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A SURVEY ON TENT SPACES AND THEIR APPLICATION TO WEIGHTED INEQUALITIES

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1. Prehistory of the tent spaces

Tent spaces arise because of questions concerning Carleson measures. Carleson measures were introduced in connection with the corona problem and originally involved holomorphic functions in the unit disc. I shall give equivalent versions in the upper half-plane.

Consider a harmonic function in $\mathbb{R}^{n+1}_+ = \{(x,y) \mid x \in \mathbb{R}^n , y > 0\}$ with trace on $y = 0$ given by an L^p function $f(x)$. Then

(1) $$u(x,y) = \int P(x-z, y)\, f(z)\, dz ,$$

where P is the Poisson kernel

(2) $$P(x,y) = \frac{c_n\, y}{(|x|^2 + y^2)^{(n+1)/2}} ,$$

and Carleson [4] asked which measures $\mu \geqq 0$ satisfy

(3) $$\left(\iint |u(x,y)|^p\, d\mu(x,y)\right)^{1/p} \leqq A\left(\int |f(x)|^p\, dx\right)^{1/p}, \quad 1 < p < \infty .$$

He showed that it was necessary and sufficient that there exists a constant A such that for any cube Q with sides of length δ (always parallel to the coordinate axes),

(4) $$\mu(Q \times [0,\delta]) \leqq A|Q| .$$

Shortly there after Duren [10] considered a generalization of (3). He wanted to classify the measures μ for which

$$\left(\iint |u(x,y)|^q\, d\mu(x,y)\right)^{1/q} \leqq A\left(\int |f(x)|^p\, dx\right)^{1/p} , \quad 1 < p \leqq q < \infty$$

where $q > p$, and showed that the answer was an appropriate generalization of (4), namely that there exists a constant A such that

(5) $$\mu(Q \times [0,\delta]) \leqq A|Q|^{q/p} ,$$

for all cubes Q as above.

It was also known that the substitute for the integral inequality when $p = 1$ was

(6) $$\int |u(x,y)|\ d\mu(x,y) \leq A||f||_{H^1(\mathbb{R}^n)} ,$$

where $H^1(\mathbb{R}^n)$ is the Hardy space with equivalent definitions, but most simply thought of as the subspace of $L^1(\mathbb{R}^n)$ formed of those f for which

(7) $$u^+(x) = \sup_{y>0} |u(x,y)|$$

is in $L^1(\mathbb{R}^n)$. C. Fefferman [11] characterized the dual of $H^1(\mathbb{R}^n)$ as BMO where $f \in \mathrm{BMO}(\mathbb{R}^n)$ provided

(8) $$\sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|dx = ||f||_* < + \infty ,$$

with

(9) $$f_Q = \frac{1}{|Q|} \int_Q f(x)\ dx .$$

We can apply this result to (6) above noting that

$$\left|\iint u(x,y)\ d\mu(x,y)\right| \leq \int |u(x,y)|\ d\mu(x,y) \leq C||f||_{H^1}$$
$$= \left|\int f(x)\ P^*\mu(x)\ dx\right| ,$$

where

(10) $$P^*\mu(x) = \iint P(x-z,\ y)\ d\mu(z,y)$$

(at least formally - there are convergence questions but see [13]) and conclude that if μ is a Carleson measure, $P^*\mu$ which is called its *balayage*, belongs to $\mathrm{BMO}(\mathbb{R}^n)$.

In 1979, E. Amar and A. Bonami [2] sought to generalize this last result. This is equivalent to asking which measures μ have the property that

(11) $$\left(\int |u(x,y)|^q\ d\mu(x,y)\right)^{1/q} \leq A\left(\int |f(x)|^p\ dx\right)^{1/p}$$

for $q < p$. They did this and let me begin with the organization of Carleson measures that they used. Denote the *Carleson measures* by V^1

(12) $$V^1 = \{\mu \mid |\mu|(Q \times [0,\delta]) \leq C|Q|\} ,$$

and the *measures* considered by *Duren* as V^α

$$V^\alpha = \{\mu \mid |\mu|(Q \times [0,\delta]) \leq C|Q|^\alpha \text{ for all cubes } Q\} , \quad \alpha > 1$$

normed with

(13) $$\inf_{Q} \frac{|\mu|(Q \times [0,\delta])}{|Q|^{\alpha}} = ||\mu||_{V^{\alpha}} , \quad \alpha \geq 1 .$$

They generalized this to $\alpha < 1$ by introducing the *tent* over Ω : for an open set Ω

(14) $$T(\Omega) = \{(x,y) \mid B(x,y) \subseteq \Omega\} ,$$

and $B(x,y)$ denotes the ball of radius y about x .

Note that when Ω is a cube Q , a tent is geometrically the same as a cylinder because

(15) $$\tfrac{1}{2} Q \times [0,\delta/2] \subseteq T(Q) \subseteq Q \times [0,\delta] .$$

<u>DEFINITION</u>. $\mu \in V^{\alpha}(\mathbb{R}^{n+1}_{+})$, $0 < \alpha < 1$ if for any open set Ω ,

(16) $$|\mu|(T(\Omega)) \leq C|\Omega|^{\alpha} .$$

If we let $\alpha \to 0$ in this and then let Ω increase to $\mathbb{R}^n$ we see that

$$V^0 = \{\text{finite measures on } \mathbb{R}^{n+1}_{+}\} .$$

Amar and Bonami proved a number of interesting results about such spaces. In particular they finished the result on balayages by showing that

$$P^*\mu : V^{\alpha}(\mathbb{R}^{n+1}_{+}) \to L^{p,\infty}(\mathbb{R}^n) , \quad \frac{1}{p} = 1 - \alpha , \quad 0 < \alpha < 1$$

and $L^{p,\infty}$ is the *Marcinkiewicz space* of functions of weak type p , i.e., $f \in L^{p,\infty}(\mathbb{R}^n)$ if

$$|\{x \mid |f(x)| > y\}| \leq A/y^p ,$$

which supplements the fact that $P^* : V^1 \to BMO$ and $P^* : V^{\alpha} \to \dot{B}^{n(\alpha-1)}_{\infty,\infty}$, the homogeneous Besov space of order $n(\alpha - 1)$, for $\alpha > 1$. They further showed that this map is onto in each case, precisely, if $f \in L^{p,\infty}$ has compact support ($f \in BMO$ has compact support), then there is a measure $\mu \in V^{\alpha}$, $\alpha = 1 - 1/p$ (or $\mu \in V^1$) such that $P^*\mu = f$. They showed that V^{α} is an intermediate space between V^0 and V^1 in the sense of interpolation. This was done by showing that for the complex method of interpolation $[\ ,\ ]_{\alpha}$ or the real method $(\ ,\ )_{\alpha,\theta}$, one has

(17) $$[V^0,V^1]_{\alpha} = (V^0,V^1)_{\alpha,\infty} = V^{\alpha} .$$

They also identified a subspace W^{α} of V^{α} such that $P^* : W^{\alpha} \to L^p(\mathbb{R}^n)$, $1/p = 1 - \alpha$ (the map is onto again, without assuming f

has compact support), and finally then showed that W^α is an intermediate space for the real method

$$(18) \qquad (V^0,V^1)_{\alpha,p} = W^\alpha , \quad 1/p = 1 - \alpha .$$

The last part of the prehistory is a paper of mine that eventually appeared in the Cortona proceedings [13]. I observed there that because of Whitney's lemma for $\alpha \geq 1$, $\mu(T(\Omega)) \leq C|\Omega|^\alpha$ for every open set Ω if and only if $\mu(Q \times [0,\delta]) \leq C|Q|^\alpha$ for every cube Q, gave a counterexample of a family of measures that satisfied $\mu(Q \times [0,\delta]) \leq C|Q|^\alpha$ for all cubes, with $\alpha < 1$, but which were not generalized Carleson measures and gave a number of examples of generalized Carleson measures. I noted that the analogue of the Carleson inequality for $\alpha < 1$ was

$$\left(\int |u(x,y)|^{p\alpha}\, d\mu(x,y)\right)^{1/p} \leq C||f||_{(p,p\alpha)} ,$$

where this last norm is the Lorentz space norm of a function f defined in terms of its decreasing rearrangement f^* by

$$(19) \qquad ||f||_{(p,q)} = \left(\int_0^\infty [f^*(t)t^{1/p}]^q \frac{dt}{t}\right)^{1/q} .$$

Of course the $L^{p,q}$ form an increasing family of spaces as q varies, the smallest space is $L^{p,1}$ and the largest is the space of weak type p functions which is $L^{p,\infty}$. Finally, I noted that there was an atomic space whose dual was V^1 but I could not characterize it concretely.

2. Tent spaces

The preprint of my paper appeared at about the time that Coifman, Meyer and McIntosh proved [6] the boundedness of the Cauchy integral (in L^2) on Lipschitz curves with arbitrary Lipschitz constant. Coifman, Meyer and Stein introduced tent spaces as a means to provide another proof of the boundedness [7] and later [8] discussed their general theory.

For $1 \leq q < \infty$, set for $F(x,t)$ defined on $\mathbb{R}^{n+1}_+$

$$(1) \qquad A_qF(x) = \left(\iint_{\Gamma(x)} |F(y,t)|^q\, dy \frac{dt}{t^{n+1}}\right)^{1/q} ,$$

where $\Gamma(x) = \{(y,t) \mid |y - x| < t\}$ is an equiangular cone with vertex at x. For $q = \infty$, set

(2) $$A_\infty F(x) = \operatorname*{ess\ sup}_{\Gamma(x)} |F(y,t)| .$$

Define, for $1 \leq p < \infty$ (for the moment),

(3) $$T^p_q = \{F \mid A_q F \in L^p(\mathbb{R}^n)\} ,$$

normed with $||F||_{T^p_q} = ||A_q F||_{L^p}$. For $q = \infty$, we will need to consider C_0 rather than L^∞ in order to get a duality result and we define

(4) $$T^p_\infty = \{F \mid A_\infty F \in L^p \text{ and } ||F_\varepsilon - F||_{T^p_\infty} \to 0 \text{ as } \varepsilon \to 0\} ,$$

where $F_\varepsilon(x,t) = F(x, t+\varepsilon)$.

They proved a number of results about these spaces but two were basic.

THEOREM (Coifman, Meyer, Stein). $(T^1_\infty)^* = V^1$, *precisely, the pairing* $(F,d\mu) \to \int F(x,t)\,d\mu(x,t)$ *with* $F \in T^1_\infty$ *continuous on* $\mathbb{R}^{n+1}_+$ *and* $\mu \in V^1$ *realizes the duality of* T^1_∞ *with the Carleson measures.*

For the next result say that a is an *atom* if $\operatorname{supp} a \subseteq T(B)$ for a ball B and $||a||_\infty \leq |B|^{-1}$. It is easy to see that an atom belongs to T^1_∞ and there is also an atomic decomposition for T^1_∞.

THEOREM (Coifman, Meyer, Stein). *For any* $F \in T^1_\infty$, $F = \sum\lambda_j a_j$, *with* a_j *atoms and* $\sum|\lambda_j| < +\infty$. *In fact,*

(5) $$||F||_{T^1_\infty} \simeq \inf \{\sum|\lambda_j| \mid F = \sum\lambda_j a_j\}$$

In fact there are some technicalities to take care of because elements of T^1_∞ are continuous while the atoms need not be, but the result as quoted gives the essence of the theorem.

A number of other results were given - atomic decompositions for T^1_q, $1 < q < \infty$, duality results which say

(6) $$(T^p_q)' = T^{p'}_{q'} ,$$

and ferially, in connection with this last result, the definition of T^p_q for $p = \infty$ which requires the introduction of the C-functional For a function F on $\mathbb{R}^{n+1}_+$, let

(7) $$C_r F(x) = \sup_{x\in B} \left(\frac{1}{|B|} \iint_{T(B)} |F(y,t)|^r \, dy \frac{dt}{t}\right)^{1/r} , \quad 1 < r < \infty$$

(8) $$C_1\nu(x) = \sup_{x\in B} \frac{\nu(T(B))}{|B|} , \quad r = 1 ,$$

where ν is any measure on R_+^{n+1}. The dual of T^1 is characterized by the C functional and it is shown that each of the T_2^p spaces could also be characterized in terms of C_2,

(9) $$\|F\|_{T_2^p} \simeq \|C_2(F)\|_p ,$$

which then naturally leads them to define $T_q^\infty = \{F \mid C_qF \in L^\infty\}$. There are many other interesting results in [8] (while [7] contained also the proof of the boundedness of the Cauchy integral along Lipschitz curves); I will only mention two further results. They considered both real and complex interpolation and showed that

(10) $$\left[T_q^{p_0}, T_q^{p_1}\right]_\Theta = T_q^{p_\Theta} , \quad (T_q^{p_0}, T_q^{p_1})_{\Theta,q} = T_q^{p_\Theta} ,$$

where $\frac{1}{p_\Theta} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$. They also exploited a relation with the Hardy spaces H^p. Fix a function ϕ which satisfies

(i) ϕ has compact support (say in the unit ball);

(ii) $|\phi(x)| \leqq M$, $|\phi(x+h) - \phi(x)| \leqq M(|h|/|x|)^\varepsilon$, for some $\varepsilon > 0$;

(iii) $\int \phi(x)\,dx = 0$;

(iii)$_N$ $\int x^\gamma \phi(x)\,dx = 0$ for all $|\gamma| \leqq N$.

Write, as usual, $\phi_t(x) = t^{-n}\phi(x/t)$, for $t > 0$. Consider the operator $\Pi_\phi F(x) = \int_0^\infty (F(.,t) * \phi_t)\frac{dt}{t}$. They proved also:

THEOREM. *The operator* Π_ϕ *, defined on a dense subspace, extends to a bounded linear operator*

(a) *from* T_2^p *to* L^p , $1 < p < \infty$,

(b) *from* T_2^1 *to* H^1 ,

under the conditions (i), (ii) *and* (iii) *on* ϕ . *If* (iii)$_N$ *is satisfied with* $N \geqq n(\frac{1}{p} - 1)$, *then* Π_ϕ *extends*

(c) *from* T_2^p *to* H^p , $p \leqq 1$,

(d) *from* T_2^∞ *to* BMO ,

where the obvious extension of the definition of T_q^p *has been made for* $p < 1$.

Note that a map in the opposite direction may be set up in each case by using, e.g., the Lusin area function. There are also results given on sufficient conditions for a function $m(x,t)$ to multiply $T^p_q \to T^p_q$, but we will say more about such results when we state an extension in the next section.

3. Tent spaces and the generalized Carleson measures of Amar-Bonami

I will now describe some joint work with A. Bonami which will appear in the volume of Math. Nachrichten dedicated to Prof. Triebel. Our object was to understand the precise sense in which the tent spaces are related to the V^α spaces of Amar-Bonami. We have done this but unfortunately it is necessary to complicate everything in order to understand it. We first generalize the V^α spaces by considering for $1 < q \leq \infty$,

$$(1) \qquad V^{p,q} = \{w = \nu(x,t)\,dx\,\frac{dt}{t} \mid \exists C\,,\ \Big(\iint_{T(\Omega)} |\nu(x,t)|^q\,dx\,\frac{dt}{t}\Big)^{\frac{1}{q}} \leq C|\Omega|^{\frac{1}{q}-\frac{1}{p}} \text{ for all open sets } \Omega\}\,,$$

and for $q = 1$ we give the same condition but do not require that w be absolutely continuous,

$$(2) \qquad V^{p,1} = \{w \mid \exists C\,,\ |w|\,|T(\Omega)| \leq C|\Omega|^{1-1/p}\}\,.$$

Thus, $V^{p,1}$ is the space we met before as $V^{1-1/p}$ in Amar-Bonami. We say that a is a (p,q) *atom*, $1 \leq p \leq q \leq \infty$, if there is an open set Ω such that $\operatorname{supp} a \subseteq T(\Omega)$ and

$$(3) \qquad \|a\|_q = \Big(\iint_{T(\Omega)} |a(y,t)|^q\,dy\,\frac{dt}{t}\Big)^{1/q} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p}}$$

for $q < +\infty$ and

$$(4) \qquad \|a\|_\infty = \operatorname{ess\,sup} |a(y,t)| \leq |\Omega|^{-1/p}$$

when $q = \infty$. We are also forced to consider tent spaces based on the Lorentz spaces introduced earlier (1.19),

$$(5) \qquad T^{p,r}_q = \{F \mid A_q F \in L^{p,r}\}\,.$$

For such spaces we prove three main results:

(i) *If* $q > p$, *then* $T_q^{p,1}$ *has an atomic decomposition*

(6) $$T_q^{p,1} = \{F \mid F = \sum \lambda_k a_k \;,\; a_k \; a \; (p,q) \; atom, \; \sum|\lambda_k| < +\infty\} \;;$$

(ii) $(T_q^{p,1})^* = V^{p',q'}$; *in particular,* $(T_\infty^{p,1})^* = V^{1/p}$;

(iii) $T_q^{p,\infty} = V^{p,q}$, $q < p$.

Theorem (i) extends the range of the atomic decomposition of Coifman, Meyer and Stein at the price of allowing atoms to live on open sets. For the case considered there, the open set can be broken up into Whitney cubes and the resulting sum controlled, it could not be controlled here and the open set must be left as is. Property (ii) shows that the result of [8] does not extend to $p > 1$. The Lorentz spaces must be introduced to give a space whose dual is $V^{1/p}$, and the last result shows that the spaces considered by Amar-Bonami are exactly $T_1^{p,\infty}$ and roughly play the role of L^1 in tent space theory. In particular, the interpolation results can be reduced to the previous interpolation results (1.17), (1.18) and we obtain

(7) $$(T_q^{p_0 r_0}, T_q^{p_1 r_1})_{\theta r} = T_q^{p,r} ,$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} , \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} .$$

We also generalize the good-λ inequality connecting the A-functional and the C-functional by showing that with

(8) $$A_1\nu(x) = \iint_{\Gamma(x)} t^{-n}\, d\nu(y,t) \;,\quad C_1\nu(x) = \sup_{B \ni x} \frac{\nu(T(B))}{|B|} ,$$

then there exists a C , C' (independent of δ and ν) such that $|\{x \mid A_1\nu(x) > C\lambda \;;\; C_1\nu(x) \leq \delta\lambda\}| \leq C'\delta|\{x \mid A_1\nu(x) > \lambda\}|$. This is slightly sharper than the result in [8], where for the term on the right hand side, an A-functional on a slightly narrower cone was needed.

This result immediately implies that $||A_1\nu||_{p,r} \simeq ||C_1\nu||_{p,r}$, $1 < p < \infty$ and then, trivially, that

(9) $$||C_qF||_{p,r} \simeq ||A_qF||_{p,r} ,$$

which shows that the spaces could also be defined by the C-functional. Moreover by a technique due to Coifman and Fefferman [5] such a good-λ inequality implies a weighted inequality for weight $\omega \in A_\infty = \bigcup_{p<\infty} A_p$, where, A_p is the Muckenhoupt condition:

(10) $\omega \in A_p$ if and only if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega\, dx\right)^{1/p}\left(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{p-1}}\, dx\right)^{1/p'} < +\infty .$$

If $\omega \in A_\infty$,

(11) $$\int |A_q F(x)|^p\, \omega(x)\, dx \leqq C^p \int C_q F(x)^p\, \omega(x)\, dx ,$$

while if $p > q$ and $\omega \in A_{p/q}$,

(12) $$\int C_q F(x)^p\, \omega(x)\, dx \leqq C_1^p \int A_q F(x)^p\, \omega(x)\, dx .$$

Finally, this good-λ inequality contains several other good-λ inequalities as special cases. Lars Hedberg discussed the good-λ inequality of Muckenhoupt and Wheeden [15]. If we take $d\nu(y,t) = t^\alpha f(y)\, dy \frac{dt}{t}$,

(13) $$A_1(d\nu)(x) = R^\alpha f(x) = C_\alpha \int \frac{f(z)}{|x - z|^{n-\alpha}}\, dz ,$$

and

(14) $$C_1 d\nu(x) \simeq M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q f(y)\, dy ,$$

we see that the good-λ inequality of Muckenhoupt and Wheeden follows from ours. For another example, take $d\nu(y,t) = t|\nabla u(y,t)|^2\, dy\, dt$, where u is the Poisson integral of a function of bounded mean oscillation. One checks that $A_1(d\nu) = (Sf)^2$ is the square of the Lusin area integral, while

(15) $$C_1(d\nu) \simeq \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y) - f_B|^2\, dy ,$$

is the square of the L^2 sharp function of f and we obtain the good-λ inequality

(16) $$|\{x \mid \delta f(x) > \lambda ,\ \phi_2^{\#}(x) \leqq \delta\lambda\}| \leqq C'\delta |\{x \mid \delta f(x) > \lambda\}|$$

proved originally by Wilson [19]. In general, if one takes a Carleson measure ν , an interesting good-λ inequality should result. We have considered one other example which I will give in the next section, but other cases should be investigated.

The final thing we consider is the application of tent spaces to weighted inequalities. Weighted inequalities correspond to multiplication theorems for tent spaces, which were already proved in [8] for $T^p_\infty \to T^p_\infty$. The connection arises in the following way. For $f \in L^p(\mathbb{R}^n)$, form the *ball mean* (which is a balayage with a suitable kernel (see Section 1))

$$(17) \qquad F(y,t) = \frac{1}{b_n t^n} \int_{|y-z|\leq t} f(z)\, dz ,$$

which is also one of the competitors in the definition of the Hardy -Littlewood maximal function, $Mf(x)$. Now

$$(18) \qquad A_\infty F(x) = \sup_{(y,t)\in\Gamma_x} \frac{1}{b_n t^n} \int_{|y-z|\leq t} f(z)\, dz \leq 2^n \sup_{t>0} \frac{1}{b_n (2t)^n} \int_{|x-z|\leq 2t} f(z)\, dz = 2^n Mf(x) .$$

Thus, if $f \in L^p$, then $Mf \in L^p$ which implies that $F \in T^p_\infty$. If we multiply the ball mean by $\mu(y,t) = w(y)t^\alpha$, where we think of w as a weight, and use the lower bound on A_∞ obtained by taking the sup over the ray $\{(x,t) \mid t > 0\}$ in Γ_x, we find

$$(19) \qquad A_\infty\{w(y)t^\alpha F(y,t)\} \geq cM_\alpha f(x)w(x) .$$

A characterization of the pointwise multipliers of $T^p_\infty \to T^q_\infty$ leads to a weighted inequality for M_α from $L^p \to L^q_{dw}$. When we do this for $q \geq p$, we obtain a result of Sawyer [16] which says that

$$(20) \qquad ||M_\alpha f||_{L^q_{d\mu}} \leq A||f||_p , \quad p \leq q$$

if and only if

$$\exists A \text{ such that for every ball } B, \quad \mu(B) \leq A|B|^{\frac{q}{p}-\frac{\alpha}{n}} .$$

(When μ is the Lebesgue measure, this is the classical result of Sobolev.) However, because we have also considered $\alpha < 1$, we can characterize the inequality for $q < p$ as well.

THEOREM. *The following conditions are equivalent:*

(i) $\quad \left(\int |R^\alpha f(x)|^q\, d\mu(x)\right)^{1/q} \leq A||f||_{p,1}$.

(ii) $\quad \exists B$ *such that for every open set,*

$$(21) \qquad \left(\int_\Omega d(x, {}^c\Omega)^{\alpha q}\, d\mu(x)\right)^{1/q} \leq A|\Omega|^{1/p} .$$

I should point out that (ii) is easy to apply since if you write

$\Omega = \cup Q_k$ for the Whitney decomposition of Ω ,

(22) $$d(x, {}^c\Omega)^\alpha \asymp \sum (\text{diam } Q_k)^\alpha \, \psi_{Q_k} .$$

A simple application of this is to trace theorems. Let $d\mu(x) = dx_1 \ldots dx_k$ be Lebesgue measure on the set $\{(x_1,x_2,\ldots,x_k,0,0,\ldots,0)\} \subseteq \mathbb{R}^n$. We immediately obtain

<u>COROLLARY</u>. *If* $\frac{k}{q} = \frac{n}{p} - \alpha$, *then*

(23) $$\int_{\mathbb{R}^k} |R^\alpha f(x)| \, dx_1 \ldots dx_k \leqq A^q ||f||^q_{p,1} .$$

Such results are well-known if $\alpha > \frac{n}{p} - \frac{k}{q}$ and you can even put the L^p-norm of f on the right hand side. Such results also follow from interpolation of this result. If $q = p$ and $k = n - 1$, we have

(24) $$R^{1/p} : L^{p,1} \to L^p(dx_1 \ldots dx_{n-1}) .$$

The best other result I know of is due to Gol'dman (see [18]) and says that

(25) $$R^{1/p} : \dot{B}^0_{p,1} \to L^p(dx_1 \ldots dx_{n-1}) ,$$

but $L^{p,1}$ neither contains nor is contained in the homogeneous Besov space $\dot{B}^0_{p,1}$.

The p r o o f of the theorem is so simple that I can give it here. That (i) implies (ii) follows by choosing $f = \psi_\Omega$. The problem is to show that (ii) implies (i). Assume than that (21) holds, and we want to prove that

(26) $$R^\alpha : L^{p,1} \to L^q_{d\mu} ,$$

which is equivalent, by duality, to

(27) $$g \mapsto R^\alpha(g \, d\mu) : L^{q'}_{d\mu} \to L^{p',\infty} .$$

We have

(28) $$R^\alpha(g \, d\mu) = P^*(t^\alpha \, g(y) \, d\mu(y) \, \tfrac{dt}{t}) .$$

By the results of Amar-Bonami, it is enough to show that $t^\alpha \, g \, d\mu \, \frac{dt}{t} \in V^{1/p}$ which requires

$$\left|\int g(y) \, d(y, {}^c\Omega)^\alpha \, d\mu(y)\right| \leqq C|\Omega|^{1/p} , \qquad \forall g \in L^{q'}_{d\mu} ,$$

and hence, by Hölder, this requires $d(y, {}^c\Omega) \in L^q_{d\mu}$ and

$$||d(y, {}^c\Omega)^\alpha||_{L^q_{d\mu}} \leq C|\Omega|^{1/p} .$$

4. Other results and ideas

This idea of writing an integral operator as a balayage of a suitable measure is useful also in a context considered by Kerman and Sawyer [14]. They consider the operator

$$T_\Phi f(x) = \Phi * f ,$$

where Φ is a radial decreasing function. Let me first state their result.

THEOREM. *Let* Φ *be a nonnegative locally integrable function such that* $\int_{|y|\geq r} \Phi(y)^{p'} dy < +\infty$ *, which is radially decreasing. For* $1 < p \leq q < \infty$ *and* μ *a positive locally finite Borel measure, the following statements are equivalent:*

(i) $\exists C > 0$ *such that for* $f \geq 0$ *, measurable,*

$$\Big(\int_{\mathbb{R}^n} T_\Phi f(x)^q \, d\mu(x)\Big)^{1/q} \leq C||f||_p$$

(ii) $\exists C' > 0$ *such that* $\Big(\int_{\mathbb{R}^n} T_\Phi(\chi_Q\mu)^{p'} dx\Big)^{1/p'} \leq C'\mu(Q)^{1/q'} < +\infty$

(iii) *Let* $M_\Phi f(x) = \sup_{x \in Q} \Big(\frac{1}{|Q|} \int_{|y|\leq|Q|^{1/n}} \Phi(y)\, dy\Big) \int_Q f(y)\, dy$

be the maximal function associated with Φ . (*When* Φ *is such that* $T_\Phi = R^\alpha$, $M_\Phi = M_\alpha$ *is the fractional maximal function.*) *Then* $\exists C'' > 0$ *such that*

$$\Big(\int_Q M_\Phi(\chi_Q d\mu)^{p'} dx\Big)^{1/p'} \leq C''\mu(Q)^{1/q'} < +\infty .$$

The result has various applications to e.g. estimating eigenvalues of the Laplacian. A. Bonami and I proceed as follows. Let Θ be the distributional derivative of Φ , i.e., a Stieltjes measure such that

$$\Phi(y) = \int_{|y|}^{\infty} d\Theta(t) \ .$$

Consider the measure $t^n d\Theta(t) d\nu(x)$. Then $A_1 d\nu(x) = \Phi * f(x)$ while

$$C_1 d\nu(x) \simeq \tilde{M}_\Phi(d\nu)$$
$$= \sup_{Q \ni x} \frac{1}{|Q|} \int_{|y| \leq |Q|^{1/n}} \left[\Phi(y) - \Phi(|Q|^{1/n})\right] dy \int_Q d\nu \ ,$$

and thus the good-λ inequality holds for T_Φ , $\tilde{M}_\Phi$ and $\tilde{M}_\Phi$ is smaller than the M_Φ considered by Kerman and Sawyer. Since the good-λ inequality is the key step in their proof, it can now be carried over to $\tilde{M}_\Phi$. However, we can also prove their theorem by resorting to the atomic decomposition of T^1_∞ if we know that μ is doubling which is a defect of our method. We also get results for $L^p \to L^q_{d\mu}$ with $q < p$. I should also note that if

$$\frac{1}{\alpha^n} \int_0^\alpha \Psi(t) t^{n-1} \, dt \leq C\Psi(\alpha) \ ,$$

then M_Φ and $\tilde{M}_\Phi$ are of the same order, and that this happens when $\Phi(t) = t^{\alpha-n}$.

Our results for $q < p$ always require $L^{p,1}$ on the right hand side because we are using balayage and the spaces $V^{1/p}$. We can say something about operators mapping $L^p \to L^q_{d\mu}$ but it requires that we use balayage with measures in $W^{1/p}$ and the conditions found by Amar -Bonami for $W^{1/p}$ are not as explicit as for $V^{1/p}$. The known necessary and sufficient conditions in order that

$$R^\alpha : L^p \to L^q_{d\mu}$$

involve capacities, (see [12])

$$\mu(E)^{p/q} \leq C\dot{B}_{\alpha,p}(E) \quad \forall E \ \text{compact}$$

which are difficult to verify or conditions on cubes [17] that are also difficult to verify. By using a duality result that I will describe later, we can give an alternate formulation of the capacity of a set.

<u>THEOREM</u>. *Let* $\Sigma(E) = \inf \{ \|F\|^p_{T^p_\infty} \mid F \geq 0 \ , \int_0^\infty F(x,t) t^\alpha \frac{dt}{t} \geq \psi_E(x) \}$, *then* $\Sigma(E) \simeq \dot{B}_{\alpha,p}(E)$.

A possible interest of this is that the differential conditions $R^\alpha f \geq \psi_E$ appears implicitly in the definition of Σ and might make it easier to compute Σ .

Let me finish by mentioning two other papers touching on tent spaces. J. Alonso and M. Milman [1] have also considered duality results simultaneously with Bonami and I and also proved that $(T_\infty^{p,1})^* = V^{1/p}$, but showed as well that $(T_\infty^p)^* = W^{1/p}$, which is used by us to derive the above capacitary result. They also calculate the K-functional between T_∞^p and L^∞ , use it to give other interpolation results and begin the study of tent spaces on product domains.

Another result that should have been mentioned in Section 1 is due to Deng [9] who proved that

$$\left| \iint_{\mathbb{R}_+^{n+1}} F(x,t)\nu(x,t)\,dx\,dt \right| \leq C \int_{\mathbb{R}^n} A_p F(x) C_{p'}\nu(x)\,dx ,$$

where $1/p + 1/p' = 1$, $1 \leq p \leq \infty$. He also showed that for any $r > 1$, $C_r\nu$ belongs to the class A_1 of Muckenhoupt formed of functions ω for which there is a positive constant C such that

$$M\omega(x) \leq C\omega(x) \quad \text{for all} \quad x \in \mathbb{R}^n .$$

R E F E R E N C E S

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