde{M} f(x, t) = \sup |Q|^{-1} \int_Q |f(y)|\, dy$$

where the supremum is taken over all cubes $Q \subset R^n$ containing x and whose sides are of length at least $2^{-1}t$.

Let ϱ be a positive measure on $R^{n+1}_+ = R^n \times [0, \infty)$. L. Carleson [24] proved that $\widetilde{M}$ is bounded from $L^p(R^n)$ into $L^p(R^{n+1}_+, \varrho)$ and from $L^1(R^n)$ into $L^1_*(R^{n+1}, \varrho)$ if and only iff there exists a constant $c > 0$ such that

$$\text{(C)} \qquad \varrho\widehat{B}(x, r) \le c r^n ,$$

where $\widehat{B}(x, r) = B(x, r) \times [0, 2r)$. A measure ϱ on R^{n+1}_+ satisfying (C) will be called the Carleson measure.

We generalize this result for the functions from the Orlicz–Morrey spaces.

THEOREM 2.5 [25]. *If $\phi \in \Phi$ and ϕ^α is quasi–convex for some $\alpha \in (0,1)$, $0 \le \lambda < n$, then the inequality*

$$\sup_{x\in R^n, r>0} r^{-\lambda} \int_{\widehat{B}(x,r)} (\phi \mathrm{o} \widetilde{M} f)(y,\tau)\, d\varrho \le c_1 \sup_{x\in R^n, r>0} r^{-\lambda} \int_{B(x,r)} (\phi \mathrm{o} c_1 f)(y)\, dy$$

holds if and only if ϱ is a Carleson measure.

The case $\lambda = 0$ will be reformulated in the next theorem together with a weak type inequality. We shall consider Orlicz spaces in their standard form.

THEOREM 2.6 [25]. *Let ϱ be a positive measure on R_+^{n+1}, $\phi \in \Phi$ and suppose that ϕ^α is convex for some $\alpha \in (0,1)$. Then the following statements are equivalent:*

i) $\phi(\lambda)\, \varrho\left\{(x,t) \in R_+^{n+1} : \widetilde{M} f(x,t) > \lambda\right\} \le c_1 \int_{R^n} (\phi \mathrm{o} f)(x)\, dx$.

ii) $\int_{R^n} (\phi \mathrm{o} \widetilde{M} f)(x,t)\, d\varrho \le c_2 \int_{R^n} (\phi \mathrm{o} f)(x)\, dx$.

iii) *ϱ is a Carleson measure.*

iv) *There exists $c > 0$ such that*

$$\phi\left(\frac{\varepsilon |E|}{|B|}\right) \varrho(\widehat{B}) \le c\,\phi(\varepsilon)\,|E|$$

for every ball B and measurable $E \subset B$, and every positive ε.

If, moreover, ϕ is a Young function, then any of the above conditions implies

v) $\|\widetilde{M} f\|_{L_{\phi,\varrho}} \le c \|f\|_{L_\phi}$

Additionally, if ϕ is a Young function satisfying the Δ_2 condition, then (v) implies (i)–(iv).

For the Riesz transforms we have

THEOREM 2.7 [25], [26]. *Let ϕ and ψ be a couple of complementary Young functions, both satisfying the Δ_2 condition. Suppose w is a weight function. Then the statements below are equivalent:*

i) $\int_{R^n} (\phi \mathrm{o} R_j f)(x) w(x)\, dx \le c_1 \int_{R^n} (\phi \mathrm{o} f)(x) w(x)\, dx$

ii) $\|R_j f\|_{L_\phi(\varepsilon w)} \le c_2 \|f\|_{L_\phi(\varepsilon w)}, \quad \varepsilon > 0$

iii) $\phi(\lambda) \int_{\{x:|R_j f(x)|>\lambda\}} w(x)\, dx \le c_3 \int_{R^n} (\phi \mathrm{o} f)(x)\, w(x)\, dx\,, \quad j = 1,2,\ldots,n$.

iv) *$w \in A_{i(\phi)}$ (the Muckenhoupt class) where $i(\phi)$ is the lower index of ϕ.*

The proof of theorem 2.7 rests on the pioneering paper of R.Kerman and A. Torchinsky [27]. The generalization of results from [27] for anisotropic fractional maximal functions and Riesz potentials can be found in [28], [29].

Now we discuss the weight problem for the fractional maximal function.

$$M_\gamma f(x) = \sup_{Q\ni x} |Q|^{1-\gamma/n} \int_Q |f(y)|\, dy\,, \quad 0<\gamma<n\,.$$

THEOREM 2.8 [31]. *Let $1\le s\le p<q<\infty$. Then the inequality*

$$\|M_\gamma(f)\|_{L_v^{q\infty}} \le c_1 \|f\|_{L_w^{ps}}$$

is valid for some positive c independent of f if and only if

$$(vB)^{1/q} \|\chi_{B(x,r)} w^{-1}\|_{L_w^{p's'}} \le c_2 |B|^{1-\gamma/n}$$

for c_2 independent of B.

THEOREM 2.9 [31]. *Let $1<p<\dfrac{n}{\gamma}$, $\dfrac{1}{q}=\dfrac{1}{p}-\dfrac{n}{\gamma}$, $1<s<\infty$. Then the following estimates are equivalent:*

i) There is a constant $c>0$ such that for each $f\in L_w^{ps}$,

$$\|M_\gamma(fw^{\gamma/n})\|_{L_w^{qs}} \le c_1\|f\|_{L_w^{ps}}\,.$$

ii) There exists a constant $c_2>0$ such that

$$\|T_\gamma(fw^{\gamma/n})\|_{L_w^{qs}} \le c_2\|f\|_{L_w^{ps}}\,.$$

iii) $w\in A_s$ where $s=1+\dfrac{q}{p'}$.

The case of the Hardy–Littlewood maximal function was discussed earlier by H. M. Chung, R. Hunt, D. Kurtz [32].

Now we return to the Riesz transforms and present a complete description of the class of functions w for which the Riesz transforms map $Llog^+L(w)$ into L_w^1 or weak L^1. These problems for maximal functions were solved earlier by A. Carbery, S.–Y. A. Chang, J. Garnett [33]. For the Riesz transforms we have

THEOREM 2.10 [34]. *The following statements are equivalent:*

i) There exists a constant $c_1>0$ such that for any measurable function f supported in a cube Q we have the inequality

$$\int_Q |R_j f(x)| w(x)\, dx \le c_1 \left(\int_Q w(x)\, dx + \int_Q |f(x)| \log^+ |f(x)| w(x)\, dx \right)$$

where c_1 is independent of Q and f.

ii) There exist constants $\varepsilon > 0$ and $c_2 > 0$ such that for any cube Q and measurable subset $E \subset Q$,

$$\frac{1}{|Q|} \int_Q \exp\left(\varepsilon \frac{|R_j(\chi_E w)(x)|}{w(x)}\right) dx \le c_2 , \qquad j = 1, 2, \ldots, n .$$

THEOREM 2.11 [34]. *Let $0 < \alpha < \infty$. Then the following conditions are equivalent:*

i) There exists a constant $c_3 > 0$ such that for any $\lambda > 0$ and any measurable function f we have the estimate

$$w\{x \in R^n : |R_j f(x)| > \lambda\} \le c_3 \int_{R^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \frac{|f(x)|}{\lambda}\right)^\alpha w(x)\, dx \quad (j = 1, 2, \ldots, n).$$

ii) There exists a constant $c_4 > 0$ such that for any $\lambda > 0$ and any measurable function f we have

$$w\{x \in R^n : Mf(x) > \lambda\} \le c_4 \int_{R^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \frac{|f(x)|}{\lambda}\right)^\alpha w(x)\, dx .$$

iii) There exists a constant $\varepsilon > 0$ such that

$$\sup \frac{1}{|Q|} \int_Q \exp\left(\varepsilon \frac{1}{|Q| w(x)} \int_Q w(y)\, dy\right)^{1/\alpha} dx < \infty .$$

where the supremum is taken over all cubes Q.

Now we consider the strong maximal function

$$M_s f(x) = \sup_{x \in J} \frac{1}{|J|} \int_J |f(y)|\, dy,$$

where the supremum is taken over all rectangles with faces which are parallel to coordinate axes and contain the point x.

A collection of rectangles $\{R_j\}$ is said to satisfy *the weak overlapping condition*, if

$$\left| R_j \setminus \bigcup_{i=1}^{j-1} R_i \right| > \frac{1}{2} |R_j|$$

for each $j = 1, 2, \ldots$.

The following theorem provides a full description of the class of weight functions w for which the weak type weighted inequality for the strong maximal function is true.

THEOREM 2.12 [35]. *The weighted inequality*

$$w\{x \in R^n : M_s f(x) > \lambda\} \le c \int_{R^n} \frac{|f(x)|}{\lambda} \left(\log^+ \frac{|f(x)|}{\lambda}\right)^{n-1} w(x)\, dx$$

holds if and only if there exists a positive constants ε and c_1 such that for every collection $\{R_j\}$ of rectangles satisfying the weak overlapping condition the inequality

$$\int_{\bigcup_j R_j} \exp\left\{\varepsilon \sum_j \frac{w(R_j)}{|R_j| w(x)} \chi_{R_j}(x)\right\}^{1/(n-1)} w(x)\, dx \le c\, w\Big(\bigcup_j R_j\Big)$$

is fulfilled.

In the paper [16] it was shown that if $w \in A_1$ defined by rectangles, then the above weak type inequality holds, but there are examples of weight functions w which satisfy the above condition, but do not belong to A_1.

3. Maximal functions and potential type integrals in spaces of the homogeneous type

Let (X, ρ, μ) be the space of the homogeneous type, i.e. the space with a measure μ and equipped with a quasimetric ρ; the latter being a mapping $\rho : X \times X \to R^1_+$ such that

1) $\rho(x, y) = \rho(y, x)$ for every $x, y \in X$;

2) $\rho(x, y) = 0$ if and only if $x = y$;

3) $\rho(x, y) \le \eta(\rho(x, z) + \rho(z, y))$ for every $x, y, z \in X$

where the constant $\eta > 0$ is independent of x, y, and z.

We shall assume that all balls $B(x, r) = \{\, y \in X : \rho(x, y) < r \,\}$ are μ–measurable and that there is a constant $c > 0$ such that for every $x \in X$ and $r > 0$,

$$0 < \mu B(x, 2r) \le c\, \mu B(x, r) < \infty.$$

For a locally integrable function f on X we define the following maximal function

$$\widetilde{M}_\gamma f(x, t) = \sup (\mu B)^{\gamma - 1} \int_B |f(y)|\, d\mu, \qquad 0 \le \gamma < 1, \quad t > 0,$$

where the supremum is taken over all balls B containing the point x and with radius greater than $t/2$.

In what follows we also consider integrals of the potential type

$$T_\gamma f(x, t) = \int_X \frac{f(y)}{(\mu B(x, \varrho(x, y) + t))^{1-\gamma}}, \qquad 0 < \gamma < 1, \quad t \ge 0.$$

The functions M_γ and T_γ generalize the Hardy–Littlewood maximal function and the Riesz potential.

Let $\widehat{B}$ denote the cylinder $B \times [0, 2r)$, and let w be a weight function on X, i.e. w is locally integrable on X and positive almost everywhere in the μ–measure sense. Further, β will be a measure defined on the product of σ–algebras generated by balls from X and intervals from $[0, \infty)$.

We shall discuss the problem of a full description of pairs (w, β) for which weighted Lorentz–norm weak type inequalities and Lebesgue–norm strong type inequalities hold for operators $\widetilde{M}_\gamma$ and T_γ. Results about these operators in the Morrey spaces will be given, too. Below, we denote by $L^p(X, wd\mu)$ the Lebesgue space and by $L^{ps}(X, d\mu)$ the weighted Lorentz space defined for μ–measurable functions acting from X into R^1.

THEOREM 3.1 [41]. *Let $1 \le s \le p \le q < \infty$, $0 \le \gamma < 1$. Then the following statements are equivalent:*

i) There is a constant $c_1 > 0$ such that for any function f from $L^{ps}(X, wd\mu)$ and $\tau > 0$,

$$\beta\Big\{(x,t) \in X \times [0,\infty): \ \widetilde{M}_\gamma f(x,t) > \tau\Big\} \ \le \ c_1 \tau^{-q} \|f\|^q_{L^{ps}(X,\, d\mu)}$$

ii) There is a constant $c_2 > 0$ such that for any ball B from X,

$$(\beta\widehat{B})^{1/q} \|\chi_B w^{-1}\|_{L^{p's'}(X,\, d\mu)} \ \le \ c_2 \, (\mu B)^{1-\gamma} .$$

Note that Theorem 3.1 was established in [38] for the case $p = 1$.

Now we consider the two–weight weak type problem for the operator T_γ.

THEOREM 3.2 [41]. *Let $1 \le s \le p < q < \infty$. Then the following two conditions are equivalent:*

i) There is a positive constant c such that for any $f \in L^{ps}(X, wd\mu)$ the inequality

$$\beta\{(x,t) \in X \times [0,\infty): \ |T_\gamma f(x,t)| > \tau\} \ \le \ c\,\tau^{-q} \|f\|^q_{L^{ps}(X,\, wd\mu)}$$

holds.

ii) $\sup \beta\left(\widehat{B}(a, N(2r+t))\right)^{1/q} \|\chi_{R^N \setminus B(a,r)} w^{-1} (\mu B(a, \varrho(a,.)) + 1))^{s-1}\|_{L^{p's'}(X,\, wd\mu)} \ < c_1$,
with a positive constant c_1 is independent of $a \in X$, $r > 0$ and $t > 0$, $N = \eta + 4\eta^2$.

This theorem was proved earlier in [43] in the case when $d\beta = wd\mu \otimes \delta_0$ where δ_0 is Dirac measure supported at the origin and $p = s$. In the paper [42] there was obtained the solution to the one weight problem for the operator $T_\gamma f(x, 0) = T_\gamma f(x)$.

THEOREM 3.3 [43]. *Let* $1 < p < \frac{1}{\gamma}$, $\frac{1}{q} = \frac{1}{p} - \gamma$. *Then the necessary and sufficient condition for the validity of the inequality*

$$\left(\int_X |T_\gamma(fw^\gamma)(x)|^q\, w(x)d\mu\right)^{1/q} \le c\left(\int_X |f(x)|^p w(x)d\mu\right)^{1/p}$$

with some positive constant c *independent of* f *is*

$$\sup\left(\frac{1}{\mu B}\int_B w(x)d\mu\right)\left(\frac{1}{\mu B}\int_B w^{-1/(s-1)}(x)d\mu\right)^{s-1} < \infty\,,$$

where $s = 1 + \frac{q}{p'}$ *and the supremum is taken over all balls in* X.

The next theorem gives us the necessary and sufficient condition for the validity of two weight strong inequality for the operator $\widetilde{M}_\gamma$.

THEOREM 3.4 [44]. *Let* $1 < p \le q < \infty$, $0 \le \gamma < 1$ *and let the measure* $\beta = w^{-1/(p-1)}d\mu$ *satisfy the doubling condition, i.e.*

$$\int_{B(a,2r)} w^{-1/(p-1)}(x)\, d\mu \le c \int_{B(a,r)} w^{-1/(p-1)}(x)\, d\mu$$

for every ball $B(a,r)$. *Then the following conditions are equivalent:*

i) The inequality

$$\left(\int_{X\times[0,\infty)} \left(\widetilde{M}_\gamma f(x,t)\right)^q d\beta\right)^{1/q} \le c_1\left(\int_X |f(x)|^p\, w(x)\, d\mu\right)^{1/p}$$

holds for every function $f \in L^p(X, d\mu)$ *with a constant* c_1 *independent of* f.

ii) There exists a constant c_2 *such that*

$$\left(\int_{\widehat{B}} \left(\widetilde{M}_\gamma(\chi_B w^{-1/(p-1)})(x,t)\right)^q d\beta\right)^{1/q} \le c_2\left(\int_B w^{-1/(p-1)}(x)\, d\mu\right)^{1/p}.$$

Considering the case $w \equiv 1$ in Theorem 3.4 we have

THEOREM 3.5 [42]. *If* $1 < p < q < \infty$ *and* $0 < \gamma < 1$, *then the following statements are equivalent:*

i) The inequality

$$\left(\int_{X\times[0,\infty)} \left(\widetilde{M}_\gamma f(x,t)\right)^q d\beta\right)^{1/q} \le c_1\left(\int_X |f(x)|^p\, d\mu\right)^{1/p}$$

holds for every function $f \in L^p(X, d\mu)$.

ii) The inequality

$$\left(\int\limits_{X\times[0,\infty)} |T_\gamma f(x,t)|^q \, d\beta \right)^{1/q} \leq c_2 \left(\int\limits_X |f(x)|^p \, d\mu \right)^{1/p}$$

holds for any function $f \in L^p(X, d\mu)$.

iii) It is

$$\beta\widehat{B} \leq c_3 \, (\mu B)^{q(1/p-\gamma)}$$

for every ball B *from* X.

The constants c_1, c_2, c_3 *are independent of* f *and* B.

The equivalence of ii) and iii) in the case $d\beta = w d\mu \otimes \delta_0$ has been proved earlier in [45].

Let us mention several other results. In [46] the necessary and sufficient condition on the weight function was found which ensures continuity of the fractional maximal function with respect to the basis of the convex comparable sets in the weighted Lebesgue spaces. Under an additional condition on this basis a complete description of pairs of weight functions is given such that the fractional maximal function is continuous from one weighted Lebesgue space to the other.

Further, in [43] the Koosis problems were solved for the fractional maximal functions and fractional integrals defined on the homogeneous type spaces. In [47] there was obtained a complete description of pairs (v, w) of weights for which the potential operator in the space of the homogeneous type for a critical index acts continuously from the weighted Lebesgue space L^p_w into the space BMO_v — the weighted space of functions with bounded mean oscillation. In this case the Koosis type problem have been also solved.

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