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WEIGHTED ESTIMATES FOR CLASSICAL OPERATORS

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Abstract

In this article we discuss recent work on weighted norm inequalities for the Fourier-, Laplace- and Hardy operators. Specifically, we give weight conditions in terms of integrals and integral products which insure inequalities of the form $||Tf||_{q,u} \leq C||f||_{p,v}$, where $0 < p$, $q < \infty$, $p \geq 1$ and f in weighted Hardy- or Lebesgue spaces. In several cases the weights are also near optimal.

For certain convolution operators the results constitute examples of translation invariant operators which map weighted L^p to weighted L^q for $1 < q < p < \infty$. This contrasts with the unweighted case for which such operators are zero ([19]).

1. Introduction

Given a linear operator T, defined on some suitable subspace of L^p, then for many problems in mathematical analysis it is desirable to obtain from some known (initial) properties of the operator a norm estimate $||Tf||_q \leq C||f||_p$, for p and q in some interval. Typical tools to obtain such estimates are the interpolation theorems of Riesz-Thorin and Marcinkiewicz and its many generalizations and abstractions.

Naturally one might ask whether the initial data of T implies the optimal conclusion, in the sense that if the data implies an (L^p, L^q)-estimate of T, does it imply a weighted (L^p, L^q)-estimate for some weights u and v of which $u = v = 1$ is a special case? Moreover, are the weights in this class optimal? That is, does the weighted estimate imply that the weights belong to that class? Of course an affirmative answer to these questions yield more information

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about T and hence the given problem but it may also lead to weight classes with interesting properties.

A related question is the following: Suppose T is bounded from L^p to L^q, where p and q are in some intervals, say (p_0,p_1) respectively (q_0,q_1) is there a weight class of functions u, v such that $T : L^p_v \to L^q_u$ for p and q in extended intervals? That is, does the weight class extend the range and domain of the operator in this sense? For example, suppose T commutes with translation and suppose further that $T : L^p \to L^q$, $1 < p \leqq q < \infty$, is bounded. It is well known ([19]) that such operators cannot be bounded from L^p to L^q if $1 < q < p < \infty$, (unless $T = 0$), however - as we shall see - there exist weights u and v, for which, such operators are bounded from L^p_v to L^q_u, $1 < q < p < \infty$.

In this note we elaborate on- and answer some of these questions for certain classical operators such as the Fourier- and Laplace transform, the Hardy operator and some of their extensions. We emphasize specifically recent work regarding the Fourier transform and show (Theorem 3.1) that there exist easily computable weights for which a weighted (L^p,L^q)-estimate holds if $0 < p$, $q < \infty$, $p \geqq 1$. Moreover, in the range $1 < p \leqq q < \infty$ the weights are near optimal (Theorem 3.2). Similar results are given for the Hankel operator.

There are many weighted Fourier inequalities in the literature which have wide applications. Some are used to prove new multiplier estimates ([11], [18]) while others are applied to obtain new representation theorems of functions as Laplace transforms ([3], [4], [35]). We discuss here the results of Rooney [35], Aguilera and Harboure [1], Hirschman [18] and Flett [11] and show that their weighted Fourier estimates are special cases of Theorem 3.2.

We deduce that one of the weight classes considered here - namely the $F^*_{p,q}$ classes - are related to the Muckenhoupt weight class A_p (Corollary 3.1). Moreover we state the known A_p-weighted norm inequality for the Hardy-Littlewood maximal operator (Proposition 3.1) and use it to prove another weighted Fourier inequality. Section 3 concludes with some recently established Fourier norm estimates.

In case $p \leqq 1$, Fourier estimates are in general only possible if the domain space of the operator is the Hardy space. In the last section we define the weighted (atomic) Hardy space, characterized by Garcia-Cuerva [12], and show that certain weighted L^q-estimates of the Fourier transform imply weighted H^p-estimates. Indeed the result does apply to other operators as well.

For several of the estimates given in Section 3, weighted inequalities for the Hardy operator and its dual are required. These results and various extensions are given in the next sections with weight conditions in terms of certain integrals and integral products. Other, rather interesting weight conditions (cf. Gupta, Kufner, Triebel) in terms of solutions of certain differential equations are not discussed here. Unlike these, our weight conditions do not generalize easily to higher dimensions, however. We do state a weight characterization of the two dimensional Hardy operator (Theorem 2.4). In addition we give weighted results for certain (translation invariant) operators which map weighted L^p to L^q spaces, $q < p$ and also for Laplace transforms.

Throughout we adhere to the following notations and conventions: $\mathbb{R} = (-\infty,\infty)$, $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}_+^2 = \mathbb{R}_+ \times \mathbb{R}_+$. The Lebesgue measure of a set $E \subset \mathbb{R}$ is denoted by $|E|$ and χ_E is the characteristic function of E. Given Banach (metric) spaces X and Y, then we will also denote by $[X,Y]$ the collection of bounded linear operators T from X to Y. In particular we give conditions and operators for which $T \in [L_v^p(\Omega), L_u^q(\Omega)]$ or $T \in [H_v^p, L_u^q]$, where Ω is $\mathbb{R}$ or $\mathbb{R}_+$, $L_w^r = \{f : w^{1/r}f \in L^r\}$ with norm (metric) $||w^{1/r}f||_r = ||f||_{w,r}$ and H_v^p is the weighted (atomic) Hardy space. If $r = \infty$ the weighted norm is interpreted as the essential supremum of f. The conjugate index p' of p is defined by $p' = p/(p-1)$ for $p > 0$, with $p' = \infty$ if $p = 1$, and similarly for other letters.

A, B and C denote constants which may be different at different occurrances. S is the Schwartz class of slowly increasing functions and M the *Hardy-Littlewood maximal operator* defined by

$$(Mf)(x) = \sup_{\substack{x \in I \\ I \subset \mathbb{R}}} \frac{1}{|I|} \int_I |f(y)|\, dy ,$$

where I is an interval. Finally, inequalities are to be interpreted in the sense that if the right side is finite, so is the left and the inequality holds.

2. Operators of Hardy type

In this section we discuss weighted Hardy inequalities and provide a simple proof for the one dimensional operator in the case $1 < q < p < \infty$. Further extensions to certain convolution operators are given, which yield examples of translation invariant operators in $[L^p_v, L^q_u]$, $1 < q < p < \infty$. In addition weight conditions for which weighted estimates for the Laplace transform exist are given. The section concludes with a weighted estimate for the two dimensional Hardy operator.

Define the *Hardy operator* P and its *dual* P′ by

$$(Pf)(x) = \int_0^x f(t)\,dt \quad \text{and} \quad (P'f)(x) = \int_x^\infty f(t)\,dt .$$

If $1 < p < \infty$, then Hardy's inequality ([13]) states that

$$\int_0^\infty |(Pf)(x)/x|^p\,dx \leqq [p/(p-1)]^p \int_0^\infty |f(x)|^p\,dx$$

with a corresponding dual estimate. There are numerous generalizations of this result (cf. [26], [33], [39] and the bibliography cited there). The characterization of the weights given next is in terms of integral conditions.

<u>THEOREM 2.1</u>. *Suppose* f , u *and* v *are non negative locally integrable functions on* $\mathbb{R}_+$ *and let* $\sigma = v^{1-p'}$.

(a) *Let* $1 \leqq p \leqq q < \infty$, *then*

$$(2.1) \qquad \left(\int_0^\infty u(x)\,[(Pf)(x)]^q\,dx\right)^{1/q} \leqq A\left(\int_0^\infty v(x)\,[f(x)]^p\,dx\right)^{1/p}$$

if and only if

$$(2.2) \qquad \sup_{s>0}\,(P'u)(s)^{1/q}(P\sigma)(s)^{1/p'} \equiv B < \infty$$

with the usual modification when $p = 1$. *Moreover,* $B \leqq A \leqq (p')^{1/p'}B$.

(b) *If* $0 < q < p < \infty$, $p \geqq 1$, *then* (2.1) *holds if and only if*

$$(2.3) \qquad \left\{\int_0^\infty [(P'u)(x)^{1/q}(P\sigma)(x)^{1/q'}]^r\,\sigma(x)\,dx\right\}^{1/r} \equiv C < \infty$$

where $1/r = 1/q - 1/p$.

(c) *The results hold also for the dual operator* P′ , *that is* (a) *and* (b) *hold with* P *and* P′ *interchanged.*

Part (a) of Theorem 2.1 in the case $q = p$ was proved in [29] (and elsewhere) and the general case by Maz'ya [28] and Bradley [5]. Part (b) was proved by Maz'ya [28] in the case $1 < q < p < \infty$ and by Sinnamon [38] in the case $0 < q < 1$, $p \geq 1$. The general case $0 < q < 1$, $p \geq 1$ was proved independently also by Sawyer [36] with different (equivalent) weight condition. The simple proof given below in the case $1 < q < p < \infty$ is due to Sawyer (personal communication).

If $\sigma = v^{1-p'}$, then integrating and an interchange of order of integration shows that

$$\int_0^\infty u(x)\left(\int_0^x \sigma(y)\ g(y)\ dy\right)^q dx$$

$$= q\int_0^\infty u(x)\left[\int_0^x \left(\int_0^y \sigma(t)\ g(t)\ dt\right)^{q-1} \sigma(y)\ g(y)\ dy\right] dx$$

$$= q\int_0^\infty \sigma(y)\ g(y)\left(\int_0^y \sigma(t)\ g(t)\ dt\right)^{q-1}\left(\int_y^\infty u(x)\ dx\right) dy$$

$$= q\int_0^\infty \sigma(y)\ g(y)\left(\frac{\int_0^y \sigma(t)g(t)\ dt}{\int_0^y \sigma(t)\ dt}\right)^{q-1}\left(\int_y^\infty u(x)dx\right)\left(\int_0^y \sigma(t)\ dt\right)^{q-1} dy\ .$$

Now write $\sigma = \sigma^{1/p+(q-1)/p+(p-q)/p}$ and apply Hölder's inequality with p , $p/(q-1)$ and $p/(p-q)$, then the last expression is not larger than

$$q\left(\int_0^\infty \sigma(y)g(y)^p\ dy\right)^{1/p}\left[\int_0^\infty\left(\frac{\int_0^y \sigma(t)g(t)dt}{\int_0^y \sigma(t)dt}\right)^p \sigma(y)\ dy\right]^{(q-1)/p}$$

$$\cdot\left\{\int_0^\infty\left[\left(\int_y^\infty u(x)dx\right)^{1/q}\left(\int_0^y \sigma(t)\ dt\right)^{1/q'}\right]^r \sigma(y)\ dy\right\}^{(p-q)/p}\ .$$

The last integral product is by (2.3) dominated by C^q . If in the middle integral product we write $w(y) = \sigma(y)/(\int_0^y \sigma(t)\ dt)^p$, then the integral takes the form

$$\int_0^\infty w(y)\left(\int_0^y \sigma(t)\ g(t)\ dt\right)^p dy\ .$$

But by part (a) with $q = p$ this is not larger than

$$(p')^{p/p'}\ B^p \int_0^\infty \sigma(y)\ g(y)^p\ dy$$

whenever

$$\sup_{s>0}\left(\int_s^\infty w(y)\ dy\right)^{1/p}\left(\int_0^s \sigma(t)\ dt\right)^{1/p'} \equiv B$$

is finite. But direct integration shows that this is dominated by $(p-1)^{-1/p}$. Therefore

$$\int_0^\infty u(x)\left(\int_0^x \sigma(y)g(y)\ dy\right)^q dx$$

$$\leq C^q q(p')^{(q-1)/(pp')}(p-1)^{-(q-1)/p}\left(\int_0^\infty \sigma(y)g(y)^p\ dy\right)^{q/p}.$$

From this result follows on letting $\sigma g = f$, for then $\sigma g^p = \sigma^{1-p}f^p = vf^p$, and taking q's roots.

We now consider the convolution operator T defined by

(2.4) $$(Tf)(x) = (k*f)(x) = \int_{-\infty}^x k(x-y)f(y)\ dy$$

where $k(x) \geq 0$ and non-increasing.

Observe that if f is supported on $\mathbb{R}_+$ and $k(x) \equiv 1$, (2.4) reduces to the *Hardy operator* and for $k(x) = x^{\alpha-1}/\Gamma(\alpha)$, $0 < \alpha < 1$, the *Riemann-Liouville fractional integral operator*. Other choices of k reduce (2.4) to the more general *Erdélyi-Kober operators*.

As special cases of a more general result of [2] and [16] we obtain the following two theorems:

THEOREM 2.2. *Suppose* $1 \leq p$, $q \leq \infty$ *and* u, v *non-negative weights such that with* $k(x) \geq 0$, *non-increasing*

$$\sup_{s\in\mathbb{R}}\left(\int_s^\infty k(x-s)^{\beta q}u(x)\ dx\right)^{1/q}\cdot$$

$$\cdot\left(\int_{-\infty}^s k(s-x)^{(1-\beta)p'}v(x)^{1-p'}\ dx\right)^{1/p'} < C$$

for some $\beta \in [0,1]$ *and* $1 \leq p \leq q \leq \infty$, *or in case* $1 < q < p < \infty$, *with* $1/r = 1/q - 1/p$

(2.5) $$\int_{-\infty}^\infty\left[\left(\int_y^\infty k(x-y)^q u(x)\ dx\right)^{1/q}\left(\int_{-\infty}^y v(x)^{1-p'}dx\right)^{1/q'}\right]^r v(y)^{1-p'}dy < \infty$$

holds. Then the operator T *defined by* (2.4) *is in* $[L_v^p(R),\ L_u^q(R)]$.

THEOREM 2.3. *If for* $1 \leq p \leq q \leq \infty$

$$\text{(2.6)} \qquad \sup_{s>0} \left(\int_0^{1/s} u(x)\,dx\right)^{1/q} \left(\int_0^s v(x)^{1-p'}\,dx\right)^{1/p'} < \infty$$

and for some $\beta \in [0,1]$

$$\sup_{s>0} \left(\int_{1/s}^\infty e^{-\beta sqx}\, u(x)\,dx\right)^{1/q} \left(\int_s^\infty e^{-(1-\beta)p'x/s}\, v(x)^{1-p'}dx\right)^{1/p'} < \infty$$

holds, then the Laplace transform $L \in [L_v^p,\ L_u^q]$.

In case $1 < q < p < \infty$, $1/r = 1/q - 1/p$, *the result holds also provided the previous weight conditions are replaced by*

$$\int_0^\infty \left[\left(\int_0^{1/y} u(x)dx\right)^{1/q} \left(\int_0^y v(x)^{1-p'}dx\right)^{1/q'}\right]^r v(y)^{1-p'}\,dy < \infty$$

and

$$\int_0^\infty \left[\left(\int_{1/y}^\infty e^{-xyq}\, u(x)dx\right)^{1/q} \left(\int_y^\infty v(x)^{1-p'}dx\right)^{1/q'}\right]^r v(y)^{1-p'}dy < \infty .$$

Note that if u and $1/v$ are decreasing, then it is easily seen that in case $1 \leq p \leq q < \infty$ the single weight condition (2.6) suffices for $L \in [L_v^p,\ L_u^q]$. A weaker condition is shown to be enough in the next section.

Recall that if τ_h is the translation operator defined by $(\tau_h f)(x) = f(x-h)$, $h \in R$, then the operator T given by (2.4) satisfies $(\tau_h T)f = (T\tau_h)f$ so that T is translation invariant. Hence by Hörmander's result ([19]) $T \notin [L^p,\ L^q]$, $1 < q < p < \infty$. On the other hand, Theorem 2.2 shows that there exist weights u and v , namely those which satisfy (2.5) for which $T \in [L_v^p,\ L_u^q]$.

Characterizations of weights (in terms of integral conditions) for which a higher dimensional weighted Hardy inequality holds are in general not available. The notable exception is the two dimensional case although various n-dimensional partial results are available. We record the special two dimensional case because it is used in the proof of Theorem 3.6.

Let P_2 and P_2' be the *two dimensional Hardy operator* and its *dual:*

$$(P_2f)(x_1,x_2) = \int_0^{x_1}\int_0^{x_2} f(y_1,y_2)\,dy_1\,dy_2 ,$$

$$(P_2'f)(x_1,x_2) = \int_{x_1}^{\infty}\int_{x_2}^{\infty} f(y_1,y_2)\,dy_1\,dy_2 .$$

THEOREM 2.4. ([37]) *If* $1 < p \leqq q < \infty$, *then* $P_2 \in [L_v^p(R_+^2), L_u^q(\mathbb{R}_+^2)]$ *if and only if the following three conditions are satisfied:*

$$\sup_{\alpha,\beta>0} (P_2'u)(\alpha,\beta)^{1/q}(P_2\sigma)(\alpha,\beta)^{1/p'} < \infty ,$$

$$\left(\int_0^{\alpha}\int_0^{\beta}(P_2\sigma)(x,y)^q u(x,y)\,dy\,dx\right)^{1/q} \leqq A(P_2\sigma)(\alpha,\beta)^{1/p} ,$$

$$\left(\int_{\alpha}^{\infty}\int_{\beta}^{\infty}(P_2'u)(x,y)^{p'}\sigma(x,y)\,dy\,dx\right)^{1/p'} \leqq A(P_2'u)(\alpha,\beta)^{1/q'} ,$$

where $\sigma = v^{1-p'}$.

A different (calculus) proof of this theorem may be useful in obtaining the higher dimensional analogue and possibly shows the way in proving the case $1 < q < p < \infty$.

3. Weighted L^p-estimates

In this section we give estimates for operators of Fourier type in weighted Lebesgue spaces. The *Fourier transform* is defined by

$$(Tf)(x) = \hat{f}(x) = \int_{\mathbb{R}} e^{ix\cdot y} f(y)\,dy , \quad x \in \mathbb{R} ,$$

where f is in some suitable class for which the integral converges. The Fourier transform on $\mathbb{R}^n$ is defined similarly, only now $\mathbb{R}$ is replaced by $\mathbb{R}^n$ and $x\cdot y$ denotes the usual inner product in $\mathbb{R}^n$.

We wish to find optimal- or near optimal weights u and v for which $T \in [L_v^p, L_u^q]$, $0 < p$, $q < \infty$, $p \geq 1$ holds. Recall that the Fourier transform satisfies $||\hat{f}||_\infty \leq ||f||_1$ and $||\hat{f}||_2 = ||f||_2$ (Plancherel) so that $Tf = \hat{f}$ is an operator of type $(1,\infty)$ and $(2,2)$. The Riesz-Thorin interpolation theorem now implies the Hausdorff-Young inequality: $T \in [L^p(\mathbb{R}), L^{p'}(\mathbb{R})]$, $1 < p < 2$. Note that if $f \in L^p$, $p > 2$, then the Fourier transform does only exist as a distribution and not as a function ([40, p. 34]).

A somewhat different argument yields the Titchmarsh extension of the Hausdorff-Young theorem: Define T by $(Tf)(x) = x\hat{f}(x)$ and let $\mu(E) = \int_E x^{-2}\,dx$, $E \subset \mathbb{R}\setminus\{0\}$, then by Plancherel's theorem:

$$\int_{\mathbb{R}} |(Tf)(x)|^2 d\mu(x) = ||f||_2^2 .$$

Now the obvious weak type (1,1) inequality

$$\mu(\{x \in \mathbb{R} : |(Tf)(x)| > \lambda\}) \leqq \mu(\{x \in \mathbb{R} : |x|\ ||f||_1 > \lambda\})$$

$$= 2 \int_{\lambda/||f||_1}^{\infty} x^{-2}\,dx = 2||f||_1/\lambda$$

and the Marcinkiewicz interpolation theorem ([40]) yields

$$\text{(3.1)} \qquad \int_{\mathbb{R}} |\hat{f}(x)|^p\ |x|^{p-2}\,dx \leqq C\int_{\mathbb{R}} |f(x)|^p\,dx , \quad 1 < p \leqq 2 .$$

It is known ([9]) that if the L^p-norm on the right of (3.1) is replaced by an H^p-norm, then (3.1) also holds for $0 < p \leqq 1$. We return to this point in the next section.

A duality argument using (3.1) gives the second Titchmarsh extension namely

$$\text{(3.2)} \qquad \left(\int_{\mathbb{R}} |f(x)|^q\ u(x)\ dx\right)^{1/q} \leqq C\left(\int_{\mathbb{R}} |\hat{f}(x)|^p\ v(x)\ dx\right)^{1/p}$$

with $p = q > 2$, $u \equiv 1$ and $v(x) = |x|^{q-2}$. Various extensions of these results are obtained by further interpolation arguments. We mention here Rooney's generalization [35] of Pitt's theorem [34]: If $1 < p < \infty$, $1 < r \leqq \min(p,p')$, and $p \leqq q \leqq r'$, then (3.2) holds with $u(x) = |x|^{q/r'-1}$ and $v(x) = |x|^{p/r-1}$. In a different direction Hirschman [18] showed that if u is a weak-$L^{1/\alpha}$ function, that is, u satisfies for each $\lambda > 0$

$$\lambda^{1/\alpha}|\{x \in \mathbb{R} : u(x) > \lambda\}| \leqq C , \quad 0 < \alpha < 1 ,$$

then again, (3.2) holds with $p = q = 2$ and $v(x) = |x|^{\alpha}$. A similar but more recent result of Aguilera and Harboure [1] is that if for each measurable $E \subset \mathbb{R}$, u satisfies

$$\int_E u(x)\,dx \leq C|E|^{p-1}, \quad 1 < p < 2,$$

then (3.2) is satisfied with $p = q$ and $v \equiv 1$.

These results were further generalized by Flett [11] who considered functions ϕ and ψ (non-negative measurable) such that $|\{x \in \mathbb{R} : \phi(x) \leq y\}| \leq y$ and similarly for ψ. Then, if $1 < r \leq 2$, $r \leq p \leq q \leq r'$, (3.2) holds with $u = \phi^{q/r'-1}$ and $v = \psi^{p/r-1}$.

We now define the weight classes for the first theorem of this section.

DEFINITION 3.1. Let u and v be non negative, locally integrable functions, $0 < p$, $q < \infty$, $p \geq 1$ and write (as before) $\sigma = (1/v)^{p'-1}$. The functions u, v belong to the class $F_{p,q}$ if

$$(3.3) \qquad \sup_{s>0} (Pu)(1/s)^{1/q}(P\sigma)(s)^{1/p'} \equiv A < \infty \quad \text{for} \quad 1 \leq p \leq q < \infty,$$

and in case $0 < q < p < \infty$, $p \geq 1$, the two conditions

$$(3.4) \qquad \begin{gathered} \int_0^\infty [(Pu)(1/x)^{1/q}(P\sigma)(x)^{1/q'}]^r \sigma(x)\,dx < \infty, \\ \int_0^\infty \Big[\Big(\int_{1/x}^\infty t^{-q/2} u(t)\,dt\Big)^{1/q}\Big(\int_x^\infty t^{-p'/2} \sigma(t)\,dt\Big)^{1/q'}\Big]^r x^{-p'/2}\sigma(x)\,dx < \infty \end{gathered}$$

are satisfied with $1/r = 1/q - 1/p$. Recall that P denotes the Hardy operator discussed in Section 1.

DEFINITION 3.2. Let f be Lebesgue measurable on $\mathbb{R}$. The *distribution function* of f is defined by $D_f(y) = |\{x \in \mathbb{R} : |f(x)| > y\}|$, $y > 0$, and the *equimeasurable decreasing rearrangement* of $|f|$ by $f^*(t) = \inf\{y > 0 : D_f(y) \leq t\}$.

Clearly

$$\int_{\mathbb{R}} |f(x)|\,dx = \int_0^\infty f^*(t)\,dt$$

and it is well known that the rearrangement commutes with exponentiation: $(f^q)^* = (f^*)^q$. For properties of rearrangements of functions we refer to [25] and [40].

If the weights u and $1/v$ in $F_{p,q}$, that is, (3.3) and (3.4) are replaced by their rearrangements u^* and $(1/v)^*$ (hence σ by

$(1/v)*^{p'-1}$), then we write $(u,v) \in F^*_{p,q}$.

REMARK 3.1. If u and $1/v$ are even and decreasing on $\mathbb{R}_+$, then $u^*(x) = u(x/2)$ and $(1/v)^*(x) = (1/v)(x/2)$, $x > 0$. Therefore in this case $(u,v) \in F^*_{p,q}$ if and only if

$$\sup_{s>0} (Pu)(1/2s)^{1/q}(P\sigma)(s/2)^{1/p'} \equiv A < \infty \quad \text{if} \quad 1 \leq p \leq q < \infty \tag{3.5}$$

and for $0 < q < p$, $p \geq 1$

$$\int_0^\infty [(Pu)(1/4x)^{1/q} \; (P\sigma)(x)^{1/q'}]^r \; \sigma(x) \; dx < \infty$$

$$\int_0^\infty \Big[\Big(\int_{1/4y}^\infty t^{-q/2} \; u(t) \; dt\Big)^{1/q} \cdot \Big(\int_x^\infty t^{-p'/2} \; \sigma(t) \; dt\Big)^{1/q'}\Big]^r \; x^{-p'/2} \; \sigma(x) \; dx < \infty , \tag{3.6}$$

$1/r = 1/q - 1/p$.

The main result of this section is the following:

THEOREM 3.1. ([4]). *Suppose* $(u,v) \in F^*_{p,q}$, $0 < p$, $q < \infty$, $p \geq 1$ *and* $f \in L^p_v$.

(i) *If* $\lim_{n\to\infty} ||f_n - f||_{p,v} = 0$ *for a sequence of simple functions* $\{f_n\}$, *then* $\{\hat{f}_n\}$ *converges in* L^q_u *to a function* $\hat{f} \in L^q_u$. *The function* $\hat{f}$ *is independent of the sequence* $\{f_n\}$ *and is called the Fourier transform of* f .

(ii) *There exists a constant* $C > 0$, *such that for all* $f \in L^p_v$

$$||\hat{f}||_{q,u} \leq C||f||_{p,v} . \tag{3.7}$$

(iii) *If* $g \in L^{q'}_{1/u}$, $q > 1$ *and* $(1/v, 1/u) \in F^*_{q',p'}$, (*this condition is always satisfied if* $(u,v) \in F^*_{p,q}$ *for* $1 \leq p \leq q < \infty$), *then Parseval's formula*

$$\int_{\mathbb{R}} \hat{f}(y) \; g(y) \; dy = \int_{\mathbb{R}} f(x) \; \hat{g}(x) \; dx$$

holds.

The key element of the proof of this theorem is the estimate (3.7). This was proved by Muckenhoupt [31], [32], Jurkat and Sampson [23] and Heinig [14], independently, in the case $1 \leq p \leq q < \infty$.

(The condition $F^*_{p,q}$ in case $p = q = 2$ in [14] is replaced by another condition.) For monotone weights, see also [3]. The case $1 < q < p < \infty$ was proved by Benedetto, Heinig and Johnson [4] (see also [15]) and the case $0 < q < 1$, $p \geq 1$ by Sinnamon [38]. Conceptually the proof of (3.7) is not difficult. One applies Calderón's estimate ([6])

$$(Tf)^*(x) \leq C\left[\int_0^{1/x} f^*(t)dt + x^{-1/2}\int_{1/x}^{\infty} t^{-1/2} f^*(t)dt\right],$$

which holds for all operators of type $(1,\infty)$ and $(2,2)$, so in particular for $Tf = \hat{f}$. But this means that $(Tf)^*$ is dominated by the sum of a Hardy operator and a dual Hardy operator. Applying the weighted estimates of Theorem 2.1 for these operators together with Minkowski's inequality and properties of rearrangements one obtains (3.7). For the case $p = q = 2$, the Calderón estimate must be replaced by the inequality

$$\int_0^x [(Tf)^*(t)]^2\, dt \leq C\int_0^x \left[\int_0^{1/t} f^*(y)\, dy\right]^2 dt$$

due to Jodeit and Torchinsky ([22, Prop. 3.1]). Since this estimate is necessary and sufficient for any operator of type $(1,\infty)$ and $(2,2)$, the result follows essentially as above (see [4, Prop. 3.1]).

If u and $1/v$ are even, define $f(y) = \chi_{(0,s/2)}(|y|)\sigma(y)$, $s > 0$, where $\sigma = v^{1-p'}$. Substituting this into (3.7) one obtains on reducing the integral on the left side

$$\left(\int_0^{1/(2s)} u(x)\left[\int_0^{s/2} \cos xy\ \sigma(y)\ dy\right]^q dx\right)^{1/q} \leq C\left(\int_0^{s/2} \sigma(y)dy\right)^{1/p}.$$

But $0 < x \leq 1/(2s)$ and $0 < y \leq s/2$ implies $0 < xy \leq 1/4$ so that $\cos xy \geq \cos 1/4 > 0$. Hence we obtain for all $s > 0$, $(Pu)\left(1/(2s)\right)^{1/q}(P\sigma)(s/2)^{1/p'} \leq C$ and thus the inequality (3.7) is essentially sharp:

THEOREM 3.2. *If* $0 < p,q < \infty$ *and* (3.7) *is satisfied for all* f *with* u *and* v *even, then* (3.5) *holds. In particular, if* u *and* $1/v$ *are in addition non-increasing in* $\mathbb{R}_+$, *then* (*by* Remark 3.1) $(u,v) \in F^*_{p,q}$ *for* $1 < p \leq q < \infty$.

REMARK 3.2. The proof of (3.7) shows that this estimate holds not only for the Fourier transform, but also any operator of (weak) type $(1,\infty)$ and $(2,2)$. In Particular, one obtains this result also for the Laplace transform L. That is, if $(u,v) \in F^*_{p,q}$, $0 < q < p$, $p \geq 1$, then $L \in [L^p_v, L^q_u]$. (Compare this with Theorem 2.3.)

The estimate (3.7) holds for other operators which are not of type $(1,\infty)$ and $(2,2)$ but differ from such operators by some power function. The *Hankel transformation* H_λ defined by

$$(H_\lambda f)(x) = \int_0^\infty (xt)^{1/2} J_\lambda(xt) f(t)\, dt , \quad x > 0 , \quad \lambda \geq -\frac{1}{2} ,$$

where J_λ is the Bessel function of order λ, is such an operator. For that operator one has

THEOREM 3.3. ([10]). *Let* u *and* v *be non-negative functions on* $\mathbb{R}_+$ *such that* $u_\lambda(x) = x^{\lambda+1/2}u(x)$, $v_\lambda(x) = x^{-\lambda-1/2}v(x)$, $\lambda \geq -1/2$. *If* $(u_\lambda, v_\lambda) \in F^*_{p,q}$, $1 < p,q < \infty$, *then* $H_\lambda \in [L^p_v(\mathbb{R}_+), L^q_u(\mathbb{R}_+)]$.

This result gives in particular an estimate of the Fourier transform of a radial function on $\mathbb{R}^n$. Moreover the weights are near optimal in the range $1 < p \leq q < \infty$.

Before continuing our discussion on weighted Fourier estimates, we note that Theorem 3.1 contains the previously mentioned results.

In the case of Aguilera and Harboure, the condition

$$\int_E u(x)\, dx \leq C|E|^{p-1} , \quad 1 < p < 2 ,$$

for every measurable set $E \subset \mathbb{R}$ is equivalent to

$$\int_0^t u^*(x)\, dx \leq C|t|^{p-1} , \quad t = |E| .$$

But since

$$\left(\int_0^{1/t} u^*(x)dx\right)^{1/p}\left(\int_0^t dx\right)^{1/p'} \leq Ct^{(1-p)/p + 1/p'} = C ,$$

$(u,1) \in F^*_{p,p}$ and therefore their result follows from Theorem 3.1.

To show that Flett's result follows from Theorem 3.1, let $u = \phi^{q/r'-1}$, $1/v = \psi^{-p/r+1}$, $r \leq p \leq q \leq r'$, then it suffices to shown that $(u,v) \in F^*_{p,q}$. But since ϕ and ψ satisfy

$$|\{x : \phi(x) < y\}| \leq y \quad \text{and} \quad |\{x : \psi(x) < y\}| \leq y ,$$

we obtain on substituting

$$|\{x : u(x) > y^{q/r'-1}\}| \leq y \quad \text{and}$$
$$|\{x : (1/v)(x) > y^{1-p/r}\}| \leq y .$$

Now let $t = y^{q/r-1}$, respectively $t = y^{1-p/r}$, then in terms of the notation of Definition 3.2, this becomes $D_u(t)^{q/r'-1} \geq t$ respectively $D_{1/v}(t)^{1-p/r} \geq t$, and in terms of rearrangements $u^*(x) \leq x^{q/r'-1}$ and $(1/v)^*(x) \leq x^{1-p/r}$. But then

$$\left(\int_0^{1/s} u^*(x)\,dx\right)^{1/q}\left(\int_0^{s} (1/v)^*(x)^{p'-1}\,dx\right)^{1/p'}$$
$$\leq \left(\int_0^{1/s} x^{q/r'-1}\,dx\right)^{1/q}\left(\int_0^{s} x^{(1-p/r)(p'-1)}\,dx\right)^{1/p'} \leq C$$

for all $s > 0$ and hence $(u,v) \in F^*_{p,q}$, $r \leq p \leq q \leq r'$.

We therefore obtain not only Hörmander's multiplier estimate, ([19]) but also Flett's generalization, namely that if $1 < p \leq 2 \leq q < \infty$, $\alpha = 1/p - 1/q$ and $|\{x \in \mathbb{R} : \phi(x) \leq y\}| \leq y$, $y > 0$, $\phi \geq 0$, then for $f \in S$ (Schwartz class), $||(\hat{f}\phi^{-\alpha})^{\wedge}||_q \leq C||f||_p$, and indeed much more. (The n-dimensional version may be deduced from [4, Theorem 3.3] or [14].)

Now we establish Fourier estimates with weights in the Muckenhoupt A_p class. Although the weight conditions now are not as sharp as the ones of Theorem 3.1, some of these results are needed in Section 4, also they may be of intrinsic interest.

DEFINITION 3.3. A non-negative locally integrable function w is in A_p, $1 < p < \infty$, if for each interval $I \subset \mathbb{R}$

$$\left(\int_I w(x)\,dx\right)^{1/p}\left(\int_I \sigma(x)\,dx\right)^{1/p'} \leq A\,|I| ,$$

where $\sigma = w^{1-p'}$.

It is well known ([7], [30]) that if $w \in A_p$, $p > 1$, then $w \in A_r$ for some $r < p$. The number $p_0 = \inf_{r<p} r$ is called the critical exponent for w. Further, as is easily verified, $w \in A_p$ if and only if $\sigma \in A_{p'}$. Another property of A_p needed in the sequel is

LEMMA 3.1. ([20, Lemma 1]). *If* $w \in A_p$, $1 < p < \infty$, *then for each* $x > 0$

$$(3.8)\qquad \int_x^\infty y^{-p}\, w(y)\, dy \le Cx^{-p} \int_0^x w(y)\, dy \ .$$

Utilizing this lemma we obtain the following corollary of Theorem 3.1:

COROLLARY 3.1. *Let* u *and* v *be even, non-decreasing weight functions on* $\mathbb{R}_+$ *with* $u \in A_q$ *and* $\sigma = v^{1-p'}$, $1 < p \le q < 2$. *If for each* $s > 0$,

$$(3.9)\qquad \left(\int_0^s u(x)\, dx\right)\left(\int_0^s \sigma(x)\, dx\right)^{q/p'} \le Cs^q ,$$

then

$$(3.10)\qquad \left(\int_{\mathbb{R}} |\hat{f}(x)|^q |x|^{q-2} u(1/x) dx\right)^{1/q} \le C\left(\int_{\mathbb{R}} |f(x)|^p v(x) dx\right)^{1/p} .$$

Of course this is another generalization of Titchmarsh's extension of the Hausdorff-Young inequality.

P r o o f . Let $w(x) = |x|^{q-2}\, u(1/x)$, then w is non-increasing on $\mathbb{R}_+$. If $(w,v) \in F^*_{p,q}$, then the result follows from Theorem 3.1. But $(w,v) \in F^*_{p,q}$ is by (3.5) (and a change of variable) equivalent to

$$\sup_{s>0} \left(\int_{2s}^\infty y^{-q} u(y) dy\right)^{1/q} \left(\int_0^{s/2} \sigma(y) dy\right)^{1/p'} \equiv B < \infty \ .$$

Now by (3.8) and (3.9) the product of these two integrals is finite so the corollary is proved. |||

Observe that if in addition $v \in A_p$, then the result holds with (3.9) replaced by

$$\left(\int_0^s u(x)\, dx\right)\left(\int_0^s v(x)\, dx\right)^{-q/p} \le C \ .$$

There is a certain dual result to Corollary 3.1 for functions defined on $\mathbb{R}^n$, for which the following well known weighted L^p-inequality for the Hardy-Littlewood maximal operator is required. Now however we use the obvious n-dimensional analogue of the definition of A_p (Definition 3.3) where the intervals are replaced by cubes Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

PROPOSITION 3.1. ([7], [39]). *Suppose* M *is the Hardy-Littlewood* *maximal operator, then*

$$(3.11)\qquad \int_{\mathbb{R}^n} |(Mf)(x)|^p\, u(x)\, dx \leq C\int_{\mathbb{R}^n} |f(x)|^p\, u(x)\, dx\ ,$$

$1 < p < \infty$, *if and only if* $u \in A_p$.

Using this result one obtains the following weighted Fourier inequality for functions with vanishing moment. The proof follows very much along the ideas of recent results of C. Sadosky and R. L. Wheeden (Some weighted norm inequalities for the Fourier transform of functions with vanishing moments, (Preprint)) who also obtained weighted (L^p, L^q)-estimates, $p \leq q$.

<u>THEOREM 3.4</u>. *Suppose* $f \in S$, *the Schwartz class, and*

$$\int_{\mathbb{R}^n} f(x)\, dx = 0\ .$$

If $u \in A_p$, $1 < p < \infty$, *then*

$$(3.12)\qquad \Big[\int_{\mathbb{R}^n} |\hat{f}(x)|^p u(x/|x|^2)\, |x|^{-2n}\, dx\Big]^{1/p} \leq C\Big[\int_{\mathbb{R}^n} |f(x)x|^p\, u(x)\, dx\Big]^{1/p}\ .$$

P r o o f . Since the integral of f is zero, one obtains

$$|\hat{f}(x)| = \Big|\int_{\mathbb{R}^n} (e^{ix\cdot t} - 1)\, f(t)\, dt\Big|$$

$$\leq \Big|\int_{|x||t|\leq 1} \frac{e^{ix\cdot t} - 1}{x\cdot t}\, x\cdot t\, f(t)\, dt\Big| + 2\Big|\int_{|x||t|\geq 1} f(t)\, dt\Big|$$

$$\leq C\Big[|x| \int_{|x|\leq |t|^{-1}} |tf(t)|dt + \int_{|x|>|t|^{-1}} |f(t)|dt\Big]$$

where we used the fact that the inner product of x and y satisfirs $x\cdot y \leq |x|\ |y|$. Now by Minkowski's inequality

$$\Big(\int_{\mathbb{R}^n} |\hat{f}(x)|^p u(x/|x|^2)\, |x|^{-2n}\, dx\Big)^{1/p} \leq$$

$$\leq C\Big[\int_{\mathbb{R}^n} u(x/|x|^2)\Big(|x| \int_{|t|\leq |x|^{-1}} |tf(t)|dt\Big)^p |x|^{-2n}\, dx\Big]^{1/p}$$

$$+ C\Big[\int_{\mathbb{R}^n} u(x/|x|^2)\Big(\int_{|t|>|x|^{-1}} |f(t)|dt\Big)^p |x|^{-2n}\, dx\Big]^{1/p} \equiv C(I_0 + I_1)$$

respectively. Now let $y_i = x_i/|x|^2$, $i = 1,2,\dots,n$; then $|y| = |x|^{-1}$ and with $g(t) = |t| f(t)$, Proposition 3.1 shows that

$$I_0 = \left[\int_{\mathbb{R}^n} u(y) \Big(|y|^{-1} \int_{|t| \leq |y|} |g(t)|\,dt\Big)^p\, dy\right]^{1/p}$$

$$\leq \left[\int_{\mathbb{R}^n} u(y) (Mg)(y)^p\, dy\right]^{1/p} \leq C\Big(\int_{\mathbb{R}^n} |g(x)|^p\, u(x)\, dx\Big)^{1/p} .$$

To estimate I_1 , the same substitution as above, duality, a change of order of integration and Hölder's inequality shows that

$$I_1 = \left[\int_{\mathbb{R}^n} u(y) \Big(\int_{|t|>|y|} |f(t)|\,dt\Big)^p\, dy\right]^{1/p}$$

$$= \sup \left|\int_{\mathbb{R}^n} h(y) \int_{|t|>|y|} |f(t)|\, dt\, dy\right|$$

$$\leq \sup \int_{\mathbb{R}^n} |g(t)| \Big(|t|^{-1} \int_{|y| \leq |t|} |h(y)|\,dy\Big)\, dt$$

$$\leq \Big(\int_{\mathbb{R}^n} |g(t)|^p\, u(t)\, dt\Big)^{1/p} \sup \Big(\int_{\mathbb{R}^n} |(Mh)(t)|^{p'}\, u(t)^{1-p'}\, dt\Big)^{1/p'}$$

where as above $g(t) = |t| f(t)$ and the supremum is taken over all functions h , satisfying $\|h\|_{p', u^{1-p'}} \leq 1$. But since $u \in A_p$ implies $u^{1-p'} \in A_{p'}$, then by Proposition 3.1 the last integral is dominated by

$$\Big(\int_{\mathbb{R}^n} |h(t)|^{p'}\, u(t)^{1-p'}\, dt\Big)^{1/p'} \leq 1 .$$

This proves the theorem. |||

In a similar way, R. Johnson (personal communication) proved that (3.12) holds also with two weights and p on the left side replaced by q , $q \geq p$, provided certain weight conditions for the two weight functions are satisfied. Of course if $p = q$ and the weights are the same; his result reduces to Theorem 3.4.

The two weight-mixed-norm results of Sadowsky and Wheeden mentioned above are somewhat different. One of their results for the case $n = 1$ can be described as follows:

Let f *be a Schwartz function whose Fourier transform has compact* *support not containing the origin and satisfies*

$$\int_{\mathbb{R}} f(x)\, x^j\, dx = 0\ ,\quad j = 0,1,2,\dots\ .$$

If $1 < p \le q < \infty$, $w^{q/p} \in A_{1+q/p}$, *and* k *a positive integer, then*

$$(3.13)\qquad \Big(\int_{\mathbb{R}} |\hat f(x)|^q |x|^{-kq+a}\, w(1/x)^{q/p}\, dx\Big)^{1/q}$$

$$\le C\Big(\int_{\mathbb{R}} |f(x)|^p |x|^{kp}\, w(x)\, dx\Big)^{1/p}\ ,$$

where a *in* (3.13) *is equal to* $q/p' - 1$.

For $q = p$ and $k = 1$, this result reduces to (3.12) with $n = 1$.

Our discussion on weighted (L^p, L^q)-estimates for the Fourier transform concludes with some results of Kerman and Sawyer as well as a result of Benedetto, Heinig and Johnson.

THEOREM 3.5. ([24]). *Suppose* u *is an even, locally integrable function on* $\mathbb{R}$ *which is convex and decreases to zero on* $\mathbb{R}_+$. *Then for any* $v(x) \ge 0$, *the Fourier transform* $T \in [L_v^2(\mathbb{R}), L_u^2(\mathbb{R})]$ *if and only if for all intervals* I

$$\int_I \bar M(\chi_I v^{-1})(x)^2\, dx \le A \int_I v(x)^{-1}\, dx$$

where

$$(\bar M f)(x) = \sup_{x \in I}\ \Big(\int_0^{|I|^{-1}} u(y)^{1/2}\, dy\Big) \int_I f(y)\, dy\ .$$

The next theorem utilizes Theorem 2.4.

THEOREM 3.6. ([37, Prop. 1]). *Suppose* $u(x,y)$, $v(x,y)$ *are symmetric about the coordinate axes,* u *and* $1/v$ *are decreasing in each variable separately on* $\mathbb{R}_+^2$, *and* $\partial^2 u/\partial x \partial y \ge 0$ *on* $\mathbb{R}_+^2$. *Then the Fourier transform* $T \in [L_v^2(\mathbb{R}_+^2), L_u^2(\mathbb{R}_+^2)]$ *if and only if* $P_2 \in [L_v^2(\mathbb{R}_+^2), L_w^2(\mathbb{R}_+^2)]$ *where* $w(x,y) = x^{-2}y^{-2}u(x^{-1},y^{-1})$.

THEOREM 3.7. ([4,Theorem 3.3]). *If* $1 \le p \le q < \infty$, $q > 1$, *then the following conditions are equivalent:*

(a) $(u,v) \in F^*_{p,q}$,

(b) $T \in [L_v^p(\mathbb{R}^n), L_{u_1}^q(\mathbb{R}^n)]$,

(c) $\qquad T \in [L^p_v(\mathbb{R}^n), L^q_{u_\phi}(\mathbb{R}^n)]$,

where T *is the Fourier transform,* $|x|u_1(x) = (Pu^*)(|x|)$ *and*

$$u_\phi(t) = |t|^{-1} \int_0^{|t|} u^*(y)\phi(|t|/y)\, dy\ , \qquad \phi(s) = \ln^{n-1}(t)/(n-1)!$$

4. Fourier estimates of functions in weighted Hardy spaces

We remarked in Section 3 that Titchmarsh's extension of the Hausdorff-Young inequality (3.1) holds also for $0 < p \leq 1$, provided the L^p-norm on the right side of (3.1) is replaced by the Hardy space metric.

In order to realize a weighted extension of this for Fourier transforms and more general integral operators, we define first the weighted atomic Hardy spaces studied by Garcia-Cuerva [12].

DEFINITION 4.1. Let $w \in A_s$, $s > 1$, with critical exponent q_0 (see Definition 3.3). A real valued function a defined on $\mathbb{R}$ is called a *weighted* (r,s)*-atom*, $0 < r \leq 1$, *with weight* w if

(i) a is supported in an interval $I \subset \mathbb{R}$,

(ii) $\left(\int_I |a(x)|^s w(x)dx\right)^{1/s} \leq w(I)^{-1/r + 1/s}$ if $s < \infty$,

and

$||a||_\infty \leq w(I)^{-1/r}$

(here $w(I) = \int_I w(x)\, dx$),

(iii) $\int_{\mathbb{R}} x^k\, a(x)\, dx = 0$, $k = 0,1,2,\ldots,[q_0/r]-1$,

where [ ] denotes the greatest integer function.

The smallest closed interval containing the support of a is called the *supporting interval of* a .

The *weighted atomic Hardy space* H^r_w , $0 < r \leq 1$, with weight w consists of those distributions f of the form

$$f = \sum_j \lambda_j a_j\ ,$$

where a_j are weighted (r,s)-atoms with weight w and λ_j real numbers. The metric in H_w^r is defined by

$$||f||_{H_w^r} = \left[\inf \{ \sum_j |\lambda_j|^r : f = \sum_j \lambda_j a_j \}\right]^{1/r} .$$

In order to prove Fourier estimates for functions in these spaces, we consider the integral operator K defined by

$$(4.1) \qquad (Kf)(x) = \int_{\mathbb{R}} K(x,y) f(y)\, dy ,$$

where the kernel $K(x,y)$ is homogeneous of degree -1, that is the kernel satisfies $K(x\lambda, y\lambda) = \lambda^{-1} K(x,y)$, $\lambda > 0$. Note that with $K(x,y) = e^{iy/x} x^{-1}$, $x, y \in \mathbb{R}$, or $K(x,y) = e^{-y/x} x^{-1}$, $x, y \in \mathbb{R}_+$ we see that $(Kf)(x) = \hat{f}(1/x)/x$, respectively, $(Kf)(x) = (Lf)(1/x)/x$. Here L denotes the Laplace transform. These two operators are of specific interest here.

<u>THEOREM 4.1</u>. ([17, Theorem 3.3]). *Let* Kf *be defined by* (4.1). *If*

(i) $K(1,y) \equiv k(y)$ *has bounded derivatives of order* n, $n \geq 1$,

(ii) $u \in A_{(n+1)p}$, $v \in A_{(n+1)r}$, $1/(n+1) < r \leq p < 2/(n+1)$, *where* u *and* v *are even, and the critical exponent of* v *satisfies* $q_0 > nr$,

(iii) *for each interval* $I \subset \mathbb{R}$, $u(0,|I|)v(I)^{-p/r} \leq C < \infty$,

(iv) *for each weighted* $(r,(n+1)r)$*-atom* a *with weight* v $||Ka||_{(n+1)p,u} \leq C||a||_{(n+1)r,v}$,

then for any $f \in H_v^r$

$$(4.2) \qquad K \in [H_v^r, L_u^p(\mathbb{R}_+)] .$$

Applying this result to either the Fourier transform or the Laplace transform, we get the following result:

<u>THEOREM 4.2</u>. *Let* u *and* v *be even, non-decreasing on* $\mathbb{R}_+$. *If* $u \in A_{(n+1)p}$, $v \in A_{(n+1)r}$, $1/(n+1) < r \leq p \leq 2/(n+1)$, $n \geq 1$ *and the critical index* q_0 *of* v *satisfies* $q_0 > nr$ *then*

$$\sup_{s>0} u(0,s) v(0,s)^{-p/r} < \infty$$

implies

$$\left(\int_0^\infty |(Tf)(x)|^p |x|^{p-2} u(1/x)\, dx\right)^{1/p} \leq C||f||_{H_v^r} ,$$

where T *is the Laplace- or Fourier transform.*

P r o o f . Since Corollary 3.1 holds also for the Laplace transform in place of the Fourier transform (with $\mathbb{R}$ replaced by $\mathbb{R}_+$) it suffices to show (iv) of Theorem 4.1 with $(Kf)(x) = \hat{f}(1/x)/x$ or $(Kf)(x) = (Lf)(1/x)/x$.

Let $(n+1)p = \bar{q}$ and $(n+1)r = \bar{p}$, then $u \in A_{\bar{q}}$, and $v \in A_{\bar{p}}$. Hence by Corollary 3.1 (and the observation following the proof) shows that (3.10) holds if (3.9) is replaced by

$$\left(\int_0^s u(x)\ dx\right)\left(\int_0^s v(x)\ dx\right)^{-\bar{q}/\bar{p}} \leqq C$$

for all $s > 0$. But $\bar{q}/\bar{p} = p/r$ so that

$$(4.3) \qquad \left(\int |(Tf)(x)|^{\bar{q}}|x|^{\bar{q}-2}u(1/x)\ dx\right)^{1/q} \leqq C\left(\int |f(x)|^p v(x)\ dx\right)^{1/p} ,$$

(where T is the Laplace- or Fourier transform) holds whenever

$$\sup_{s>0} u(0,s)v(0,s)^{-p/r} < \infty .$$

A change of variable in the left integral of (4.3) shows that $||Kf||_{(n+1)p,u} \leqq C||f||_{(n+1)r,v}$ which is (iv) of Theorem 4.1. This proves the theorem. |||

<u>REMARK</u>. It is clear that there are non-constant weights u and v which satisfy the conditions of Theorem 4.2. On the other hand the weight condition is only sufficient for the weight estimate. It also would be desirable to eliminate the monotonicity conditions imposed on u and v in Theorem 4.2 which means the elimination of the monotonicity condition in Corollary 3.1.

R e f e r e n c e s

[1] N. E. AGUILERA, E. O. HARBOURE: *On the search for weighted norm inequalities for the Fourier transform.* Pac. J. Math. 104 (1) (1983), 1-14.

[2] K. F. ANDERSEN, H. P. HEINIG: *Weighted norm inequalities for certain integral operators.* SIAM J. Math. Anal. 14 (4) (1983), 834-844.

[3] J. J. BENEDETTO, H. P. HEINIG: *Weighted Hardy spaces and the Laplace transform.* Lecture Notes Math. 992, Springer-Verlag 1982 , 240-277.

[4] J. J. BENEDETTO, H. P. HEINIG, R. JOHNSON: *Weighted Hardy spaces and the Laplace transform II.* (Preprint).

[5] J. S. BRADLEY: *Hardy inequality with mixed norms*. Canad. Math. Bull. 21 (1978), 405-408.

[6] A. P. CALDERÓN: *Spaces between* L^1 *and* L^∞. Studia Math. 26 (1966), 273-299.

[7] R. R. COIFMAN, C. FEFFERMAN: *Weighted norm inequalities for the maximal functions and singular integrals*. Studia Math. 51 (1974), 241-250.

[8] R. R. COIFMAN, G. WEISS: *Extension of Hardy spaces and their uses in analysis*. Bull. Amer. Math. Soc. 83 (1977), 569-645.

[9] P. L. DUREN: *Theory of* H^p*-spaces*. Academic Press, New York, 1970.

[10] S. A. EMARA, H. P. HEINIG: *Weighted norm inequalities for the Hankel- and K-transformations* (to appear).

[11] T. M. FLETT: *Some elementary inequalities with applications to the Fourier transforms*. J. Lond. Math. Soc. 7 (2), (1973), 376-384.

[12] J. GARCIA-CUERVA: *Weighted* H^p*-spaces*. Diss. Math. CLXII, Warszawa (1979), 1-63.

[13] G. H. HARDY, J. E. LITTLEWOOD, G. POLYA: *Inequalities*. Cambridge, 1952.

[14] H. P. HEINIG: *Weighted norm inequalities for classes of operators*. Indiana Univ. Math. J. 33 (4), (1984), 573-583.

[15] H. P. HEINIG: *Estimates for operators in mixed weighted* L^p *spaces*. Trans. Amer. Math. Soc. 287 (2), (1985), 483-493.

[16] H. P. HEINIG: *Weighted norm inequalities for certain integral operators II*. Proc. Amer. Math. Soc. 95 (3), (1985), 387-395.

[17] H. P. HEINIG: *Fourier operators on weighted Hardy spaces*. (Preprint).

[18] I. I. HIRSCHMAN Jr.: *Multiplier transformations II*. Duke Math. J. 28 (1961), 45-56.

[19] L. HÖRMANDER: *Estimates for translation invariant operators in* L^p*-spaces*. Acta Math. 103 (1960), 93-140.

[20] R. HUNT, B. MUCKENHOUPT, R. WHEEDEN: *Weighted norm inequalities for the conjugate function and Hilbert transform*. Trans. Amer. Math. Soc. 176 (1973), 227-251.

[21] M. JODEIT Jr., K. K. SHAW: *Hardy kernels and* $H^1(R)$. (Preprint).

[22] M. JODEIT Jr., A. TORCHINSKY: *Inequalities for Fourier transforms*. Studia Math. 37 (1971), 245-276.

[23] W. JURKAT, G. SAMPSON: *On rearrangement and weight inequalities for the Fourier transform*. Indiana Univ. Math. J. 32 (2) (1984), 257-270.

[24] R. KERMAN, E. T. SAWYER: *Weighted norm inequalities for potentials with applications to Schrödinger operators, Fourier transforms and Carleson measures*. Bull. Amer. Math. Soc. 12 (1) (1985), 112-116.

[25] S. G. KREIN, Ju. I. PETUNIN, E. M. SEMENOV: *Interpolation of linear operators*. Transl. Math. Monographs 54, Amer. Math. Soc. 1982.

[26] A. KUFNER: *Weighted Sobolev spaces*. Teubner, Leipzig, 1980, Wiley, New York 1985.

[27] A. KUFNER, O. JOHN, S. FUČÍK: *Function spaces*. Noordhoff, Leyden, 1977.

[28] W. MAZJA: *Einbettungssätze für Sobolevsche Räume*. Teil 1, Teubner-Texte zur Mathematik, Teubner, Leipzig, 1979.

[29] B. MUCKENHOUPT: *Hardy's inequality with weights*. Studia Math. 44(1972), 31-38.

[30] B. MUCKENHOUPT: *Weighted norm inequalities for the Hardy maximal function*. Trans. Amer. Math. Soc. 165 (1972), 207-226.

[31] B. MUCKENHOUPT: *Weighted norm inequalities for the Fourier transform*. Trans. Amer. Math. Soc. 276 (1983) 729-742.

[32] B. MUCKENHOUPT: *A note on two weight function conditions for a Fourier transform norm inequality*. Proc. Amer. Math. Soc. 88 (1) (1983), 97-100.

[33] B. MUCKENHOUPT: *Weighted norm inequalities for classical operators*. Proc. Symp. Pure Math. 35 (1) (1979), 69-83.

[34] H. R. PITT: *Theorems on Fourier- and power series*. Duke Math. J. 3 (1937), 747-755.

[35] P. G. ROONEY: *Generalized H_p-spaces and Laplace transforms*. Proc. Conf. Oberwolfach, 1968, Birkhäuser, Basel, 146-156.

[36] E. T. SAWYER: *Weighted Lebesgue- and Lorentz norm inequalities for the Hardy operator*. Trans. Amer. Math. Soc. 281 (1) (1984), 329-337.

[37] E. T. SAWYER: *Weighted inequalities for the* n-*dimensional Hardy operator*. Studia Math. 82 (1985), 1-16.

[38] G. SINNAMON: *An integral condition for the weighted Hardy inequality*. (Preprint).

[39] E. W. STEDULINSKY: *Weighted inequalities and degenerate elliptic partial differential equations*. Lect. Notes Math. 1074, Springer-Verlag, 1984.

[40] E. M. STEIN, G. WEISS: *Introduction to Fourier analysis on Euclidean spaces*. Princeton, 1971.

[41] F. A. SYSOEVA: *The generalization of a certain Hardy type inequality*. (Russian), Izv. Vyssh.Uchebn. Zaved. Mat. 6 (49) (1965), 140-143.