Julian Musielak

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SOME APPLICATIONS OF GENERALIZED ORLICZ SPACES IN APPROXIMATION THEORY AND FOURIER SERIES

J. Musielak
Poznań, Poland

1. Basic notions

As W. Orlicz introduced and investigated at the beginning of 1930^{th} spaces, which were later called Orlicz spaces, this was done in connection with their applications to orthogonal expansions. Later development by H. Nakano (1950, 1951), W. A. I. Luxemburg and A. C. Zaanen (1955, 1960), and by M. A. Krasnoselskiĭ and Ya. B. Rutickiĭ (1958) was also connected with some applications, mostly in integral transforms and integral equations. Here, I should like to point out some applications in approximation theory and Fourier series, obtained lately.

First, some auxiliary notions will be recalled, as *modular space*, and *generalized Orlicz space* (see also [2]). A functional $\rho : X \to \overline{R}_+ = [0,\infty]$ on a real vector space X is called a *pseudomodular*, if

1^o $\rho(0) = 0$,

2^o $\rho(-x) = \rho(x)$,

3^o $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for $x, y \in X$, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

If, moreover, $\rho(x) = 0$ implies $x = 0$, ρ is called a *modular*. If in place of 3^o there holds the stronger condition

3^{oo} $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for $x, y \in X$, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$,

then ρ is called a *convex pseudomodular* (*modular*). The vector space

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0+\}$$

is called a *modular space*. The functional

$$|x|_\rho = \inf \{u > 0 : \rho(x/u) \leq u\}$$

is an *F-seminorm* in X_ρ (*F-norm*, if ρ is a modular). In case ρ is convex,

$$||x||_\rho = \inf \{u > 0 : \rho(x/u) \leq 1\}$$

is a *seminorm* in X_ρ (*norm*, if ρ is a modular), equivalent to $|\cdot|_\rho$. Convergence in $|\cdot|_\rho$ (or $||\cdot||_\rho$) of a sequence (x_n) , $x_n \in X_\rho$, to zero is equivalent to the condition $\rho(\lambda x_n) \to 0$ for every $\lambda > 0$. If (x_n) satisfies the weaker condition $\rho(\lambda_0 x_n) \to 0$ for a

$\lambda_0 > 0$, then it is called *modular convergent* (or *ρ-convergent*) to zero, which is denoted $x_n \overset{\rho}{\to} 0$.

Let $\phi : [a,b) \times R_+ \to R_+ = [0,\infty)$ satisfy the following conditions:

(i) $\phi(t,u)$ is a nondecreasing continuous function of u with $\phi(t,0) = 0$, $\phi(t,u) > 0$ for $u > 0$ and $\phi(t,u) \to \infty$ as $u \to \infty$, for all $t \in [a,b)$,

(ii) $\phi(t,u)$ is Lebesgue measurable with respect to t for all $u \geqq 0$.

If needed, we extend ϕ (b-a)-periodically to $\phi : R \times R_+ \to R_+$. If we put

$$(1) \qquad \rho(f) = \int_a^b \phi(t,|f(t)|)\, dt$$

for every measurable function f , finite almost everywhere, then ρ is a modular in the vector space X of all measurable, finite a.e. functions f in the interval $[a,b)$, with equality a.e. If $\phi(t,u)$ is a convex function of $u \geqq 0$ for all $t \in [a,b)$, then ρ is a convex modular. The modular space X_ρ is then called a *generalized Orlicz space* and is denoted by $L^\phi(a,b)$. If $\phi(t,u) = \phi(u)$ is independent of the variable t , then $L^\phi(a,b)$ is called an *Orlicz space.*

Now, let X be the vector space of all sequences $x = (t_n)$, and let (ϕ_n) be a sequence of functions such that $\phi_n(u)$ are nondecreasing, continuous functions of $u \geqq 0$ with $\phi_n(0) = 0$, $\phi_n(u) > 0$ for $u > 0$ and $\phi_n(u) \to \infty$ as $u \to \infty$, for all n . Let us write

$$(2) \qquad \rho(x) = \sum_n \phi_n(|t_n|) ,$$

then ρ is a modular in X (if ϕ_n are all convex, it is a convex modular). The respective modular space X_ρ is called a *generalized Orlicz sequence space* and is denoted by ℓ^ϕ . If $\phi_n(u) = \phi(u)$ is independent of n , then ℓ^ϕ is called an Orlicz sequence space.

2. Application in approximation theory

We shall consider (see [3]) the problem of approximation of elements f of the generalized Orlicz space $L^\phi(a,b)$ by means of integral transforms T_w defined as follows:

$$(3) \qquad (T_w f)(s) = \int_a^b K_w(t-s, f(t))\, dt ,$$

where $w \in \mathcal{W}$, $\mathcal{W}$ is a set of indices and $K_w : [a,b) \times R_+ \to R$ are integrable in $[a,b)$ with respect to the first variable, $K_w(t,0) = 0$. If needed, all functions under consideration are extended from $[a,b)$ to R, $(b-a)$-periodically. We suppose that there is given a filter $\mathfrak{M}$ of subsets of the set $\mathcal{W}$, and convergence in w will be always understood in the sense of this filter. The approximation problem under consideration is to estimate the error of approximation of f by $T_w f$, i.e. the expression $\rho(\lambda(T_w f - f))$ for sufficiently small $\lambda > 0$.

First, some auxiliary notions. Let

$$L_w(t) = \sup_{u \neq v} \frac{|K_w(u) - K_w(v)|}{|u-v|} , \quad L(w) = \int_a^b L_w(t)\, dt .$$

$K = (K_w)_{w \in \mathcal{W}}$ is called a *semisingular kernel,* if it satisfies the following two conditions:

1° $0 < \ell = \inf_{w \in \mathcal{W}} L(w) \leq \sup_{w \in \mathcal{W}} L(w) = L < \infty$,

2° there exist a $t_0 \in (a,b)$ of the form $t_0 = n(b-a)$, n is an integer, and a $\delta > 0$ such that

$$\int_a^{t_0-\delta} L_w(t)dt \to 0 \quad \text{and} \quad \int_{t_0+\delta}^b L_w(t)dt \to 0 \quad \text{in } w .$$

A semisingular kernel is called *singular,* if

$$r(w) = \sup_{u \neq 0} \left| \frac{1}{u} \int_a^b K_w(t,u)dt - 1 \right| \to 0 \quad \text{in } w .$$

We need still the notion of τ-boundedness of ϕ. Namely, ϕ is called *τ-bounded,* if there exist constants $k_1, k_2 \geq 1$ and a function $F : R \times R \to R_+$ measurable and $(b-a)$-periodic with respect to the first variable such that

$$\phi(t-v, u) \leq k_1\phi(t, k_2u) + F(t,v) \quad \text{for all } t, v \in R, \quad u \in R_+,$$

where

$$h(v) = \int_a^b F(t,v)\, dt$$

is bounded in R and $h(v) \to 0$ as $v \to 0$ and $v \to b-a$. We write

$H = \sup_{v \in R} h(v)$. If ϕ is convex, then we may take always $k_1 = 1$. If $\phi(t,u) = \phi(u)$ does not depend on t , then it is always τ-bounded, with $k_1 = k_2 = 1$ and $F = 0$.

Finally, the *ϕ-modulus of continuity* of f is defined by

$$\omega_\phi(f,\delta) = \sup_{|v| \leq \delta} \int_a^b \phi\big(t, |f(t+v) - f(t)|\big) dt$$

for $\delta > 0$. If $f \in L^\phi(a,b)$, then there exists a $\lambda_0 > 0$ such that $\omega_\phi(\lambda_0 f, \delta) \to 0$ as $\delta \to 0+$.

There holds the following approximation theorem.

THEOREM 2.1. *Let ϕ be convex and τ-bounded, $\int_a^b \phi(t,u)dt < \infty$ for all $u > 0$, and let $K = (K_w)_{w \in \mathcal{W}}$ be a singular kernel. Then T_w given by* (3) *are such that $T_w : L^\phi(a,b) \to L^\phi(a,b)$ for all $w \in \mathcal{W}$ and $T_w f \xrightarrow{\rho} f$ in w . Moreover, ρ from* (1) *satisfies the following inequality*

$$\rho\big(\lambda(T_w f - f)\big) \leq \frac{1}{2}\omega_\phi(2\lambda L f, \delta) + \frac{1}{4\ell}\Big[2\rho(4\lambda L k_2 f) + H\Big]\Big\{\int_a^{t_0-\delta} L_w(t)dt + \int_{t_0+\delta}^b L_w(t)dt\Big\} + \frac{1}{2}\rho(2\lambda r(w)f)$$

for all $f \in L^\phi(a,b)$ and $w \in \mathcal{W}$.

Let us remark that the right-hand side of this inequality can be made arbitrarily small taking w "large" according to the filter $\mathfrak{M}$.

Similar problem may be put and solved in the space ℓ^ϕ (see [4]). Only an outline will be given here. Namely, let $K_w = (K_{w,i})_{i=0}^\infty$, $w \in \mathcal{W}$, where $K_{w,i} : R_+ \to R_+$. The transforms T_w are now defined by $T_w x = \big((T_w x)_i\big)_{i=0}^\infty$ for $x = (t_i)_{i=0}^\infty$, where

$$(4) \qquad (T_w x)_i = \sum_{j=0}^{i} K_{w,i-j}(|t_j|) , \quad w \in \mathcal{W} .$$

Let

$$L_{w,i} = \sup_{u \neq v} \frac{|K_{w,i}(u) - K_{w,i}(v)|}{|u - v|} , \quad L(w) = \sum_{i=0}^\infty L_{w,i} .$$

The kernel $K = (K_w)_{w \in \mathcal{W}}$ is called *semisingular*, if

1° $\sup_{w \in \mathcal{W}} L(w) = L < \infty$, 2° $\frac{L_{w,j}}{L(w)} \to 0$ in w for $j = 1,2,3,\dots$.

A semisingular kernel is called *singular*, if

$$\frac{1}{c} K_{w,0}(c) - 1 \to 0 \quad \text{in} \quad w \quad \text{for every} \quad c > 0 .$$

The sequence $(\phi_i)_{i=0}^{\infty}$ is called τ_+*-bounded*, if there exist constants k_1, $k_2 \geq 1$ and an infinite matrix $(\varepsilon_{i,j})$, $\varepsilon_{i,j} \geq 0$ for $i, j = 0,1,2,\dots$, $\varepsilon_{i,0} = 0$, $\varepsilon_j = \sum_{i=0}^{\infty} \varepsilon_{i,j} \to 0$ as $j \to \infty$, $\varepsilon = \sup_{j \geq 0} \varepsilon_j < \infty$, such that

$$\phi_{i+j}(u) \leq k_1 \phi_i(k_2 u) + \varepsilon_{i,j} \quad \text{for} \quad u \geq 0 \quad \text{and} \quad i, j = 0,1,2,\dots$$

If ϕ_i are convex, one can take $k_1 = 1$. If $\phi_i(u) = \phi(u)$ do not depend on i, the sequence is always τ_+-bounded with $k_1 = k_2 = 1$, $\varepsilon_{i,j} = 0$.

Let us still write

$$x_w^{(j)}(c) = (\underbrace{0,0,\dots,0}_{j+1 \text{ times}}, K_{w,1}(c), K_{w,2}(c), \dots)$$

for any $c > 0$. Then there holds

THEOREM 2.2. *Let all ϕ_i be convex and let $(\phi_i)_{i=0}^{\infty}$ be τ_+-bounded. Let $K = (K_w)_{w \in \mathcal{W}}$ be a singular kernel. Moreover, let*

$$\rho(\lambda x_w^{(j)}(c)) \to 0 \quad \textit{in} \quad w \quad \textit{for all} \quad \lambda > 0 , \quad c > 0 , \quad j = 0,1,2,\dots .$$

Then $T_w : \ell^{\phi} \to \ell^{\phi}$ for all $w \in \mathcal{W}$ and $T_w x \overset{\rho}{\to} x$ in w for every $x \in \ell^{\phi}$, where T_w are given by (4).

Let us remark, that similarly as in Theorem 2.1, an estimation of the error of approximation may be also given.

3. Application in Fourier series

Let $a(f) = (a_n(f))_{n=2}^{\infty}$ and $b(f) = (b_n(f))_{n=2}^{\infty}$ be the Fourier coefficients of a 2π-periodic function f, integrable in $(0,2\pi)$. Starting with the famous results of S. Bernstein, 1914, and of A. Zygmund, 1928, one can find a series of theorems on absolute convergence of the Fourier series and their generalizations under assumptions on moduli of continuity and variation of the function f (see e.g. [6]). Among else, there is considered convergence:

(5) $$\sum_{n=2}^{\infty} n^{\beta}\left(|a_n(f)|^{\gamma} + |b_n(f)|^{\gamma}\right) < \infty , \quad \beta \geq 0 , \quad \gamma > 0 .$$

If we put $\phi_n(u) = n^\beta |u|^\gamma$, the condition (5) becomes equivalent to the following one: $a(f) \in \ell^\phi$ and $b(f) \in \ell^\phi$, where $\phi = (\phi_n)_{n=2}^\infty$. Denoting by $\omega_p(f,\delta)$ the p-th modulus of continuity of a function $f \in L^p_{2\pi}$, $1 < p \leqq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, the following result is obtained (see [5]), applying the Hausdorff-Young inequality:

THEOREM 3.1. *Let* $(\phi_n(u))_{n=2}^\infty$ *be a nondecreasing sequence for every* $u > 0$ *and let the functions* $\overline{\phi}_n(u) = \phi_n(u^{1/q})$ *be concave,* $f \in L^p_{2\pi}$. *Then*

$$\max\left[\rho(a(f)),\rho(b(f))\right] \leqq \frac{1}{2}\sum_{k=1}^\infty \rho_k(f) ,$$

where

$$\rho_k(f) = 2^k\phi_{2^k}\left(2^{-k/q}\,\omega_p(f, \frac{1}{2^k})\right) .$$

In particular, if $\sum_{k=1}^\infty \rho_k(\lambda f) < \infty$ *for a* $\lambda > 0$, *then* $a(f), b(f) \in \ell^\phi$.

From this theorem follows a number of known results from the theory of Fourier series, e.g. taking $f \in \mathrm{Lip}\,(\alpha,p)$ we obtain that if $\gamma \leqq q$, $\alpha\gamma p > \gamma + p(1 + p - \gamma)$, then (5) holds. However, one may write the theorem also in the setting of a continuity statement. Namely, $\rho'(x) = \sum_{k=1}^\infty \rho_k(x)$ is a pseudomodular in the space of all sequences. Thus, from Theorem 3.1 follows

COROLLARY 3.1. *Under the assumptions of* Theorem 3.1, $a : f \to a(f)$ *and* $b : f \to b(f)$ *are linear, continuous operators from the modular space* X_ρ, *to the generalized Orlicz sequence space* ℓ^ϕ , *both provided with modular convergence.*

Now, let $\overset{b}{\underset{a}{V_r}}(f)$ be the r-th variation on the sense of L.C. Young of a continuous function f in $[a,b]$, and let $V_r^{2\pi}$ be the space of all continuous, 2π-periodic functions f with $\overset{2\pi}{\underset{0}{V_r}}(f) < \infty$. Denoting by $\omega(f,\delta)$ the modulus of continuity of f , $1 < p \leqq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, the following result is obtained (see [1]):

THEOREM 3.2. *Let* $(\phi_n(u))_{n=2}^\infty$ *be a nondecreasing sequence for every* $u > 0$ *and let the functions* $\overline{\phi}_n(u) = \phi_n(u^{1/q})$ *be concave* , $f \in C_{2\pi}$. *Then*

$$\max\left[\rho(a(f)),\rho(b(f))\right] \leqq \frac{1}{2}\sum_{k=1}^\infty \tilde{\rho}_k(f) ,$$

where

$$\tilde{\rho}_k(f) = 2^k \phi_{2^k}\{2^{-k}(\overset{4\pi}{\underset{-2\pi}{V_r}}(f))^{r/p}\ \omega^{r/p}(f,\frac{\pi}{2^k})\} .$$

In particular, if $\sum_{k=1}^{\infty} \tilde{\rho}_k(\lambda f) < \infty$ *for a* $\lambda > 0$ *, then* $a(f), b(f) \in \ell^{\phi}$.

Taking $f \in \mathrm{Lip}\,\alpha$, $f \in V_r$ with some $1 \leq r \leq p$ in the above theorem one obtains that if $\gamma \leq q$, $\alpha\gamma(p - r) > p(1 + \beta - \gamma)$, then (5) holds, which again includes known results.

In order to obtain from Theorem 3.2 a continuity result, one must still introduce the notion of two-modular convergence (generalizing that of two-norm convergence by A. Alexiewicz), because $\tilde{\rho}_k$ is not a pseudomodular. If two pseudomodulars $\rho^{(1)}$ and $\rho^{(2)}$ are given in a real vector space X , then a sequence of $f_n \in X$ is called $(\rho^{(1)},\rho^{(2)})$*-convergent to zero,* if it is $\rho^{(1)}$-bounded (i.e. $\rho^{(1)}(\varepsilon_n f_n) \to 0$ for every $\varepsilon_n \to 0$) and $\rho^{(2)}$-convergent to zero. Now, let $X_{\rho^{(2)}}$ be the modular space of functions $f \in V_r$ generated by the pseudomodular

$$\rho^{(2)}(f) = \sum_{k=1}^{\infty} 2^k\ \phi_{2^k}\{2^{-k}\ \omega^{r/p}(f,\frac{\pi}{2^k})\},$$

and let

$$\rho^{(1)}(f) = (\overset{4\pi}{\underset{-2\pi}{V_r}}(f))^{r/p} .$$

Then we have

<u>COROLLARY 3.2</u>. *Under the assumptions of* Theorem 3.2, $a : f \to a(f)$ *and* $b : f \to b(f)$ *are linear, continuous operators from the space* $X_{\rho^{(2)}}$ *with* $(\rho^{(1)},\rho^{(2)})$*-convergence to the generalized Orlicz sequence space* ℓ^{ϕ} *with modular convergence.*

Let us still remark that all the above results may be transferred also to the case of the Haar orthonormal system. Also, r-th variation may be replaced by Φ-variation with an increasing, continuous function, $\Phi(0) = 0$, such that $u^p \leq C\Phi(u)\Psi(u)$ for all $u \geq 0$, Ψ - an increasing function, $C > 0$.

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