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APPROXIMATION SOLVABILITY OF NONLINEAR EQUATIONS

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1. Introduction

One approach to the study of equations in infinite dimensional spaces is to try to obtain a solution as a limit of solutions of related problems which can be solved. We shall be concerned with two types of "approximation solvability". The first will be when we approximate the problem by related finite dimensional problems, the Galerkin method being typical. Here the theory of Approximation proper (A-proper for short) mappings is important. The second will involve the study of various iterative processes, a typical situation being that of obtaining a fixed point of a nonexpansive map as the strong limit of the fixed points of contraction mappings.

A unifying feature of our results is the frequent use, in various circumstances, of the notion of the asymptotic centre of a given sequence. Apart from obtaining new results we are able to considerably simplify proofs of some known results. The notion of asymptotic centre is related to the ball measure of noncompactness β and we make some observations on this. One such is that if $\{x_n\}$ is a sequence in a Hilbert space with $\|x_n\| = 1$ and which converges weakly to 0, then $\beta\{x_n\} = 1$.

Our work is divided into several parts. We begin with some preliminary remarks concerning results about the asymptotic centre. We then prove that accretive type and contractive type mappings are A-proper. Apart from our use of the asymptotic centre our results are new in that we prove the A-properness of contractive type mappings under a weak boundary condition, known as weakly inward. In the final section we prove various strong convergence theorems. These involve iterates of pseudocontractive and nonexpansive maps.

Many of the results here are new or are improvements of known results, while others have been published or are to be published elsewhere.

2. Preliminary notions

Let X be a real Banach space with dual space X^* and let (x,f) denote the value of $f \in X^*$ at $x \in X$. The (normalized) duality map $J : X \to X^*$ is defined by $J(x) = \{f \in X^* : (x,f) = \|x\|^2, \|f\| = \|x\|\}$, or, equivalently, $J(x)$ is the subdifferential of the convex function $\frac{1}{2}\|x\|^2$ (e.g. [22], p.44). The set $J(x)$ is nonempty in any Banach space and is a singleton when X^* is strictly convex. If X^* is uniformly convex, then J is uniformly continuous on bounded subsets of X(Kato [16]). If the norm of X is uniformly Gateaux differentiable, and we write X is (UG), J is uniformly demicontinuous on bounded sets, that is uniformly continuous from X with its norm topology to X^* with the weak* topology (e.g. Lemma 2.2 of [32]).

The notion of the asymptotic centre [10] of a bounded sequence $\{x_n\}$ will feature in many of our arguments. For z in X let $\phi(z) = \limsup_{n\to\infty} \|x_n - z\|$. Then, as is easily verified, ϕ is a continuous, convex function and $\phi(z) \to \infty$ as $\|z\| \to \infty$. Therefore, when X is reflexive, ϕ attains its infimum over every closed, convex set C. Let $K = \{v \in C : \phi(v) = \inf \phi(z),\ z \text{ in } C\}$. Then K is called the asymptotic centre of $\{x_n\}$ relative to C. Clearly K is a closed, bounded, convex set. It is known (Lim [21]) that K consists of a single point if and only if X is uniformly convex in every direction (U.C.E.D.) in the sense of Day, James and Swaminathan [7]. X is U.C.E.D. if, for every nonzero z in X and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1,\ x - y = \lambda z \ \text{ and } \ \|\tfrac{1}{2}(x+y)\| > 1 - \delta$$

imply that $|\lambda| < \varepsilon$.

We have observed [37] that the functional ϕ is related to the ball measure of noncompactness (β) of the sequence $\{x_n\}$. Recall that, given a bounded set D, $\beta(D)$ is defined by

$$\beta(D) = \inf\{r > 0 : D \text{ can be covered by finitely many balls of radius } r\}.$$

If we insist that the centres of these balls lie in some prescribed set E, we write $\beta_E(D)$.

Now let $\{x_n\}$ be a bounded sequence and suppose that $\|x_n - z\|$ has a <u>limit</u> $\phi(z)$ for every z in X. For example, if X is separable, there is always a subsequence for which this holds. This follows by a diagonalization argument and is proved in Lemma 1.1 of Reich [30]. Then, if v is in K, the asymptotic centre of $\{x_n\}$ relative to X,

$$\phi(v) = \beta(\{x_n\}).$$

Indeed, it is clear that $\beta(\{x_n\}) \leq \phi(v)$. If $\beta(\{x_n\}) = \beta$ and $\beta < \phi(v)$, given $\varepsilon > 0$ so that $\beta + \varepsilon < \phi(v)$, there would exist finitely many balls of radius $\beta + \varepsilon$ containing $\{x_n\}$. One of these would contain infinitely many points of the sequence, say

$$\|x_{n_k} - w\| \leq \beta + \varepsilon.$$

As the limit of a subsequence is the limit of the sequence this would give $\phi(w) \leq \beta + \varepsilon < \phi(v)$, a contradiction.

Let B denote the unit ball in X. Given a sequence $\{x_n\} \subset B$ an interesting question is whether $\beta_X\{x_n\} = \beta_B\{x_n\}$. If X is Hilbert space the answer is yes. In general the answer is no, for example, it is no in every L^p space. Nussbaum [24] studied this problem and said a space had the Ball Intersection Property (B.I.P) if equality held. His results show that the asymptotic centre of such a sequence $\{x_n\}$ relative to X need not lie in B. Also, as Lim [21] has shown, if X is not a Hilbert space, there is always a sequence $\{x_n\}$ such that the asymptotic centre does not intersect $\overline{co}\{x_n\}$, the closed convex hull of $\{x_n\}$.

<u>Remark 1.</u> If X has a weakly sequentially continuous duality map, then X has the B.I.P. for every weakly convergent sequence.

For, let $x_n \rightharpoonup x$ ("$\rightharpoonup$" denotes weak convergence). Then for any w in X,

$$\limsup_{n\to\infty} \|x_n - x\|^2 = \limsup_{n\to\infty} \left(x_n - x, J(x_n - x)\right)$$
$$= \limsup \left(x_n - w, J(x_n - x)\right) ,$$

since $J(x_n - x) \rightharpoonup 0$. Therefore,

$$\phi(x) \leq \phi(w).$$

It follows that if $\beta_X\{x_n\} = \beta$ and if $\|x_m - w\| \leq \beta + \varepsilon$, then also $\|x_m - x\| \leq \beta + \varepsilon$; hence $\beta_B\{x_n\} \leq \beta_X\{x_n\}$.

Another simple observation is

<u>Remark 2.</u> Let X be a Banach space (or a dual space) and let $\{x_n\}$ be a sequence with $\|x_n\| = 1$ and with $x_n \rightharpoonup 0$ (or weak*). Suppose there is a point v in the asymptotic centre of $\{x_n\}$ relative to X. Then $v \in B$ and $\frac{1}{2} \leq \phi(v) \leq 1$. The constants are best possible. If X is a Hilbert space, the asymptotic centre consists of the unique point 0 and $\beta\{x_n\} = 1$.
Indeed $\phi(v) \leq \phi(0) = 1$. Also, since $x_n \rightharpoonup 0$, $\|-v\| \leq \liminf \|x_n - v\| \leq \phi(v)$; in particular $\|v\| \leq 1$. Therefore we have,

$$\|x_n\| \leq \|x_n - v\| + \|v\| ,$$

so that,

$$\phi(0) \leq \phi(v) + \|v\| \leq 2\phi(v)$$

and this proves the first part.
Two examples show that the constants are best possible. First take $X = (c_0)$, the space of sequences that converge to zero with the supremum norm. Let x_n

be the sequence with 1 in the nth place and zeros elsewhere. Then, as is readily shown, $\phi(v) = 1$. Next take $X = \ell^\infty$, the space of all bounded sequences with the same norm and let $\{x_n\}$ be the same sequence. This time $\phi(v) = \frac{1}{2}$ since $v = (\frac{1}{2},\frac{1}{2},\frac{1}{2}, \ldots)$ lies in ℓ^∞ (but <u>not</u> in (c_0)). In a Hilbert space H, we have, for any y in H,

$$\|x_n - y\|^2 = \|x_n\|^2 - (x_n,y) - (y,x_n) + \|y\|^2$$

so $\lim_{n\to\infty} \|x_n - y\|^2$ exists and equals $1 + \|y\|^2$. Thus $\inf \phi(y) = \phi(0) = 1$ (and 0 is the unique point). By our earlier comments, $\beta\{x_n\} = 1$.

3. <u>A-properness of accretive and contractive type operators</u>

The class of Approximation-proper (A-proper) [27] mappings arises in a natural way when one tries to determine when solutions of equations in infinite dimensional spaces can be obtained as the limit of solutions of related finite dimensional problems. The A-proper maps are defined in terms of projection schemes. For our purposes we shall be interested in mappings $A : X \to X$ and we shall suppose that X is a <u>$(\pi)_1$ space</u>. This means that there is a sequence $\{P_n\}$ of linear projection operators, each of norm 1, with finite dimensional range X_n where $X_n \subset X_{n+1}$, and such that $P_n x \to x$ as $n \to \infty$. Every separable space with a monotone Schauder basis is a $(\pi)_1$ space. A map $A : X \to X$ is said to be <u>A-proper</u> if $P_n A : X_n \to X_n$ is continuous for each n and whenever $\{x_n\}$ is a bounded sequence, $x_n \in X_n$, such that $P_n A x_n$ has a subsequence convergent to a point f, there is a further subsequence x_{n_k} (say) that converges to a point x and $Ax = f$. A weaker notion is that of A being <u>pseudo-A-proper</u>: under the same hypotheses the weaker conclusion is that there exists x with $Ax = f$.

We shall begin by proving that strongly accretive operators are A-proper. Two types of argument have been employed previously, the direct one, essentially due to Browder and De Figueiredo [4] who required that X have a weakly continuous duality map, which hypothesis is known to be too restrictive outside of Hilbert space. The second method used the fact that strongly accretive operators can be shown to be surjective by using the connection with differential equations and semigroups, see Theorem 3.1 G of [27].

Our proof is direct and relies on the following property of duality mappings.

Proposition 3.1. Let X be a (UG) Banach space and let $\{x_n\}$ be a bounded sequence. Let K be the asymptotic centre of $\{x_n\}$ (relative to X). Then for each v in K,

$$\liminf_{n\to\infty} \big(w, J(x_n - v)\big) \leq 0 \quad \text{for every } w \in X.$$

Proof. As J(x) is the subdifferential of $\frac{1}{2}\|x\|^2$ we have

$$\big(z - v,\ J(x_n - z)\big) \leq \tfrac{1}{2}\|x_n - v\|^2 - \tfrac{1}{2}\|x_n - z\|^2, \text{ for all } z \in X.$$

For $t > 0$ and $w \in X$ let $z_t = v + tw$. Then

$$\begin{aligned}\liminf_{n\to\infty} \big(tw,\ J(x_n - z_t)\big) &\leq \tfrac{1}{2}\liminf \|x_n - v\|^2 - \tfrac{1}{2}\limsup \|x_n - z_t\|^2 \\ &\leq 0, \text{ for } v \in K.\end{aligned}$$

Cancel $t > 0$ and let $t \to 0$. As J is uniformly demicontinuous on bounded sets, the conclusion follows.

Remark. This is a continuation of some interesting results of Reich ([30], [33]). In particular, he proves that $\mathrm{LIM}\big(w, J(x_n - v)\big) = 0$, where LIM is a Banach limit. Proposition 3.1 is a consequence of this but our direct proof is simpler.

Corollary [36]. If X is separable, there is a subsequence $\{x_k\}$ such that $J(x_k - v) \rightharpoonup 0$.

Proof. As noted previously, a result of Reich [30] shows that there is a subsequence $\{x_k\}$ such that $\phi(z) = \lim_{k \to \infty} \|x_k - z\|$ for all z in X. In the above proof we can then obtain $\limsup \left(w, J(x_k - v)\right) \leq 0$ and replacing w by $-w$ gives the result.

We now come to the A-properness of accretive operators. A map $A : D(A) \subset X \to X$ is called <u>accretive</u> if for each $x,y \in D(A)$ there exists $f \in J(x - y)$ such that $(Ax - Ay, f) \geq 0$, or equivalently (e.g. [16]),

$$\|x - y\| \leq \|x - y + \alpha(Ax - Ay)\| \text{, for all } \alpha > 0.$$

Thus $(I + \alpha A)^{-1}$ is <u>nonexpansive</u> on $R(I + \alpha A)$, the <u>range</u> of $I + \alpha A$. An accretive map A is called <u>m-accretive</u> if $R(I + \alpha A) = X$ for some (equivalently all) $\alpha > 0$ and is called <u>strongly accretive</u> (with constant c) if $A - cI$ is accretive.

<u>Theorem 3.1.</u> Let X be a $(\pi)_1$ space with X^* uniformly convex. Let $A : X \to X$ be accretive. Then $\lambda I + A$ is A-proper for all $\lambda > 0$ <u>if and only if</u> A is demicontinuous.

<u>Remark.</u> The "if" part was proved in [36]; the "only if" part has been given in various lectures of the author but appears here for the first time. We only sketch the proof of the "if" part.

<u>Proof.</u> Write A_λ for $\lambda I + A$ and suppose that $P_j A_\lambda x_j \to f$. Since $P_j^* Jx = Jx$ for $x \in X_j$ (see, for example [4]), for all $y \in X_j$ we have

$$\left(P_j A_\lambda x_j - A_\lambda y \text{ , } J(x_j - y)\right) \geq \lambda \|x_j - y\|^2 .$$

Then, for all $y \in \bigcup X_n$,

$$\text{(1)} \qquad \limsup \left(f - A_\lambda y \text{ , } J(x_j - y)\right) \geq \limsup \lambda \|x_j - y\|^2 .$$

If we could set $y = v$, the left hand side of (1) would be zero and the result would follow. Given $\varepsilon > 0$ we take $y = P_m v$ for m large and show that $\phi(P_m v) < \varepsilon$. The uniform continuity of J and the demicontinuity of A are used here. Hence $\phi(v) = 0$ and $x_j \to v$ with $A_\lambda v = f$. For the converse, if $\lambda I + A$ is A-proper for all $\lambda > 0$, then $R(\lambda I + A) = X$ since we can solve all the finite dimensional problems by continuity of $P_n A$ and coercivity of A_λ (implied by accretivity of A). Thus A is m-accretive. Now suppose $x_n \to x$; we wish to show that $Ax_n \rightharpoonup Ax$. By a result of Fitzpatrick, Hess, and Kato [12], A is locally bounded at each interior point of its domain. Thus, Ax_n is bounded so we can suppose, for a subsequence, that $Ax_n \rightharpoonup f$. Now

$$(Ax_n - Az, J(x_n - z)) \geq 0 \text{ , for all } z \in X,$$

so that

$$(f - Az, J(x - z)) \geq 0.$$

Thus $$\|x - z\| \leq \|x - z + (f - Az)\| .$$

As I + A is surjective, there exists z such that

$$z + Az = x + f \text{ and so } x = z \text{ and } Ax = f.$$

As this argument applies to any subsequence the demicontinuity is proven.

We can also prove the A-properness of more general mappings.

<u>Definition 1.</u> A map T is called <u>semiaccretive</u> if there exists a map $S : X \times X \to X$, such that $Tx = S(x,x)$, with the properties

(i) for each fixed $x \in X$, $y \mapsto S(y,x)$ is a demicontinuous accretive map;

(ii) for each $y \in X$, $x \mapsto S(y,x)$ is completely continuous, that is, $x_n \rightharpoonup x$ implies $S(y,x_n) \to S(y,x)$.

T is called strongly semiaccretive (with constant C) if in (i), $S(\cdot,x)$ is strongly accretive with uniform constant C.

The simplest examples of semiaccretive maps are sums of accretive and completely continuous maps.

<u>Definition 2.</u> A continuous map T is called a <u>k-ball contraction</u> if, for every bounded set D, $\beta(T(D)) \leq k\beta(D)$. T is called <u>condensing</u> if

$\beta(T(D)) < \beta(D)$ whenever $\beta(D) \neq 0$.

Clearly a map is compact if and only if it is a 0-ball contraction. There are other measures of noncompactness, for example the Kuratowski measure defined in terms of coverings by sets of a given diameter, and a corresponding class of k-set contractions. We refer to [22] and [35] for more information.

The following theorem proves that the two types of mappings defined above can be related using the theory of A-proper maps. It extends a result of [37] in which an hypothesis stronger than (ii) of Definition 1 was used. That result was extensions of earlier results of Toland [34], Webb ([34], appendix), Petryshyn [26] and Milojević [23] which dealt with continuous accretive maps. We can deal with demicontinuous maps provided we assume X^* is uniformly convex.

Theorem 3.2. Let X be a $(\pi)_1$ space in which the projections are compatible, that is $P_jP_k = P_j$ for $k > j$, and with X^* uniformly convex and let $T : X \to X$ be demicontinuous and strongly semiaccretive with constant c. Let $B : X \to X$ be a k-ball contraction and let $F : X \to X$ be condensing. Then

(i) $T + B$ is A-proper if $k < c$;

(ii) $T + B + F$ is A-proper if $1 + k \leqslant c$.

The proof of Theorem 2 relies on the following fact.

Proposition 3.2. Under the hypotheses of Theorem 2 let $\{x_n\}$ be a bounded sequence. Then there exists a subsequence such that $\beta\{P_jTx_j\} \geqslant c\beta\{x_j\}$.

Assuming, temporarily, that this has been shown we give the proof of the theorem.

Proof of Theorem 3.2.

(i) Let $x_n \in X_n$ be a bounded sequence with

$$P_n T x_n + P_n B x_n \to f.$$

Then for a suitable subsequence,

$$\begin{aligned} c\beta\{x_j\} \leq \beta\{P_j T x_j\} &= \beta\{P_j B x_j\} \\ &\leq \beta\{B x_j\} \quad \text{(e.g. Lemma 1 of [35])} \\ &\leq k\beta\{x_j\}. \end{aligned}$$

As $k < c$ this implies that $\beta\{x_j\} = 0$, so (for a further subsequence) $x_j \to x$. Then $Bx_j \to Bx$ and $P_j B x_j \to Bx$. Also $Tx_j \rightharpoonup Tx$. Since, for every f in the dense set $\bigcup X_n$,

$$\lim_{j\to\infty} (P_j T x_j, f) = \lim_{j\to\infty} (Tx_j, f) = (Tx, f)$$

it follows that $P_j T x_j \rightharpoonup Tx$ and so $Tx + Bx = f$.

(ii) The proof is exactly similar.

<u>Proof of Proposition 3.2.</u> We can obviously suppose that $\{P_n T x_n\}$ is bounded. As X is separable, there is a subsequence x_k such that $\psi(w) = \lim_{k\to\infty} \|P_k T x_k - w\|$ is well defined for all $w \in X$. Let w be such that $\psi(w) = \beta\{P_k T x_k\}$. For simplicity of notation we can suppose that the same subsequence is weakly convergent, say $x_k \rightharpoonup x$. As $P_k S(\cdot, x)$ is strongly accretive and continuous, there exists $z_k \in X_k$ with $P_k S(z_k, x) = P_k w$. Moreover $\{z_k\}$ is bounded. By Theorem 3.1, $S(\cdot, x)$ is A-proper so there is a subsequence, again denoted by z_k, with $z_k \to z$ and $S(z, x) = w$. For each fixed $j, S(z_j, x_k) \to S(z_j, x)$ as $k \to \infty$, so we can find a subsequence x_{k_j} such that $S(z_j, x_{k_j}) - S(z_j, x) \to 0$. Hence $P_j S(z_j, x_{k_j}) \to w$ as $j \to \infty$. Using the fact that $P_j^* Jy = Jy$ for $y \in X_j$ [4], for each j we have

$$(P_j S(x_{k_j}, x_{k_j}) - P_j S(z_j, x_{k_j}), J(x_{k_j} - z_j)) \geq c\|x_{k_j} - z_j\|^2.$$

Thus

$$c\|x_{k_j} - z_j\| \leq \|P_j S(x_{k_j}, x_{k_j}) - P_j S(z_j, x_{k_j})\|.$$

As $z_j \to z$ and $P_j S(z_j, x_{k_j}) \to w$, given $\varepsilon > 0$, there exists N such that

$$c\,\|x_{k_j} - z\| \leqslant \|P_j S(x_{k_j}, x_{k_j}) - w\| + \varepsilon\ , \text{ for } j \geqslant N.$$

Using the compatibility of the projections and the facts that $\|P_j\| \leqslant 1$ and $P_j x \to x$, we obtain

$$c\,\|x_{k_j} - z\| \leqslant \psi(w) + 2\varepsilon\ , \text{ for } j \text{ sufficiently large.}$$

As ε is arbitrary the proof is complete.

Remark. If we do not assume compatibility of the projections we seem to need a strengthening of (ii) of Definition 1, as was done in [37].

These theorems can be used in the study of the fixed point of pseudo-contractive mappings (e.g. [20]), where U is pseudocontractive if and only if I-U is accretive. We shall mention some strong convergence results for these later. We now consider contractive type mappings where one can prove A-properness results directly without recourse to any connection with accretive operators.

Definition 3. A map $T : D \to X$ is called a generalized contraction (Kirk [18], [19]) if, for each $x \in D$ there is $\alpha(x) < 1$ such that $\|Tx-Ty\| \leqslant \alpha(x)\,\|x-y\|$ for all y in D. T is nonexpansive if $\alpha(x) \equiv 1$, and a strict contraction if $\alpha(x) \equiv k < 1$.

An interesting result of Kirk's [19] is that, if T is continuously Fréchet differentiable on an open, bounded, convex set D then T is a generalized contraction if and only if $\|T'_x\| < 1$ for all x in D.

If $T : X \to X$ is a generalized contraction, it was shown by Wong [40] that I - T is A-proper. This was generalized by Fitzpatrick [11] to the case when T map a closed, convex set D into itself. However, he was only able to prove a weaker version of A-properness, namely that I - T be A-proper at 0.

This means that if $x_n - P_nTx_n \to 0$ for a bounded sequence $x_n \in X_n$, there is a convergent subsequence $x_k \to x$ with $x - Tx = 0$. (We define pseudo-A-proper at 0 similarly.) Both of these results used ideas similar to Kirk's [18]. We can give a simple proof which has the merit of allowing greater generality. Given a closed convex set C and $x \in C$ we define the inward set $I_C(x)$ to be the set $\{z : z = (1 - \alpha)x + \alpha y$ for some $y \in C$ and $\alpha > 1\}$. A map $T : C \to X$ is called <u>inward</u> if $Tx \in I_C(x)$ and <u>weakly inward</u> if $Tx \in \overline{I_C(x)}$. Since $I_C(x) = X$ if x is an interior point of C, these conditions are boundary conditions. Such maps have been studied by Halpern - Bergman [15], Reich [28], Caristi [6], and others. Caristi [6] proved the interesting fact that T is weakly inward if and only if $\lim_{h \to 0} h^{-1}d((1-h)x + hTx, C) = 0$ for all $x \in C$, where $d(y,C)$ denotes the distance from the point y to the convex set C.

This condition had been used by Martin [see e.g. [22]] and others in the study of differential equations in Banach spaces, which can be used to prove fixed point theorems.

Our result is as follows [39].

<u>Theorem 3.3.</u> Let C be a closed, convex subset of a reflexive $(\pi)_1$ space X. Let $T : C \to X$ be a weakly inward generalized contraction. Then $I - T$ is A-proper at 0. Moreover, each map P_nT has a fixed point x_n in $C_n = P_nC$ and $x_n \to x$ the unique fixed point of T.

<u>Proof.</u> Suppose $x_n \in X_n$ is a bounded sequence with $x_n - P_nT(x_n) \to 0$. Let v be in the asymptotic centre of $\{x_n\}$ relative to C. Then,

$$\|x_n - T(v)\| \leq \|x_n - P_nT(x_n)\| + \|P_nT(x_n) - P_nT(v)\| + \|P_nT(v) - T(v)\| .$$

Since $\|P_n\| = 1$ and $P_ny \to y$ as $n \to \infty$, this yields

$$\phi(T(v)) \leq \alpha(v)\phi(v).$$

We claim that $\phi(v) = \inf\{\phi(z) : z \in \overline{I_C(v)}\}$.

For, noting that $C \subset I_C(v)$, as ϕ is continuous, if this were false there would be $w \in I_C(v)$ with $\phi(w) < \phi(v)$. Thus $w = (1 - \alpha)v + \alpha y$ for some $y \in C$ and $\alpha > 1$. Convexity of ϕ yields $\phi(y) \leq \frac{1}{\alpha}\phi(w) + (1 - \frac{1}{\alpha})\phi(v) < \phi(v)$, and this contradicts the definition of v.

Now, since $T(v) \in \overline{I_C(v)}$ and $\alpha(v) < 1$ we must have $\phi(v) = 0$ so that $x_n \to v = T(v)$.

For the last part we observe that P_nT is weakly inward on C_n so by the theorem of Halpern and Bergman [15] for compact weakly inward maps, each P_nT has a fixed point x_n. The earlier argument completes the proof.

That $\phi(v) = \inf\{\phi(z) : z \in \overline{I_C(v)}\}$ was used by Lim [21]. However, he worked in uniformly convex spaces so that the asymptotic centre was a unique point and proved a fixed point theorem for nonexpansive mappings.

The same ideas as used in the proof of Theorem 3.3 give the following result for nonexpansive mappings. We shall say that X has the f.p.p. if for every closed, bounded, convex set K in X, every nonexpansive self map on K has a fixed point. It is known that X uniformly convex ([3], [14]) or X^* uniformly convex [1], or normal structure [17] or even asymptotic normal structure [2], imply X has the f.p.p.

Theorem 3.4. Let C be a closed, convex, set in a reflexive $(\pi)_1$ space X with the f.p.p. Let $T : C \to X$ be nonexpansive and weakly inward. Then $I - T$ is pseudo-A-proper at 0.

Proof. Let x_n, v be as in Theorem 3.3. Then, exactly as there shows that $\phi(T(v)) \leq \phi(v)$ so T maps the asymptotic centre of $\{x_n\}$ relative to C into itself and thus has a fixed point.

Theorem 3.5. Let C and X be as in Theorem 3.4. Suppose $T : C \to X$ is nonexpansive and inward on C. Then $I-T$ is pseudo-A-proper at 0. If also C is bounded the same holds if T is weakly inward on C.

Proof. Let T be inward on C; we prove that T is inward on K where K is as in the proof of Theorem 3.4. Indeed, for v in K,

$Tv = (1-\alpha)v + \alpha y$ for some $y \in C$ and $\alpha > 1$, and

$\phi(Tv) \leqq \phi(v)$ as earlier. By convexity of ϕ,

$\phi(y) \leqq \phi(v)$ so that $y \in K$. A special case of Theorem 2.6

of Caristi [6] proves that f has a fixed point in K. When f is weakly inward and C is bounded, the same result of Caristi shows that f has a fixed point in C.

4. Strong Convergence of Approximants

We intend to prove in this section some theorems on the convergence of approximants to fixed points. A typical result is that if T is a nonexpansive self map of a closed, bounded, convex set C with X^* uniformly convex, and if for $0 < k < 1$ and $x_0 \in C$ we let x_k be the unique fixed point of the strict contraction T_k defined by $T_k x = kTx + (1 - k)x_0$, then x_k converges to a fixed point of T as $k \to 1$.

We need the following property of m-accretive mappings.

Proposition 4.1. Let X be reflexive with the f.p.p. and let C be a closed, convex subset of X and let $A : C \to X$ be accretive and such that $(I + \alpha A)(C) \supset C$ for some $\alpha > 0$. Suppose $\{x_n\}$ is a bounded subset of C and $Ax_n \to 0$. Then there exists $x \in C$ with $Ax = 0$. Moreover x belongs to K, the asymptotic centre of $\{x_n\}$ relative to C.

Proof. As A is accretive, $(I + \alpha A)^{-1}$ is nonexpansive on its domain. We prove that $(I + \alpha A)^{-1}$ maps K into K. Indeed, let $y = (I + \alpha A)^{-1}v$ so

$v = y + \alpha Ay$, $y \in C$. By accretivity of A we have

$$\begin{aligned} \|x_n - y\| &\leq \|x_n - y + \alpha(Ax_n - Ay)\| \\ &\leq \|x_n - v\| + \alpha\|Ax_n\|. \end{aligned}$$

Thus $\phi(y) \leq \phi(v)$ so $y \in K$. This proves that $(I + \alpha A)^{-1}$ has a fixed point x in K, so $Ax = 0$.

Corollary. Under the same hypotheses let A be m-accretive and suppose $\{x_n\}$ is bounded with $Ax_n \to f$. Then there exists x in K with $Ax = f$.

Proof. Define A_f by $A_f x = Ax - f$. As A_f is m-accretive and $A_f x_n \to 0$ the Proposition applies.

Remark. Proposition 4.1 and its Corollary are valid for multivalued mappings; an obvious modification of the given proofs. Proposition 4.1 is known at least in the version given in the Corollary. It is given by Reich [29] and essentially by Kirk, Schöneberg [20]. Webb [38] gave a simple proof which assumed X separable and X, X^* uniformly convex. These hypotheses could be weakened: separability is not needed, X(UG) and (UCED) suffice. Another method of proof is given in Calvert - Gupta [5] by using the connection between accretive operators and semigroups of nonexpansive mappings. That proof uses a common fixed point theorem for a family of commuting nonexpansive maps so conceivably the hypotheses on X might need to be stronger than the f.p.p.

Webb's proof was modified to prove that accretive, demicontinuous operators are pseudo-A-proper [38]. In fact we can weaken the hypotheses used there to prove the following new version.

Theorem 4.1. Let X be a $(\pi)_1$ space with X^* uniformly convex and let $A : X \to X$ be accretive and demicontinuous. Then A is pseudo-A-proper.

Proof. Suppose $\{x_n\}$ is bounded, $x_n \in X_n$, and $P_n Ax_n \to f$. By replacing Ax by $Ax - f$ we can suppose $f = 0$. Let K be the asymptotic centre of $\{x_n\}$ relative to X and let $v \in K$. For every $z \in X_n$ we have

$$\big(P_n Ax_n - P_n Az \; , \; J(x_n - z)\big) \geq 0.$$

By standard finite-dimensional results (for example [4]), there exists $z_n \in X_n$ with $z_n + P_n Az_n = P_n v$. Moreover $\{z_n\}$ is bounded. By Theorem 3.1, $I + A$ is A-proper, so there is a subsequence $z_k \to z$ with $z + Az = v$. Thus $z = (I + A)^{-1}v$. Setting $z = z_k$ in the above inequality gives

$$\big(P_k Ax_k - P_k v + z_k \; , \; J(x_k - z_k)\big) \geq 0$$

so that

$$\begin{aligned} \|x_k - z_k\| &\leq \|x_k - z_k + P_k Ax_k - P_k v + z_k\| \\ &\leq \|x_k - v\| + \|v - P_k v\| + \|P_k Ax_k\| . \end{aligned}$$

Since $z_k \to z$, it follows that

$$\phi(z) \leq \phi(v).$$

This proves that $(I + A)^{-1}$ maps K into K and so has a fixed point x (since X^* uniformly convex implies X has the f.p.p. [1]) that is, $Ax = 0$.

Our result on strong convergence is a consequence of combining Proposition 4.1 with Proposition 3.1.

Theorem 4.2. Let X be reflexive with the f.p.p. and (UG). Let C be a closed, convex set and $A : C \to X$ be accretive and such that $(I + \alpha A)(C) \supset C$ for some $\alpha > 0$. Let $\{x_n\} \subset C$ be bounded and suppose

$$\lambda_n x_n + Ax_n = \lambda_n g_n \, ,$$

where $\lambda_n > 0$ and $\lambda_n \to 0$, $g_n \to g$.

Then $x_n \to v$ and $Av = 0$.

Proof. Since $\lambda_n \to 0$, $Ax_n \to 0$ so there exists $v \in K$ with $Av = 0$ by

Proposition 4.1. We have

$$\begin{aligned}\lambda_n \|x_n - v\|^2 &= \lambda_n\big(x_n - v\ ,\ J(x_n - v)\big) \\ &= \big(\lambda_n g_n - Ax_n - \lambda_n v\ ,\ J(x_n - v)\big) \\ &\leqslant \big(\lambda_n(g_n - v) - Av\ ,\ J(x_n - v)\big),\end{aligned}$$

by accretivity of A.

Since $Av = 0$, cancelling λ_n gives,

$$\begin{aligned}\|x_n - v\|^2 &\leqslant \big(g_n - v\ ,\ J(x_n - v)\big) \\ &\leqslant \big(g - v\ ,\ J(x_n - v)\big) + \big(g_n - g\ ,\ J(x_n - v)\big).\end{aligned}$$

Applying Proposition 3.1 we see that

$$\liminf \|x_n - v\|^2 \leqslant 0.$$

Thus $x_{n_k} \to v$. This shows that, for any $v \in K$ with $Av = 0$ there is a subsequence $x_k \to v$ (say). Now suppose $x_k \to v$ and $x_m \to w$ where $w \in K$ and $Aw = 0$. We observe that, for v(and for w)

$$\big(x_n - g_n\ ,\ J(x_n - v)\big) \leqslant 0 \quad \text{for all } n.$$

Thus, $\big(x_k - g_k\ ,\ J(x_k - w)\big) \leqslant 0$ and $\big(x_m - g_m\ ,\ J(x_m - v)\big) \leqslant 0$. Passing to the limits and adding gives

$$\|v - w\|^2 = \big(v - w\ ,\ J(v - w)\big) \leqslant 0 \text{ and so } v = w.$$

Thus $A^{-1}(0) \cap K$ consists of a single point v and $x_n \to v$.

Remark. Theorem 4.2 is similar to Theorem 1 of Reich [31] (see his remarks following that theorem); his theorem essentially deals with the case $g_n \equiv g$. The proofs are somewhat different but our idea is similar to his.

Corollary. If A is m-accretive and

$$\lambda_n x_n + Ax_n = f + \lambda_n g_n$$

then $$x_n \to x \text{ with } Ax = f.$$

The Corollary extends a result of [38] where g_n was taken to be constant and more restrictions were imposed on the space.

We apply this theorem to prove a strong convergence theorem for pseudocontractive maps. Recall that $U : C \to X$ is said to be <u>pseudocontractive</u> if $A = I - U$ is accretive.

<u>Theorem 4.3.</u> Let X be reflexive, (UG) and with the f.p.p. Let C be a closed, bounded, convex subset of X and $U : C \to X$ pseudocontractive. Suppose that U is Lipschitz continuous with constant L and suppose that

(i) U is weakly inward on C.

Let $\alpha > 0$ be such that $\alpha L/(1 + \alpha) < 1$. Then, for $0 < k < 1$, and $x_0 \in C$, there exists y_k in C with

$$(1 - k + \alpha)y_k = \alpha U(y_k) + (1 - k)x_0$$

and y_k converges to a fixed point of U as $k \to 1$.

<u>Proof.</u> Let $A = I - U$ so that A is accretive. For $y \in C$, the equation $x + \alpha Ax = y$ is equivalent to $x = (\alpha U(x) + y)/(1 + \alpha)$. The map $x \mapsto (\alpha U(x) + y)/1 + \alpha$ is a strict contraction and satisfies either (i) or (ii) so has a fixed point ([6], [13]). Thus, $F = (I + \alpha A)^{-1}$ is a nonexpansive self map on C. For $x_0 \in C$ and $0 < k < 1$, there exists a unique fixed point x_k of the strict contraction $kF + (1 - k)x_0$. Let $y_k = Fx_k$. Then $(1 - k)y_k + \alpha Ay_k = (1 - k)x_0$ and applying Theorem 4.2 proves the result.

<u>Remarks.</u> The existence part of this theorem is contained in results of Caristi [6] and a combination of Gatica & Kirk [13] and Kirk and Schöneberg ([20], Theorem 4). If $F = (I + \alpha A)^{-1}$ is known, then x_k (and hence y_k) can

be obtained by iteration.

Although nonexpansive maps can be regarded as a special case of the above result the following theorem concerns more familiar approximants.

Theorem 4.4. Let X, C be as in Theorem 4.3 and let $T : C \to X$ be nonexpansive and satisfy (i). Then, for $0 < k < 1$, and $x_0 \in C$ there exists $x_k \in C$ with $x_k = kTx_k + (1 - k)x_0$ and x_k converges to a fixed point of T as $k \to 1$.

Proof. As in Theorem 4.3, if we set $A = I - T$, A is accretive and $(I + A)(C) \supset C$. The fact that x_k exists is a consequence of a theorem of Caristi [6]. Since then, $(1 - k)x_k + kAx_k = (1 - k)x_0$, the result follows from Theorem 4.2.

This result extends Corollary 1 of Reich [31] who assumes that T map C into itself.

We shall conclude with a convergence theorem for a general iterative process studied by Dotson and Mann [9]. They proved that a nonexpansive mapping has a fixed point if a certain sequence of iterates is bounded. Their proof uses uniform convexity of the space. We give a simpler proof which does not require uniform convexity: the f.p.p. suffices. Moreover, a stronger result is given for generalized contractions; the iterates converge to the fixed point.

Let $A = [a_{nk}]$ be an infinite real matrix satisfying

(i) $a_{nk} \geq 0$ and $a_{nk} = 0$ for $k > n$;

(ii) $\sum_{k=1}^{n} a_{nk} = 1$ for each n;

(iii) $\lim_{n \to \infty} a_{nk} = 0$ for each k.

The iterative process studied by Dotson-Mann is as follows. Let C be a closed, convex set in X and $f : C \to C$. Given $x_1 \in C$ let

$$y_n = \sum_{k=1}^{n} a_{nk} x_k \ , \ x_{n+1} = f(y_n), \ n = 1,2,3, \ \ldots \ .$$

This includes the Picard iterates and other schemes as special cases, see e.g. [9].

<u>THEOREM 4.5.</u> Let X be reflexive with the f.p.p. Let $f : C \to C$ be nonexpansive and let the iterates y_n, x_n be as above. If there exists $x_1 \in C$ such that either of the sequences $\{x_n\}$, $\{y_n\}$ is bounded, then f has a fixed point. If f is a generalized contraction then these sequences converge to the fixed point of f.

<u>Proof.</u> It is easy to see that if one of the sequences is bounded so is the other, so we suppose both are bounded. Given z in X let $\phi(z) = \limsup_{n \to \infty} \|x_n - z\|$ and let $\psi(z) = \limsup_{n \to \infty} \|y_n - z\|$. As ϕ and ψ are continuous, convex functions and $\phi(z) \to \infty$ as $\|z\| \to \infty$ (same for ψ), each attains its infimum over the closed convex set C. Let $K = \{v \in C : \phi(v) = \inf \phi(z) : z \in C\}$; K is called the <u>asymptotic centre</u> of $\{x_n\}$ relative to C (e.g. [10]) and is a closed, bounded, convex subset of X. We shall show that f maps K into itself. We begin by showing that, for any $w, \psi(w) \leq \phi(w)$. Indeed, let $\varepsilon > 0$ and let N be chosen sc that $\|x_k - w\| \leq \phi(w) + \varepsilon$, for $k \geq N$. Then, for $n > N$,

$$\|y_n - w\| \leq \sum_{k=1}^{N} a_{nk} \|x_k - w\| + \sum_{k=N+1}^{n} a_{nk} \|x_k - w\| \ .$$

Since $a_{nk} \to 0$ as $n \to \infty$ for each k, there exists $N_1 > N$ such that the first term is less than ε for $n \geq N_1$. Thus, for $n > N_1$,

$$\|y_n - w\| \leq \varepsilon + \sum_{K=N+1}^{n} a_{nk}(\phi(w) + \varepsilon)$$

$$\leq \phi(w) + 2\varepsilon.$$

As ε is arbitary, this proves $\psi(w) \leq \phi(w)$. Next, we prove that $\phi(f(w)) \leq \alpha(w)\psi(w)$ (where $\alpha(w) = 1$ if f is nonexpansive).

Indeed, since

$$\|x_{n+1} - f(w)\| = \|f(y_n) - f(w)\| \leq \alpha(w)\|y_n - w\|,$$

the result is immediate.

Now let $v \in K$. As we obviously have $\phi(v) \leq \phi(f(v))$ we obtain

$$\phi(v) \leq \phi(f(v)) \leq \alpha(v)\psi(v) \leq \alpha(v)\phi(v).$$

When f is a generalized contraction, $\alpha(v) < 1$ and this implies $\phi(v) = 0$ so $x_n \to v = f(v)$. When f is nonexpansive we have $\phi(f(v)) = \phi(v)$ so f maps K into K and so has a fixed point.

Remark. As shown in [9], if f is nonexpansive and has a fixed point then the iterates defined above are bounded. More recently we discovered that Professor S. Reich has proved this result (Pacific J. Math. 60 (1975), 195-198), using essentially the same idea.

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