ON DIFFERENTIABILITY OF THE EXTREMALS OF VARIATIONAL INTEGRALS

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In these lectures *) we shall be concerned with the differentiability properties of the extremals of multiple integrals in the Calculus of Variations and, more generally, with the regularity properties of weak solutions of nonlinear elliptic systems that arise as natural extensions of Euler equations or equations in variation.

Our aim is to describe some results and methods that have been used. Proofs are given only in simple situations and are omitted most of the time. For more information we refer to the original papers quoted, as well as to the notes [36].

Because of the time and space restrictions, many contributions are not even mentioned; in particular we say very little on the functionals with general polynomial growth, on the regularity theory for diagonal systems and its connections with the problem of regularity of weakly harmonic mappings and H-surfaces, on Liouville's type theorems and, finally, on applications.

Anyway we hope that these lectures can be a somehow useful introduction to a field which still offers so many open problems, especially in connection with differential geometry and mathematical physics.

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I. Introduction

Let Ω be a bounded connected open set with smooth boundary in the Euclidean n-dimensional space $\mathbb{R}^n$, $n \geq 2$. We shall denote by $x = (x_1,\ldots,x_n)$ points in $\mathbb{R}^n$.

Let $u(x) = (u^1(x),\ldots,u^N(x))$ be a vector valued function defined in Ω with values in $\mathbb{R}^N$, $N \geq 1$. We shall denote by Du the gradient of u, i.e. the set $\{D_\alpha u^i\}$, $\alpha = 1,\ldots,n$, $i = 1,\ldots,N$ where $D_\alpha = \frac{\partial}{\partial x_\alpha}$.

A <u>variational integral</u> is a functional of the type

$$(0.1) \qquad \mathcal{F}[u;\Omega] = \int_\Omega F(x,u(x),Du(x))\,dx$$

where $F(x,u,p)$ is a map from $\Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$. Dependence on higher order derivatives could be also permitted, but in the sequel we shall confine ourselves to the simplest case (0.1).

Variational integrals arise in different fields of mathematics and in applications (for example in differential geometry and in the theory of elasticity) and two of the classical problems are:

a) 20^{th} Hilbert's problem: existence of minimum points in class K of admissible functions;

b) 19^{th} Hilbert's problem: the differentiability properties of such minimum points.

In the sequel we shall mainly consider the problem of regularity of such minimum points or more generally of <u>stationary points</u>. But let us start briefly with the problem of existence.

1. Existence

Surely one of the simplest and classical ways of proving the existence of a minimum point for $\mathcal{F}[u;\Omega]$ in a class K of competing (or admissible) functions is using the <u>direct methods of the Calculus</u>

of Variations.

The idea is very simple and well known. The set K is not a priori equipped with a topology. So the problem of minimizing $\mathcal{F}$ on K can be seen as the problem of introducing a topology on K for which both K [or more precisely the (or one of the) minimizing sequences in K] is sequentially compact and $\mathcal{F}$ is sequentially lower semi-continuous (s.l.s.c.) on K . Note that in order to grant that $\mathcal{F}$ be s.l.s.c. we need in general a rich topology, while for the compactness of K the topology must not be too rich: so the two requests are one against the other. But a satisfactory compromise can be reached for example for a large class of variational integrals working on the Sobolev spaces. In fact we have

THEOREM 1.1. Suppose that

(i) $F(x,u,p) \geqq 0$,

(ii) F *is measurable in* x *for all* (u,p) *and continuous in* u *for all* p *and almost all* x ,

(iii) F *is convex in* p *for all* u *and almost every* x .

Then the functional $\mathcal{F}[u;\Omega]$ *in* (0.1) *is* s.l.s.c. *with respect to the weak convergence in* $H^{1,m}_{loc}(\Omega,\mathbb{R}^N)$ *for* $1 \leqq m < +\infty$.

In order to prove the existence of a minimum point in $K \subset \subset H^{1,m}(\Omega,\mathbb{R}^N)$ it is now enough to impose a condition that ensures compactness of the minimizing sequences (or of K).

For example in the case of the Dirichlet problem, i.e. of the problem of minimizing $\mathcal{F}[u;\Omega]$ among the questions with prescribed value u_0 at the boundary, it is sufficient to assume that: a) for an extension $\tilde{u}_0$ of u_0 in Ω we have $\mathcal{F}[\tilde{u}_0;\Omega] < +\infty$; b) for some $m > 1$

$$(1.1) \qquad F(x,u,p) \geqq \lambda|p|^m , \quad \lambda > 0 .$$

This is the case, for example, if

$$(1.2) \qquad \lambda|p|^m \leqq F(x,u,p) \leqq \mu|p|^m , \quad m > 1 , \quad \lambda > 0 ,$$

and u_0 is the trace on $\partial\Omega$ of an $H^{1,m}$ function. Now remembering that $H^{1,m}(\Omega,\mathbb{R}^N)$ is a reflexive Banach space for $m > 1$, proving the existence of a minimum point is a very simple exercise.

The range of applicability of the above method and of Theorem 1.1 is quite large and on the other hand quite well known, so we shall not insist on that point. We only mention that Theorem 1.1 under stronger regularity assumptions on F is proved in [95, Theorem 1.8.2]. There is a very large literature on the semicontinuity theorems, starting from the results by L. Tonelli and C. B. Morrey till nowadays; we refer to [19] for the proof of Theorem 1.1 and to [36] for a sketch of it and for further references.

The convexity assumption in (iii), Theorem 1.1, is natural in the scalar case $N = 1$, actually it is essentially necessary (classical proofs of this fact are available, we refer to [6], [75] for proofs under sufficiently weak assumptions); but it is very far from being necessary in the vector valued case $N > 1$. It should be substituted with the <u>quasi-convexity condition</u> of C. B. Morrey [95, Sec. 4.4]:

for a.e. $x_0 \in \Omega$ and for all s_0, ξ_0, $\phi \in C_0^\infty(\Omega, \mathbb{R}^N)$,

$$\frac{1}{|\Omega|} \int_\Omega F(x_0, s_0, \xi_0 + D\phi(x))\,dx \geq F(x_0, s_0, \xi_0) ,$$

which generally is a weaker condition than the convexity and reduces to it for $N = 1$. Although uneasy to handle, the quasi-convexity condition arises in a natural way in many problems, especially in elastostatics see [5], [6], [7], [8]. For example, if $n = N$ any convex function of the invariants of the Jacobian matrix of u is a quasi-convex function.

Semicontinuity theorems under the quasi-convexity condition plus quite strong assumptions were proved in [88], [81], [95, Sec. 4.4] and in [5], [8]. Recently the works [29], [76], [1] have given a strong contribution to the question. Let us state the main theorem of [1] without proof:

<u>*THEOREM 1.2.*</u> *Let* $F(x,u,p)$ *be measurable in* x *and continuous in* (u,p). *Assume moreover that*

$$0 \leq F(x,u,p) \leq 1 + \lambda(|u|^m + |p|^m) , \quad m \geq 1 .$$

Then the functional (0.1) *is weakly* s.l.s.c. *on* $H^{1,m}(\Omega, \mathbb{R}^N)$ *if and only if* F *is quasi-convex.*

The p r o o f of this theorem is quite complicated. It is simpler to prove instead

<u>*THEOREM 1.3.*</u> *Let* $F(x,u,p)$ *be measurable in* x *and continuous in* (u,p). *Assume moreover that*

$$|F(x,u,p)| \leq 1 + \lambda(|u|^m + |p|^m) \ , \quad m \geq 1 \ ,$$

and that F *is quasi-convex. Then* $\mathcal{F}[u;\Omega]$ *is weakly* s.l.s.c. *on* $H^{1,q}(\Omega,\mathbb{R}^N)$ *for* $q > m$.

See [29] and [76] for a simpler proof.

Unfortunately the assumption $q > m$ is crucial: the result fails if $q = m$ as an example of L.Tartar and F.Murat [100] shows. Note that in view of Theorem 1.2 it would be true provided $F \geq 0$. But Theorem 1.3 permits, by combining two results in [22], [39], to obtain the existence of a minimum point, as P. Marcellini and C. Sbordone have shown [76]. We shall go back to this question in Sec. 4, Chap. II.

Since we want to avoid any complications due to the boundary data, from now on " u <u>is a minimum for</u> $\mathcal{F}$ <u>in</u> Ω " means that for all $\phi \in C_0^\infty(\Omega,\mathbb{R}^N)$ with $\mathrm{supp}\,\phi \subset\subset \Omega$

$$\mathcal{F}[u;\ \mathrm{supp}\,\phi] \leq \mathcal{F}[u + \phi;\ \mathrm{supp}\,\phi] \ .$$

2. The problem of regularity

As we have seen, by enlarging the spaces of competing functions, it is possible to prove quite simply the existence of generalized solutions to minimum problems for variational integrals; but we pay for this simplicity by the new problem of showing, if possible, the differentiability (in the classical sense) of the generalized solutions.

It would be very difficult to quote all the many contributions to the regularity problem, and even more difficult to record the many influences that methods and results have had in different fields of mathematics. Let us recall that they start at the beginning of this century, have run till nowadays and that many problems remain still open.

We can anyway distinguish, at least from the point of view we are adopting, two main steps:

a) "<u>from</u> C^1 <u>on</u>". The concluding result can be stated as: *any* C^1 *stationary point of "regular" multiple integrals in the Calculus of Variations is as regular as the data permit.*

The starting point of this result (apart from Hilbert's work for $n = 1$) is probably due to S. Bernstein in 1904 who proved that each solution of class C^3 of a nonlinear elliptic analytic second order equation in the plane is an analytic function. Through the fundamental contributions of L. Lichtenstein, E. Hopf, I. G. Petrovskiĭ, J. Leray, J. Schauder, R.Caccioppoli, K.O.Friedrichs, H. Lewy, O. A. Ladyzhenskaya, F. John, L. Nirenberg (among others) we arrive at the result stated in a) - see C. B. Morrey [89], 1954.

This step, seen a posteriori, has mainly to do with the linear theory of elliptic system and we shall try to describe the main points in the next two sections.

The theory of boundary value problems for linear elliptic system received relevant contributions during the fifties and culminated in the work of S. Agmon - A. Douglis - L. Nirenberg. Actually even later contributions should be mentioned, but we shall omit them, since they are not really related to what follows.

But except for the two-dimensional case (the results of C. B. Morrey 1938-39 [85], see also [86] [87]) no real progress was made in the direction of filling the gap

b) "<u>from</u> $H^{1,m}$ <u>to</u> C^1 " until the famous result of E. De Giorgi in 1957 [17] (see also J. Nash [101]) who proved that any weak solution of a second order linear elliptic equation with measurable coefficients is Hölder-continuous, deducing in this way that any extremal of a functional of the type

$$\int_\Omega F(Du)dx$$

is as regular as the data permit.

The paper [17] opened a new stage, which reached its culmination in the works by G. Stampacchia, C. B. Morrey and O. A. Ladyzhenskaya and N. N. Uraltseva. Under suitable growth conditions on F and on the derivatives of F plus the ellipticity condition step b) was accomplished, thus solving the 19^{th} Hilbert's problem for a large class of functionals, in the scalar case $N = 1$.

This theory can now be considered as classical; we refer to the two books [73] [95]. Anyway we shall return to it in the next sections.

Besides a result by J. Nečas [102] for a class of higher order equations in dimension 2, no result was obtained during the years 1957-68 for the case $N > 1$. Many new proofs of De Giorgi's result were given, but none of them could be extended to cover the case of systems.

In 1968 E. De Giorgi [18] showed that his result for equations could not be extended to systems. By modifying De Giorgi's example

E. Giusti and C. Miranda [55] showed that functionals of the type

$$\int_\Omega A_{ij}^{\alpha\beta}(u)D_\alpha u^i D_\beta u^j dx$$

with analytic coefficients $A_{ij}^{\alpha\beta}$ satisfying

$$A_{ij}^{\alpha\beta}\ \xi_\alpha^i\ \xi_\beta^j \geqq |\xi|^2$$

may have singular minima for large dimension n, and the same holds for weak solutions of elliptic quasilinear systems of the type

$$\int_\Omega A_{ij}^{\alpha\beta}(u)D_\alpha u^i D_\beta \phi^j dx = 0 \qquad \forall\phi \in C_0^\infty(\Omega,\mathbb{R}^N)$$

in dimension $n \geq 3$. Similar exaples were presented in the meantime by Maz'ya [77] and now different extensions and improvements in various directions are available [4], [24], [25], [32], [47], [62], [106], [107]; we especially point out the examples in [32], [106]. We shall discuss some of these counterexamples to the regularity in the next sections. But already now we can state that vector valued minima or extremals of regular integrals or weak solutions to nonlinear elliptic systems are in general non-smooth. There is hope only for "partial regularity", i.e. regularity except an a closed singular set hopefully of small dimension.

Results on the partial regularity of solutions to nonlinear elliptic systems, essentially systems of the type of systems in variation for multiple integrals of the kind

$$\int_\Omega F(x,Du)dx\ ,$$

were obtained by C. B. Morrey [96], E. Giusti, C. Miranda [56], E. Giusti [52], L. Pepe [112] during the years 1968-71. The method used relies on an indirect argument, very similar to the one introduced by E. De Giorgi and J. F. Almgren for proving the regularity of parametric minimal surfaces. We refer to [96], [56], [36] for a description of the main idea and to [36] for an account of the results.

During the years 1975-79 elliptic systems of diagonal form, i.e. of the type

$$-\ D_\beta\left(A^{\alpha\beta}(x,u)D_\alpha u^i\right) = f_i(x,u,Du)$$

with

$$|f(x,u,p)| \leqq a|p|^2 + b\ ,\quad A^{\alpha\beta}\xi_\alpha\xi_\beta \geqq |\xi|^2$$

have been particularly studied, mainly in connection with the problem of the regularity of weak harmonic maps between Riemannian manifolds and of the regularity of H-surfaces (we refer to [60] for an account of methods and results), and more recently in connection with the theory of stochastic differential games (see for example [9]). Under suitable assumptions, regularity everywhere has been proved. In the sequel we shall not mention these results with a few exceptions.

In 1978 a new argument of direct type was introduced by M. Giaquinta and E. Giusti [38], and improved in [43], [44], for proving partial regularity of solutions of nonlinear elliptic systems. In this way results of partial regularity were obtained for solutions of a large class of nonlinear elliptic systems [38], [43], [44] as well as for minimum points of certain regular functionals [39], [40]. But the regularity problem for general functionals

$$\int_\Omega F(x,u,Du)dx$$

is still an open question. In the sequel we shall be mainly concerned with these results and with some ideas which lead to them.

We conclude this section with a simple remark on the first step of stage a). Let u be a minimum point of

$$\mathcal{F}[u;\Omega] = \int_\Omega F(x,u,Du)dx$$

and suppose that $u \in C^1(\Omega,\mathbb{R}^N) \cap H^{2,2}_{loc}(\Omega,\mathbb{R}^N)$. Then, as is well known, u is a solution to the <u>Euler system</u> in the weak formulation, i.e.

$$\int_\Omega \left(F_{p^i_\alpha}(x,u,Du)D_\alpha\phi^i + F_{u^i}(x,u,Du)\phi^i\right)dx = 0 \qquad \forall\phi \in C^\infty_0(\Omega,\mathbb{R}^N).$$

Now choosing $\phi = D_s\psi$, $\psi \in C^\infty_0(\Omega,\mathbb{R}^N)$ and integrating by parts, we deduce that the derivatives of u are solutions of the so-called <u>system in variation</u>

$$\int_\Omega \left[D_s F_{p^i_\alpha}(x,u,Du)D_\alpha\psi^i - F_{u^i}(x,u,Du)D_s\psi^i\right]dx = 0 \quad \forall\psi \in C^\infty_0(\Omega,\mathbb{R}^N),$$

i.e.

$$\begin{aligned}(2.1)\qquad \int_\Omega \Big[&F_{p^i_\alpha p^j_\beta}(x,u,Du)D_\beta D_s u^j + F_{p^i_\alpha u^j}(x,u,Du)D_s u^j + \\ &+ F_{p^i_\alpha x_s}(x,u,Du) + \delta_{\alpha s}F_{u^i}(x,u,Du)\Big]D_\alpha\psi^i = 0 .\end{aligned}$$

If we now read the coefficients as functions of x, the system in variation shows its character of a linear system for $D_s u^j$ with continuous coefficients.

3. Linear systems

Two results from the linear theory of elliptic systems are relevant in order to accomplish step a) of Section 2.

Let us consider the linear system

$$(3.1) \qquad - D_\beta\left(A_{ij}^{\alpha\beta}(x) D_\alpha u^i\right) = - D_\beta f_\beta^j$$

and assume that it is elliptic, i.e.

$$(3.2) \qquad A_{ij}^{\alpha\beta}(x)\xi_i\xi_j\eta_\alpha\eta_\beta \geq |\xi|^2|\eta|^2 \quad \forall \xi , \eta .$$

Then we have

<u>*THEOREM 3.1.*</u> *Assume that* $A_{ij}^{\alpha\beta} \in C^k(\Omega)$ *and* $f_\beta^j \in H^k_{loc}(\Omega)$, $k \geq 0$. *Then any weak solution* $u \in H^{1,2}_{loc}(\Omega,\mathbb{R}^N)$ *to system* (3.1) *belongs to* $H^{k+1}_{loc}(\Omega,\mathbb{R}^N)$.

<u>*THEOREM 3.2.*</u> *Assume that* $A_{ij}^{\alpha\beta} \in C^{k,\gamma}(\Omega)$ *and* $f_\beta^j \in C^{k,\gamma}(\Omega)$, $k \geq 0$, $0 < \gamma < 1$. *Then any weak solution* $u \in H^{1,2}_{loc}(\Omega,\mathbb{R}^N)$ *to system* (3.1) *belongs to* $C^{k+1,\gamma}(\Omega)$.

Since not only the results but also the way of proving them is relevant for our purpose, let us hint at the proof. For the sake of simplicity we shall assume from now on the stronger ellipticity condition

$$(3.3) \qquad A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j \geq |\xi|^2 \quad \forall \xi .$$

<u>Hilbert space regularity</u>. There are several ways of proving Theorem 3.1 *). But probably the simplest proof and surely the most suitable for our purpose is the one by L. Nirenberg [110] who replaced the mollifiers by the difference quotients.**) This proof is nowadays well known, but let us sketch it, assuming moreover the coefficients $A_{ij}^{\alpha\beta}$ to be constant.

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*) One method employs F. John's construction of the fundamental solution; another employed by F. John is the method of spherical means; a third one introduced by K. O. Fridrichs employs mollifiers and a priori estimates of higher derivatives (we should at this point mention also R. Caccioppoli, J. Leray, O. A. Ladyzhenskaya). Finally, still another method has been used by P. D. Lax.

**) we mention that difference quotients were already used by L.Lichtenstein.

Let $\eta \in C_0^\infty(B_{2R}(x_0))$ *), $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $B_R(x_0)$, $|D\eta| \leq 2/R$. Inserting $\phi = \eta^2 u$ in the weak formulation of (3.1), i.e. in

$$\int_\Omega A_{ij}^{\alpha\beta} D_\alpha u^i D_\beta \phi^j dx = \int_\Omega f_\beta^j D_\beta \phi^j dx \quad \forall \phi \in C_0^\infty(\Omega, \mathbb{R}^N) \,,$$

we immediately obtain by simple tricks

$$\int_{B_R} |Du|^2 dx \leq \frac{c}{R^2} \int_{B_{2R}} |u|^2 dx + c \int_{B_{2R}} |f|^2 dx \,, \tag{3.4}$$

which is called the <u>Caccioppoli inequality</u>**) and plays a fundamental role in the theory of elliptic systems.

Now differencing equation (3.1) we deduce

$$\int_\Omega A_{ij}^{\alpha\beta} D_\alpha [u^i(x+h) - u^i(x)] D_\beta u^j dx = \int_\Omega [f_\beta^j(x+h) - f_\beta^j(x)] D_\beta \phi^j dx$$

and therefore

$$\int_{B_r} \left| D \frac{u(x+h) - u(x)}{h} \right|^2 dx \leq \frac{c}{R^2} \int_{B_{2R}} \left| \frac{u(x+h) - u(x)}{h} \right|^2 dx + \tag{3.5}$$

$$+ c \int_{B_{2R}} \left| \frac{f(x+h) - f(x)}{h} \right|^2 dx \,.$$

It is simple to show that (3.5) implies $Du \in H^1_{loc}$ and

$$\int_{B_R} |D^2 u|^2 dx \leq \frac{c}{R^2} \int_{B_{2R}} |Du|^2 dx + \int_{B_{2R}} |Df|^2 dx \leq$$

$$\leq c(R) \int_{B_{4R}} |u|^2 dx + c||f||^2_{H^{1,2}} \,.$$

By induction, Theorem 3.1 then follows in the case of constant coefficients. Moreover, we have, if $f \equiv 0$,

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*) $B_R(x_0)$ denotes the ball of radius R around x_0.

**) More precisely, inserting $\phi = \eta^2(u - u_{2R})$,

$$u_{2R} = u_{x_0,2R} = \fint_{B_{2R}(x_0)} u\,dx = \frac{1}{|B_{2R}|} \int_{B_{2R}(x_0)} u\,dx \,,$$

we have

$$\int_{B_R} |Du|^2 dx \leq \frac{c}{R^2} \int_{B_{2R}} |u - u_{2R}|^2 dx + c \int_{B_{2R}} |f|^2 dx \,.$$

$$||u||_{H^k(B_R)} \leqq c(R,k)||u||_{L^2(B_R)} .$$

Extensions to systems with variable coefficients need only formal changes.

The p r o o f of Theorem 3.1 under the ellipticity condition (3.2) needs more care. Still by inserting $\phi = \eta^2 u$ one deduces

$$\int A_{ij}^{\alpha\beta} D_\alpha(u^i\eta) D_\beta(u^j\eta)dx \leqq c\int |u|\ |D\eta|\ |D(u\eta)|\ dx +$$
$$+ c\int |u|^2 |D\eta|^2 dx + \int |f|\ |D(u\eta)|\ dx .$$

Now, by means of Fourier transform and using (3.2), one sees that

$$\int |D(u\eta)|^2 dx \leqq \int A_{ij}^{\alpha\beta} D_\alpha(u^i\eta) D_\beta(u^j\eta)dx$$

and the proof can be easily completed in the case of constant coefficients. For variable coefficients one uses Korn's device (compare with the proof of Gårding inequality), one freezes the coefficients at a point and looks at the remainders as a small perturbation (assuming the coefficients at least continuous). Remark that this procedure does not work for example, for quasilinear systems ($A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(u)$) while inequality (3.4) still holds assuming the strong ellipticity condition (3.3). This is the reason why almost nothing is known when considering nonlinear systems satisfying the ellipticity condition (3.2).

Hölder regularity. As we have remarked, if u is a weak solution of

$$\int A_{ij}^{\alpha\beta} D_\alpha u^i D_\beta \phi^j dx = 0 \quad \forall\phi \in C_0^\infty(\Omega,\mathbb{R}^N) \tag{3.6}$$

then $u \in H^k_{loc}$ for all k and

$$||u||_{H^k(B_R,R^N)} \leqq c(R,k)||u||_{L^2(B_{2R})} .$$

Then we have for all $\rho < R/2$, using also Sobolev imbedding theorem,

$$\int_{B_\rho(x_0)} |u|^2 dx \leqq c\rho^n \sup_{B_\rho(x_0)} |u|^2 \leqq c(R)\rho^n ||u||^2_{H^k(B_{R/2})} \leqq$$
$$\leqq c(R)\rho^n \int_{B_R(x_0)} |u|^2 dx .$$

Now it is easily seen from the equation, by using a dilatation argument, that $c(R) = \text{const } R^{-n}$, i.e.

$$\int_{B_\rho(x_0)} |u|^2 dx \leqq c\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |u|^2 dx \tag{3.7}$$

for all $\rho < R/2$. Since (3.7) is obvious for $R/2 < \rho < R$ we have (3.7) for all $\rho < R$. Since now any derivative of n is also a solution of (3.6) we can state

<u>*PROPOSITION 3.3.*</u> *Let* u *be a solution of system* (3.6). *For* $x_0 \in \Omega$, $\rho \leq R < \mathrm{dist}(x_0, \partial\Omega)$ *we have*

$$(3.8) \qquad \int_{B_\rho(x_0)} |Du|^2 dx \leq c\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |Du|^2 dx .$$

We are now ready to prove the following

<u>*THEOREM 3.4.*</u> *Let* u *be a weak solution to*

$$(3.9) \qquad - D_\alpha\left(A_{ij}^{\alpha\beta}(x) D_\beta u^j\right) + D_\alpha f_i^\alpha = 0 \;, \quad i = 1,\ldots,N \;,$$

and suppose that $A_{ij}^{\alpha\beta}$, $f_i^\alpha \in C^0$. *Then* $u \in C^{0,\gamma}_{\mathrm{loc}}$ *for all* $\gamma < 1$.

Remark that Theorem 3.4 applies to system (2.1) at the end of Sec. 2 and permits to conclude that any extremal of class $C^1 \cap H^{2,2}$ actually has Hölder-continuous first derivatives. Therefore the coefficients of system (2.1) are Hölder-continuous.

P r o o f . Let $B_R(x_0) \subset\subset \Omega$. In $B_R(x_0)$, u is a weak solution of

$$- D_\alpha\left(A_{ij}^{\alpha\beta}(x_0) D_\beta u^j\right) + D_\alpha F_i^\alpha = 0 \;,$$

$$F_i^\alpha = f_i^\alpha + \left[A_{ij}^{\alpha\beta}(x_0) - A_{ij}^{\alpha\beta}(x)\right] D_\beta u^j \;.$$

Let v be the solution to the Dirichlet problem

$$\begin{cases} \displaystyle\int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x_0) D_\beta v^j D_\alpha \phi^i dx = 0 & \forall \phi \in H_0^1(B_R(x_0), \mathbb{R}^N) \;, \\ v - u \in H_0^1(B_R(x_0), \mathbb{R}^N) \;. \end{cases}$$

By Proposition 3.3 we have

$$(3.10) \qquad \int_{B_\rho(x_0)} |Dv|^2 dx \leq c\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |Dv|^2 dx .$$

On the other hand,

$$\int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x_0) D_\beta(u^j - v^j) D_\alpha \phi^i dx = \int_{B_R(x_0)} F_i^\alpha D_\alpha \phi^i dx$$
$$\forall \phi \in H_0^1(B_R(x_0), \mathbb{R}^N)$$

and inserting $\phi = u - v$ we obtain

$$(3.11) \qquad \int_{B_R(x_0)} |D(u-v)|^2 dx \leqq$$

$$\leqq c\{ \int_{B_R(x_0)} |f|^2 dx + \int_{B_R(x_0)} |A(x_0) - A(x)|^2 |Du|^2 dx\} .$$

Putting together (3.10), (3.11) we deduce

$$(3.12) \qquad \int_{B_\rho} |Du|^2 dx \leqq c\left[\left(\frac{\rho}{R}\right)^n + \omega(R)\right] \int_{B_R} |Du|^2 dx + c \int_{B_R} |f|^2 dx$$

where $\omega(R)$ is the modulus of continuity of the coefficients $A_{ij}^{\alpha\beta}$.

Now we have:

let $\phi(t)$ be a non-negative and non-decreasing function; if

$$\phi(\rho) \leqq A\left[\left(\frac{\rho}{R}\right)^\alpha + \varepsilon\right]\phi(R) + BR^\beta$$

for all $\rho \leqq R \leqq R_0$ with $A, B, \alpha, \beta, \varepsilon$ positive constants, $\alpha > \beta$ and $\varepsilon < \varepsilon_0 = \varepsilon_0(A,\alpha,\beta)$, then for all $\rho \leqq R \leqq R_0$

$$\phi(\rho) \leqq c[R^{-\beta}\phi(R) + B]\rho^\beta$$

with $c = c(\alpha,\beta,A)$.

Since $\int_{B_R} |f|^2 dx \leqq \sup |f|^2 \omega_n R^n$, from (3.12) we deduce taking $\phi(\rho) = \int_{B_\rho} |Du|^2 dx$:

$$(3.13) \qquad \int_{B_\rho(x_0)} |Du|^2 dx \leqq c(\varepsilon)\rho^{n-\varepsilon}\left[R^{\varepsilon-n} \int_{B_R} |Du|^2 dx + \sup_{B_R} |f|^2\right]$$

$$\forall \varepsilon > 0 .$$

Observing now that (3.13) holds for all x_0 , the result follows from the Dirichlet growth theorem of C. B. Morrey [95, Theor.3.5.2].

Q.E.D.

Remark that actually we have only used that $f \in L^\infty$ and that it would be sufficient to assume that

$$R^{-n} \int_{B_R(x_0)} |f - f_{x_0,R}|^2 dx \leqq \text{const. independent of } x_0 \text{ and } R .$$

The proof we have given appears in C. B. Morrey [89], an analogous argument was also used in [84]. We have to remark that Morrey's proof uses in a strong way potential theory; the proof we have given is due to S. Campanato [13]. When passing on to prove that if the coefficients and the data f_i^α in (3.9) are Hölder-continuous then the first derivatives of the solutions are Hölder-continuous, Morrey's proof in [89] becomes less transparent. On the other hand, Campanato's approach [13] is very simple and useful.

The first result is the following characterization of Hölder-continuous functions, see [12], which replaces the Dirichlet growth theorem.

THEOREM 3.5. Let Ω *be a smooth open set. Then* $u \in C^{0,\alpha}(\Omega)$ *if and only if for all* $x_0 \in \Omega$ *and all* $R < R_0$ *we have*

$$\int_{B_R(x_0)\cap\Omega} \left|u - \fint_{B_R(x_0)\cap\Omega} u\right|^2 dx \leq cR^{n+2\alpha}\ , \quad 0 < \alpha \leq 1\ .$$

See [12], [69], or [36] where also further references are given, for the proof.

In the same way we proved inequality (3.8), or using Poincaré inequality on the left hand side and Caccioppoli inequality on the right hand side we easily deduce

PROPOSITION 3.6. Let u *be a weak solution to system* (3.6). *For* $x_0 \in \Omega$, $\rho < R < \operatorname{dist}(x_0,\partial\Omega)$ *we have*

$$\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 dx \leq c\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |u - u_{x_0,R}|^2 dx\ . \tag{3.14}$$

Moreover, (3.14) *also holds when replacing* u *with* $D^k u$.

By the same method as in the proof of Theorem 3.4 we can now prove (see [13]) the following

THEOREM 3.7. Suppose $A_{ij}^{\alpha\beta}$, $f_i^\alpha \in C^{0,\mu}$ *and let* u *be a weak solution to system* (3.8). *Then* $Du \in C^{0,\mu}_{loc}$.

P r o o f . Splitting u as in the proof of Theorem 3.4 and using (3.14) (with u replaced by Du) instead of (3.8) we obtain for $B_\rho \subset B_R \subset \Omega$

$$\int_{B_\rho} |Du - (Du)_\rho|^2 dx \leq c_1\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |Du - (Du)_R|^2 dx + \tag{3.15}$$

$$+ c_2\int_{B_R} |f - f_R|^2dx + c_3 \sup[A_{ij}^{\alpha\beta}]_{C^{0,\mu}} R^{2\mu} \int_{B_R(x_0)} |Du|^2dx .$$

We know from the proof of Theorem 3.4 that

$$\int_{B_R} |Du|^2dx \leq cR^{n-\varepsilon}$$

and, because of the assumptions,

$$\int_{B_R} |f - f_R|^2dx \leq cR^{n+2\mu} .$$

Hence, by the same algebraic argument as in the proof of Theorem 3.4 and of Theorem 3.5, we deduce that $Du \in C^{0,\gamma}_{loc}$ *) for $\gamma < \mu$. In particular, Du is locally bounded, therefore from (3.15) we deduce

$$\int_{B_\rho} |Du - (Du)_\rho|^2dx \leq c_1\left(\frac{\rho}{R}\right)^{n+2}\int_{B_R} |Du - (Du)_R|^2dx + c_4R^{n+2\mu} ,$$

which concludes the proof.

Q.E.D.

Theorem 3.2 now follows by induction, by differentiating the system.

4. From the functional to the system in variation

Consider the multiple integral

$$(4.1) \qquad \mathcal{F}[u;\Omega] = \int_\Omega F(x,u,Du)dx$$

and suppose, for simplicity, that

$$|p|^2 \leq F(x,u,p) \leq c|p|^2 .$$

The possibility of differentiating $\mathcal{F}$ in the direction of a function $\phi \in H^{1,2}_0$ **) at a minimum point u depends strongly on the growth

*) Note that the L^2-norm plus the Hölder seminorm is a norm equivalent to the Hölder norm.

**) As is well known, functionals of type (4.1) in general do not possess derivatives in the sense of Fréchet, compare for example [119]. On the other hand, we may hope to get information on u (a priori estimate) only by choosing suitable variations, which, of course, must involve u ; therefore we need to have the possibility of making variations at least in the same class to which u belongs (plus zero boundary conditions, in order not to change the boundary value of u as requested at the end of Sec. 1).

conditions we have for the derivatives of F .

EULER SYSTEM. Formally the derivative of $\mathcal{F}$ in the direction of ϕ at u should be

$$\int_\Omega [F_{p_\alpha^i}(x,u,Du)D_\alpha\phi^i + F_{u^i}(x,u,Du)\phi^i]dx \, . \tag{4.2}$$

Noting that $D\phi \in L^2$ and that, because of the Sobolev imbedding theorem, $\phi \in L^{2^*}$, in order for (4.2) to have a meaning we must assume that

$$F_{p_\alpha^i}(x,u,Du) \in L^2 \; ; \quad F_{u^i}(x,u,Du) \in L^{2^{*\prime}} \; ;$$

$$2^{*\prime} = \text{the dual exponent of } 2^* \quad (= \frac{2n}{n-2}) \quad (n \geq 3) \; .$$

This is granted, taking into account the Sobolev imbedding theorem, for example by the following growth conditions

$$\begin{cases} |F_p(x,u,p)| \leq \mu\left[\chi_1(x) + |u|^{\frac{n}{n-2}} + |p|\right] , \\ |F_u(x,u,p)| \leq \mu\left[\chi_2(x) + |u|^{\frac{n+2}{n-2}} + |p|^{1+\frac{2}{n}}\right] , \\ \chi_1 \in L^2 , \quad \chi_2 \in L^{2n/(n+2)} , \end{cases} \tag{4.3}$$

if $n \geq 3$, or

$$\begin{cases} |F_p(x,u,p)| \leq \mu\left[\chi_1 + |u|^{q/2} + |p|\right] , \\ |F_u(x,u,p)| \leq \mu\left[\chi_2 + |u|^{q-1} + |p|^{2(1-\frac{1}{q})}\right] , \\ \chi_1 \in L^2 , \quad \chi_2 \in L^{q/(q-1)} , \; 1 < q < +\infty , \end{cases} \tag{4.3'}$$

if $n = 2$.

Now it is easy to verify that

> conditions (4.3), (4.3′) are also sufficient for the differentiability of $\mathcal{F}[u;\Omega]$ in the direction of $\phi \in H_0^1(\Omega,R^N)$.

While conditions (4.3), (4.3′), which we shall call <u>controllable growth conditions</u>, are "natural" if there is no explicit dependence on u in F , i.e. $F(x,u,p) = F(x,p)$, they are quite unnatural in the general case, as it is unnatural to assume that F_u increases of the same order, with respect to p , as F_p . For instance, for the simple functional

$$\int_\Omega a(u)|Du|^2 dx \ , \quad N = 1 \ , \quad 0 < m \leq a(u) \ , \quad a'(u) \leq M$$

we have

$$|F_p| \sim |p| \ , \quad |F_u| = a'(u)|p|^2 \sim |p|^2 \ .$$

Hence it is more suitable to assume

$$(4.4) \qquad \begin{cases} \text{for } |u| \leq R \text{ and } V = (1 + |p|^2)^{1/2} \ , \\ |F_u| \leq \mu(R)V^2 \ , \\ |F_p| \leq \mu(R)V \ . \end{cases}$$

Conditions (4.4) are not sufficient to ensure the differentiability of $\mathcal{F}$ in the direction of $\phi \in H_0^1$, but this is true if we work in $H^1 \cap L^\infty$ instead of H^1 . We shall refer to (4.4) as the <u>uncontrollable growth conditions</u>.

Concluding, we are able to consider <u>extremals</u> of $\mathcal{F}[u;\Omega]$ (and the Euler system for $\mathcal{F}$) in the following two situations:

a) controllable growth conditions hold, $u \in H^{1,2}(\Omega,\mathbb{R}^N)$; u satisfies

$$(4.5) \qquad \int_\Omega \left[F_{p_\alpha^i}(x,u,Du)D_\alpha\phi^i + F_{u^i}(x,u,Du)\phi^i\right]dx = 0 \qquad \forall\phi \in H_0^1(\Omega,\mathbb{R}^N),$$

b) uncontrollable growth conditions hold, $u \in H^{1,2} \cap L^\infty(\Omega,\mathbb{R}^N)$; u satisfies (4.5) for all $\phi \in H_0^1 \cap L^\infty(\Omega,\mathbb{R}^N)$.

Analogous meaning should be given to the notion of <u>weak solutions</u> to nonlinear elliptic system of the type

$$(4.6) \qquad - D_\alpha A_i^\alpha(x,u,Du) + B_i(x,u,Du) = 0 \ , \quad i = 1,\dots,N$$

with the obvious analogy.

As we shall see, this distinction really corresponds to a different behaviour of the solutions.

Analogous considerations can be carried out in the more general situation

$$|p|^m \leq F(x,u,p) \leq c|p|^m \ , \quad m > 1 \ .$$

<u>SYSTEM IN VARIATION.</u> According to controllable or uncontrollable growth conditions, we also need different assumptions in order to deduce the system in variation.

Let us start with extremals of the simple integral

$$\int_\Omega F(Du)dx$$

or more generally with weak solutions to a system

$$(4.7) \qquad - D_\alpha A_i^\alpha(Du) = 0 \;, \quad i = 1,\dots,N \;,$$

assuming controllable growth conditions and ellipticity, i.e.

$$|A_i^\alpha(p)| \leq c|p| \;, \quad |A^\alpha_{ip^j_\beta}(p)| \leq L \;,$$

$$A^\alpha_{ip^j_\beta}(p)\xi^i_\alpha \xi^j_\beta \geq \lambda|\xi|^2 \;, \quad \forall \xi \quad (\lambda > 0) \;.$$

Then, differencing system (4.7), i.e. using the quotient method. As in Section 3, one easily gets that

a weak solution $u \in H^{1,2}(\Omega,\mathbb{R}^N)$ to system (4.7) has square integrable second derivatives, satisfying

$$(4.8) \qquad \int_\Omega A^\alpha_{ip^j_\beta}(Du) D_\beta(D_s u^j) D_\alpha \phi^i dx = 0 \;, \quad \forall \phi \in H^1_0(\Omega,\mathbb{R}^N) \;.$$

The identity (4.8) can be rewritten as a quasilinear system for the vector valued function

$$U = (U^j_s) = (D_s u^j)$$

as

$$(4.8') \qquad \int_\Omega \delta_{\ell s} A^\alpha_{ip^j_\beta}(U) D_\beta U^j_s D_\alpha \phi^i_\ell dx = 0 \quad \forall \phi \in H^1_0(\Omega,\mathbb{R}^N) \;.$$

An analogous result can be obtained for general elliptic systems like (4.6) under controllable growth conditions (i.e. differentiation with respect to u decreases the order of growth in p by one), see [95], [36].

On the other hand, it is not true in general that a weak solution of the simple quasilinear system

$$\int_\Omega A^{\alpha\beta}_{ij}(x,u) D_\alpha u^i D_\beta \phi^j dx = 0 \quad \forall \phi \in H^1_0(\Omega,\mathbb{R}^N)$$

with smooth coefficients satisfying

$$A^{\alpha\beta}_{ij}\xi^i_\alpha \xi^j_\beta \geq |\xi|^2 \quad \forall \xi$$

has square integrable second derivatives.

Under uncontrollable growth conditions we are able to prove that a weak solution u to system (4.6) has square integrable second derivatives verifying the system in variation only provided u is assumed to be continuous. We refer to [95], [36] for further information and proofs.

Anyway, let us remark that the results of this section together with the ones in Section 3 completely prove the result of step a) in Section 2 : any C^1-extremal of a regular multiple integral in the Calculus of Variations is as regular as the data permit $^{*)}$.

But in general we are only able to find extremals or weak solutions in $H^{1,2}$, therefore, as we have already stated, there is a gap, step b) in Sec. 2, to be filled. The rest of these lectures will be dedicated to some contributions in that direction, i.e. from H^1 to C^1 .

5. Regularity for equations and counterexamples for systems

Let $u \in H^{1,2}(\Omega)$, $N = 1$, be an extremal of the functional

$$(5.1) \qquad \int_\Omega F(Du)dx$$

where F is a smooth convex function,

$$F_{p_\alpha p_\beta}(Du)\xi_\alpha\xi_\beta \geq |\xi|^2 \ , \quad |F_{p_\alpha p_\beta}(Du)| \leq L \ .$$

As we have seen $Du \in H^1$ and any derivative $D_s u$ satisfies the equation

$$(5.2) \qquad \int F_{p_\alpha p_\beta}(Du)D_\beta(D_s u)D_\alpha\phi dx = 0 \quad \forall\phi \in H^1_0 \ .$$

Equation (5.2) can be seen as a linear elliptic equation with coefficients $A^{\alpha\beta}(x) = F_{p_\alpha p_\beta}(Du)$ in $L^\infty(\Omega)$. The gap between H^1 and C^1 is filled, and hence the 19^{th} Hilbert's problem completely solved (in this case), by the following famous result by E. De Giorgi.

THEOREM 5.1. *Let* $v \in H^1(\Omega)$ *be a weak solution to*

$$\int_\Omega a^{\alpha\beta}(x)D_\alpha v D_\beta\phi = 0 \quad \forall\, \phi \in H^1_0(\Omega) \ ,$$

where $a^{\alpha\beta} \in L^\infty(\Omega)$ *and* $a^{\alpha\beta}\xi_\alpha\xi_\beta \geq \nu|\xi|^2 \quad \forall\xi \quad (\nu > 0)$. *Then* $u \in$

$^{*)}$ The same result could be obtained using Schauder type estimates for the Euler equation in the strong formulation (4.6), i.e. the Hölder regularity theory for non-variational systems with smooth coefficients. But in the sequel it is more convenient to work with systems in divergence form, and therefore with the system in variation.

$\in C^{0,\gamma}_{loc}(\Omega)$ *for some positive* γ .

Unfortunately Theorem 5.1 is not true in the vector valued case and $n \geq 3$, see E. De Giorgi [18]; weak solutions to quasilinear systems of the type of systems in variation like (4.8) are singular for $n \geq 3$ [55]; finally, extremals (and therefore minimum points) of functionals of the type (5.1) for $n \geq 5$, $N > 1$, need not be C^1 [106], [107] *) . As we have stated, other counterexamples to the regularity are available, but these already say that in the vector valued case we should look for partial regularity results and for conditions under which everywhere regularity holds.

The situation gets worse when passing on to consider general functionals

$$(5.3) \qquad \int_\Omega F(x,u,Du)\,dx$$

and quasilinear or nonlinear systems under uncontrollable growth conditions, and that even in the scalar case.

First of all, as we have seen, we can consider only bounded extremals. Actually, "extremals" even in the scalar case can be unbounded. Moreover, still in the scalar case, typical phenomena of elliptic equations (as uniqueness in the small) fail for unbounded solutions.

Therefore (in the case of uncontrollable growth conditions) we are led to consider definitely $H^1 \cap L^\infty$ as the natural class where to start with weak solutions; and the most convincing argument is the following result by O. A. Ladyzhenskaya and N. N. Ural'tseva [73].

<u>*THEOREM 5.2.*</u> *Weak solutions (i.e. bounded) of nonlinear equations,* $N = 1$ *, under uncontrollable growth conditions are smooth.*

Actually in [73] it is proved (compare the next chapter for a stronger result) that minimum points of functional (5.1), $N = 1$, bounded at the boundary of Ω are bounded. Therefore Theorem 5.2 applies also to minimum points (not to extremals in general!).

Of course, for systems we cannot expect regularity in the general situation, but the situation is extremaly unpleasant. For example: a) [73], [62]: $u(x) = x|x|^{-1}$ is a weak solution to

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*) An example of a minimum point $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^2$, $\mathbb{R}^3$ for an elliptic functional of the type (5.1) is missing: it would be very interesting to produce such an example.

$$- \Delta u = u|Du|^2$$

and an extremal $u \in H^1 \cap L^\infty$ for the functional

$$\int_\Omega a(|u|)|Du|^2 dx$$

provided $a(t)$ is a smooth function with $a'(1) = -2a(1)$.

b) [25]: for $n = N = 2$ the function $u(x) = (\sin \log \log|x|^{-1}, \cos \log \log|x|^{-1})$ is a discontinuous weak solution of the system

$$- \Delta u^1 = 2 \frac{u^1 + u^2}{1 + |u|^2} |Du|^2 ,$$

$$- \Delta u^2 = 2 \frac{u^2 - u^1}{1 + |u|^2} |Du|^2 .$$

Note that systems in a) and b) are even diagonal.

c) [28]: in dimension $n = 2$, $N > 1$, functionals (5.3) may have bounded and discontinuous extremals.

The above examples show that (at the first rude approach) regularity depends not only on the boundedness of u (as in the scalar case) but also on a smallness condition on the bound for $|u|$. We refer to [58], [59], [60], [36] for more information.

II. Direct methods for the regularity

De Giorgi's result of regularity, as well as its generalizations, have as their starting point the Euler equation of the functional in question. Therefore it requires at least:

a) some smoothness of the function $F(x,u,p)$, moreover suitable growth conditions, not only on F, but also for its partial derivatives $F_p(x,u,p)$ and $F_u(x,u,p)$, and also

b) under natural conditions we need to start with bounded minimum points,

c) it does not distinguish between true minima and simple extremals,

d) it needs the ellipticity condition.

Of course the smoothness of F and the convexity (or ellipticity) condition on F are necessary if one wants to prove the differentiability of the minima (this is already the case in dimension

$n = 1$). But if we look only for the continuity (in the sense of Hölder) of the minima, those assumptions seem (and are) superfluous.

In this chapter we want to describe some works of M. Giaquinta and E. Giusti which show that the "first" stage of regularity can be obtained by working directly with the functional F instead of working with its Euler equation [39], and that even for weak solutions the first stage of regularity (Hölder regularity) depends on a "minimality property" of weak solutions to elliptic systems.

In this direction we should mention one classical result in dimension $n = 2$, due to C. B. Morrey [85] in 1938.

<u>*THEOREM*</u>. *Suppose that*

$$|p|^2 \leq F(x,u,p) \leq M|p|^2$$

and that $n = 2$, $N \geq 1$. *Let* $u \in H^{1,2}$ *be a minimum point for the functional*

$$\int_\Omega F(x,u,Du)dx \ .$$

Then u *is locally Hölder-continuous.*

We note explicitly that F is not assumed smooth, nor convex with respect to p .

<u>1. The scalar case</u>

Let us consider the multiple integral

$$\mathcal{F}[u;\Omega] = \int_\Omega F(x,u,Du)dx$$

with $N = 1$, where

(i) $F(x,u,p) : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a Carathéodory function, i.e. measurable in x and continuous in (u,p) . Thus $F(x,u,p)$ is measurable for measurable $u(x)$ and $p(x)$.

(ii) There exist positive constants a and b and a real number $m > 1$ such that

$$(1.1) \qquad |p|^m - b(|u|^\alpha + 1) \leq F(x,u,p) \leq a|p|^m + b(|u|^\alpha + 1)$$

where $m \leq \alpha < m^* = \dfrac{mn}{n - m}$ *) .

Let u be a minimum point for $\mathcal{F}$; we recall that this means, in our terminology, that for every $\phi \in H^{1,m}(\Omega)$ with $\text{supp}\, \phi \subset\subset \Omega$

we have

(1.2) $\qquad \mathcal{F}[u;\ \mathrm{supp}\ \phi] \leq \mathcal{F}[u + \phi;\ \mathrm{supp}\ \phi]$.

Then we have, see [39],

<u>*THEOREM 1.1.*</u> u *is locally bounded in* Ω .

Because of Theorem 1.1 we are now justified in assuming, instead of (1.1) the weaker condition

(1.3) $\qquad |p|^m - b(M) \leq F(x,u,p) \leq a(M)|p|^m + b(M)$

for $x \in \Omega$, $|u| \leq M$ and $p \in \mathbb{R}^n$.

Thus we have, see [39],

<u>*THEOREM 1.2.*</u> *Let* (1.3) *hold and let* $u \in H^{1,m}_{\mathrm{loc}} \cap L^{\infty}_{\mathrm{loc}}$ *be a minimum point for* $\mathcal{F}[u;\Omega]$. *Then* u *is Hölder-continuous in* Ω .

We refer to [39] for the proof of Theorem 1.1 and we restrict ourselves to prove Theorem 1.2. The proof uses the following characterization of Hölder-continuous functions due to E. De Giorgi [17] for $m = 2$, compare [73] for $m > 1$.

<u>De Giorgi's classes</u> $\mathcal{B}_m(\Omega,M,\gamma,\delta,\frac{1}{q})$. The symbol $\mathcal{B}_m(\Omega,M,\gamma,\delta,\frac{1}{q})$ denotes the class of functions $u(x)$ in $H^{1,m}$ with $\max\limits_{\Omega} |u| \leq M$ such that for u and $-u$ the following inequalities are valid in an arbitrary ball $B_\rho \subset \Omega$ for arbitrary $\sigma \in (0,1)$:

$$\int_{A_{k,\rho-\sigma\rho}} |Du|^m dx \leq \gamma\{ \frac{1}{\sigma^m \rho^{m(1-n/q)}} \max |u(x) - k|^m + 1\}|A_{k,\rho}|^{1-\frac{m}{q}}$$

for $k \geq \max\limits_{B_\rho} u - \delta$, where

$$A_{k,\rho} = \{x \in B_\rho : u(x) > k\} , \quad 1 < m \leq n , \quad q > n \geq 2 .$$

We have, see [17], [73],

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*) Here we shall restrict ourselves to the case $1 < m < n$. In fact, when $m > n$, every function in $H^{1,m}$ is trivially Hölder-continuous; and we shall consider the case $m = n$ in Sec. 2 of this Chapter.

THEOREM 1.3. *Let* $u \in \mathcal{B}_m(\Omega, M, \gamma, \delta, \frac{1}{q})$. *Then* u *is locally Hölder-continuous and for* $B_\rho \subset B_{\rho_0}$ *we have*

$$\operatorname*{osc}_{B_\rho} u \leq c\left(\frac{\rho}{\rho_0}\right)^\alpha$$

for some positive α .

P r o o f of Theorem 1.2. Let $x_0 \in \Omega$ and $B_R = B_R(x_0) \subset \Omega$. Let $w = \max(u-k, 0)$ and let $\eta(x) \in C_0^\infty(B_s)$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_t , $|D\eta| \leq 2(s-t)^{-1}$, $t < s < R$.

Using the minimality of u , condition (1.2), we have

$$\mathcal{F}[u; \operatorname{supp} \eta w] \leq \mathcal{F}[u - \eta w; \operatorname{supp} \eta w]$$

and using (1.3),

$$\int_{A_{k,s}} |Du|^m dx \leq \gamma_1 \{ \int_{A_{k,s}} (1-\eta)^m |Du|^m dx + \int_{A_{k,s}} w^m |D\eta|^m dx + |A_{k,s}| \},$$

hence

$$\int_{A_{k,t}} |Du|^m dx \leq \gamma_2 \{ \int_{A_{k,s} \setminus A_{k,t}} |Du|^m dx + (s-t)^{-m} \int_{A_{k,R}} (u-k)^m dx + |A_{k,R}| \} .$$

Now we fill the hole (compare [128]) i.e. we add to both sides γ_2 times the left-hand side, obtaining

$$\int_{A_{k,t}} |Du|^m dx \leq \frac{\gamma_2}{1+\gamma_2} \int_{A_{k,s}} |Du|^m dx + \gamma_3 \{ (s-t)^{-m} \int_{A_{k,R}} (u-k)^m dx + |A_{k,R}| \} . \tag{1.4}$$

Now we have

LEMMA 1.4. *Let* $f(t)$ *be a non-negative bounded function defined for* $0 \leq T_0 \leq t \leq T_1$. *Suppose that for* $T_0 \leq t < s \leq T_1$ *we have*

$$f(t) \leq A(s-t)^{-\alpha} + B + \theta f(s)$$

where A , B , α , θ *are non-negative constants, and* $\theta < 1$. *Then there exists a constant* c , $c = c(\alpha, \theta)$, *such that for every* ρ , R , $T_0 \leq \rho < R \leq T_1$ *we have*

$$f(\rho) \leq c[A(R-\rho)^{-\alpha} + B] . \tag{1.5}$$

Let us postpone the proof.
Applying Lemma 1.4 we deduce from (1.4)

$$\int_{A_{k,\rho}} |Du|^m dx \leq \gamma_4 \{(R-\rho)^{-m} \int_{A_{k,R}} (u-k)^m dx + |A_{k,R}|\} .$$

The same inequality holds for $-u$, since it minimizes the functional

$$\overline{\mathcal{F}}[v;\Omega] = \int_{\Omega} \overline{F}(x,v,Dv)dx$$

with $\overline{F}(x,v,p) = F(x,-v,-p)$ satisfying the same growth condition (4.3). The result then follows from Theorem 1.3.

Q.E.D.

P r o o f of Lemma 1.4: Define

$$t_0 = \rho , \quad t_{i+1} - t_i = (1-\tau)\tau^i(R-\rho) , \quad 0 < \tau < 1 .$$

By iteration

$$f(t_0) \leq \theta^k f(t_k) + \left[\frac{A}{(1-\tau)^\alpha}(R-\rho)^{-\alpha} + B\right] \sum_{i=0}^{k-1} \theta^i \tau^{-i\alpha} .$$

We now choose τ such that $\tau^{-\alpha}\theta < 1$ and let $k \to \infty$. Then we get (1.5) with $c = (1-\tau)^\alpha(1-\theta\tau^{-\alpha})^{-1}$.

Q.E.D.

REMARK 1.5. We mention that a result of the type of Theorem 1.2 appears in [26] under strong assumptions on F. In the case that F does not depend on u and is convex in p, the proof of [26] relies on the following observation. We have

$$\int_{A_{k,R}} F(x,Du)dx \leq \int_{A_{k,R}} F\Big(x,\ Du - D\big(\eta^m(u-k)\big)\Big)dx .$$

Writting

$$Du - D\big(\eta^m(u-k)\big) = (1-\eta^m)Du + \eta^m\left[-\frac{m}{\eta}D\eta(u-k)\right] ,$$

using the convexity and (1.3) we then deduce

$$\int_{A_{k,R}} (1-\eta^m)F(x,Du)dx \leq \int_{A_{k,R}} \eta^m F\big(x, \frac{m}{\eta}D\eta(u-k)\big)dx \leq$$

$$\leq \int_{A_{k,R}} [a(M)|D\eta^m|m^m|u-k|^m + b(M)]dx ,$$

which implies the Hölder-continuity.

2. The vector valued case. Estimates for the gradient

As we have seen, in the vector valued case $N > 1$, we have no hope (except in small dimensions) of proving Hölder regularity. But a basic regularity result still holds for the minima: it is an L^q-estimate, $q > m$, for the gradient.

Results of this kind were proved first by B. V. Boyarskiĭ [11] and N. G. Meyers [79] for solutions of linear elliptic equations; and by N. G. Meyers, A. Elcrat [82], M. Giaquinta, G. Modica [43] for classes of nonlinear elliptic systems; we refer to [36] for a discussion.

Besides their intrinsic interest, they are an essential tool in the study of regularity of solutions of nonlinear elliptic systems, following the method introduced in [38] (see also [35], [39], [43], [44], [45]).

In this section we state an L^q-estimate for the minima, due to M. Giaquinta and E. Giusti [39], again without assuming regularity on F nor convexity in p, and in the next section we shall present further results.

Let us consider the variational integral

$$\mathcal{F}[u;\Omega] = \int_\Omega F(x,u,Du)\,dx$$

with $N \geq 1$, and assume

(i) $F(x,u,p) : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$ is a Carathéodory function

and for the sake of simplicity,

(ii) $|p|^m \leq F(x,u,p) \leq a|p|^m$.

Then we have

<u>*THEOREM 2.1.*</u> *Let* $u \in H^{1,m}_{loc}(\Omega,\mathbb{R}^N)$ *be a minimum point for* $\mathcal{F}[u;\Omega]$. *Then there exists an exponent* $q > m$ *such that* $u \in H^{1,q}_{loc}(\Omega,\mathbb{R}^N)$. *Moreover, for every* $R < \mathrm{dist}(x_0,\partial\Omega)$ *we have*

$$(2.1)\qquad \left(\fint_{B_{R/2}(x_0)} |Du|^q dx\right)^{1/q} \leq c\left(\fint_{B_R(x_0)} |Du|^m dx\right)^{1/m}$$

c *being a constant depending only on* a, N, n, m.

P r o o f . Let $x_0 \in \Omega$, $0 < t < s < R < \mathrm{dist}(x_0,\partial\Omega)$. With the usual choice of η , we have from the minimality of u

$$\mathcal{F}[u;\ \mathrm{supp}\ \eta(u-u_R)] \leq$$
$$\leq \mathcal{F}[u-\eta(u-u_R);\ \mathrm{supp}\ \eta(u-u_R)] \ ;$$

hence using (ii),

$$\int_{B_s} |Du|^m dx \leq \gamma_1 \{ \int_{B_s \setminus B_t} |Du|^m dx + (s-t)^{-m} \int_{B_s} |u-u_R|^m dx \} \ .$$

Now arguing as in the proof of Theorem 1.2, i.e. filling the hole and applying Lemma 1.4, we deduce

$$\int_{B_{R/2}} |Du|^m dx \leq \gamma_2 R^{-m} \int_{B_R} |u-u_R|^m dx \ .$$

Using the Sobolev-Poincaré inequality

$$\int_{B_R} |u-u_R|^m dx \leq c(n,m,N) \left(\int_{B_R} |Du|^r dx \right)^{m/r} \ , \quad r = \frac{n\cdot m}{n+m}$$

and dividing by R^n we finally get

$$\left(\fint_{B_{R/2}} |Du|^m dx \right)^{1/m} \leq \gamma_3 \left(\fint_{B_R} |Du|^r dx \right)^{1/r} \ . \tag{2.2}$$

The result then follows at once from (2.2) by applying the following

PROPOSITION 2.2. Let Q *be a domain in* $\mathbb{R}^n$, $g \in L^t_{loc}(Q)$, $f \in L^s(Q)$, $s > t$. *Suppose that*

$$\fint_{B_R(x_0)} g^t dx \leq b \left(\fint_{B_{2R}(x_0)} g dx \right)^t + \fint_{B_{2R}} f^t dx$$

for each $x_0 \in Q$ *and each* $R < \min(\frac{1}{2}\,\mathrm{dist}(x_0,\partial Q), R_0)$ *, where* R_0 , b *are constants* $R_0 > Q$, $b > 1$. *Then* $g \in L^q_{loc}(Q)$ *for* $q \in [t, t+\varepsilon)$ *and*

$$\left(\fint_{B_R} g^q dx \right)^{1/q} \leq c \{ \left(\fint_{B_{2R}} g^t dx \right)^{1/t} + \left(\fint_{B_{2R}} f^q dx \right)^{1/q} \}$$

for $B_{2R} \subset Q$, $R < R_0$ *, where* c *and* ε *are positive constants depending only on* b , t , n , s .

Proposition 2.2 is due to M. Giaquinta, G. Modica [43], and represents the local version of a result by F. W. Gehring [30]. We omit the not simple proof and we refer to [43] or to [34], [36], [124] for some extensions. In [36] the reader will find a discussion of this and some related results.

In the special case $F = F(x,p)$ convex in p, the result of Theorem 2.1 can be obtained by using the trick described in Remark 1.5 and Proposition 2.2, compare with [3].

It is worth remarking that Theorem 2.1 does not hold for extremals of the functionals $\mathcal{F}[u;\Omega]$, even when assuming that $F(x,u,p)$ is convex in p and $N = 1$, as the example in [26] shows, see also [28]. When $N > 1$, the result is in general false for elliptic systems, even if we assume that u is bounded, see example b) in Sec. 5, Chap. I (and example c) in Sec. 5, Chap. I) and it is necessary to suppose that u is "small" ([38], [43]).

The p r o o f of Proposition 2.2 shows that the exponent $q > m$ can be taken in an interval $(m, m+\sigma)$, with σ independent of m for m close to n. Therefore we have

<u>*COROLLARY 2.3.*</u> *There exists a* $\sigma > 0$ *depending only on* a *in* (ii), n *and* N *such that if* $m > n - \sigma$ *and* F *satisfies* (ii), *then every minimum point for* $\mathcal{F}[u;\Omega]$ *is Hölder-continuous in* Ω.

In particular Corollary 2.3 extends Morrey's result stated at the beginning of this chapter. For elliptic systems, results of this type appear in [128], [118], [120], [126], [43].

3. Quasi-minima

Consider the multiple integral

$$\mathcal{F}[u;\Omega] = \int_\Omega F(x,u,Du)\,dx \tag{3.1}$$

where $F(x,u,p) : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying

$$|p|^m - b|u|^\gamma - g_1(x) \leq F(x,u,p) \leq a|p|^m + b|u|^\gamma + g_2(x) \tag{3.2}$$

with

$$1 < m < n\,, \quad \gamma < m^* = \frac{mn}{n-m}\,.$$

Until now we have dealt with minimum points of $\mathcal{F}$; we introduce now the following

<u>*DEFINITION*</u>. $u \in H^{1,m}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ *is a quasi-minimum for* $\mathcal{F}$ *in* Ω *with a constant* A *if*

$$\mathcal{F}[u;\ \mathrm{supp}\ \phi] \leqq A\mathcal{F}[u+\phi;\ \mathrm{supp}\ \phi]$$

for all ϕ *with* $\mathrm{supp}\ \phi \subset \Omega$.

The constant A may of course depend on u .

Then the results of Section 1 and 2 hold also for quasi-minima, as a simple inspection of the proofs shows; more precisely, we have

THEOREM 3.1. *Let* u *be a quasi-minimum for* $\mathcal{F}$ *in* Ω *and assume* (3.2) *holds. Then we have*

(i) *if* $N = 1$ *and* $g_1, g_2 \in L^s(\Omega)$ *for some* $s > \frac{n}{m}$, *then* u *is locally Hölder-continuous; in particular, it is locally bounded;*

(ii) *if* $N \geqq 1$ *and* $g_1, g_2 \in L^s(\Omega)$ *for some* $s > 1$, *then there exists an exponent* $r > m$ *such that* $u \in H^{1,r}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$.

Quite a lot of results for solutions of elliptic (linear and nonlinear) systems can be re-read in terms of quasi-minima.

1. Of course, any minimum point for $\mathcal{F}$ is a quasi-minimum. It is not difficult to verify that moreover, any minimum point for $\mathcal{F}$ is a quasi-minimum for

$$\int_\Omega \{|Du|^m + b|u|^\gamma + (g_1 + g_2 + b)\}dx \ .$$

In particular, for $m = 2$, $b = 0$, $g_1, g_2 = 0$ we obtain that any minimum point for

$$\int_\Omega F(x,u,Du)dx \ , \quad |p|^2 \leqq F(x,u,p) \leqq a|p|^2 \ ,$$

is a quasi-minimum for the Dirichlet integral.

Any weak solution to the linear elliptic system with L^∞ coefficients

$$-D_\beta(A^{\alpha\beta}_{ij}(x)D_\alpha u^i) = 0 \ , \quad j = 1,\dots,N \ , \quad A^{\alpha\beta}_{ij}\xi^i_\alpha\xi^j_\beta \geqq |\xi|^2 \quad \forall \xi$$

is a quasi-minimum for the Dirichlet integral. To see that, it is sufficient to test with $u - v$ with $\mathrm{supp}(u - v) \subset \Omega$. In particular, for $N = 1$ we obtain De Giorgi's result *) we have stated in Sec. 2, Chap. I.

More generally, the Hölder regularity of weak solution to nonlinear elliptic equations (see [73]) and the L^p-estimates for the gradient of general nonlinear elliptic systems (compare [38], [43],

*) Remark that this is only a re-reading in terms of quasi-minima of De Giorgi's result, the proof being essentially the same.

[82]) can be obtained as consequences of Theorem 3.1 *). We have in fact the following result.

2. Let u be a weak solution to

$$\int_\Omega \left(A_i^\alpha(x,u,Du)D_\alpha\phi^i + B_i(x,u,Du)\phi^i\right)dx = 0 \qquad \phi \in C_0^\infty(\Omega,R^N) \; ; \tag{3.3}$$

(A) assume that the controllable growth conditions and ellipticity in the following weak form hold:

$$A_i^\alpha(x,u,p)p_\alpha^i \geqq |p|^m - L|u|^\gamma - f(x) \; , \qquad \gamma < m^* \; ,$$

$$|A(x,u,p)| \leqq L|p|^{m-1} + L|u|^\sigma + g(x) \; , \quad \sigma = \gamma\frac{m-1}{m} \; ,$$

$$|B(x,u,p)| \leqq L|p|^\tau + L|u|^\delta + h(x) \; , \qquad \tau = \frac{\gamma-1}{\gamma}m \; , \; \delta = \gamma - 1 \; .$$

Then, inserting $\phi = u - v$, we get that u is a quasi-minimum for

$$\int_\Omega \left[|Du|^m + |u|^\gamma + (f + g^{\frac{m}{m-1}} + h^{\frac{\gamma}{\gamma-1}} + 1)\right]dx \; .$$

(B) Assume that the uncontrollable growth conditions hold:

$$\begin{aligned} &A_i(x,u,p)p_\alpha^i \geqq |p|^m - L_0 - L_1 f(x) \; , \\ &|A(x,u,p)| \leqq L_2|p|^{m-1} + L_3 + L_4 g(x) \; , \\ &|B(x,u,p)| \leqq a|p|^m + L_5 h(x) + L_6 \; , \\ &L_i = L_i(M) \; , \quad a = a(M) \; , \quad |u| \leqq M \; . \end{aligned} \tag{3.4}$$

(B_1) Suppose moreover that $N = 1$. Then we get that u is a quasi--minimum for

$$\int_\Omega \left[|Du|^m + (f + g^{\frac{m}{m-1}} + h) + 1\right]dx \; . \tag{3.5}$$

This can be shown by inserting $(u - w)^+ e^{\lambda(u-w)}$ and $(w - u)^+ e^{\lambda(w-u)}$ as test functions ϕ with $w = v$ for $|v| \leqq M$, $w = -M$ for $v < -M$, $w = M$ for $v > M$, for any v with $\mathrm{supp}(u - v) \subset\subset \Omega$.

(B_2) The L^p-estimate, as we have already remarked, would not be

---

*) This is not completely true, since in this way we are not able to handle the limit case corresponding to the value γ (below) $\gamma = m^*$. The Hölder regularity ($N = 1$) and the L^p-estimates ($N \geqq 1$) nevertheless hold even for $\gamma = m^*$, compare [73], [43] and [36].

true in the vector valued case under (3.4) even if assuming $|u|$ bounded; more precisely, it is in general not true if $a(M)M > 1$, compare [62]. But if we assume that $2a(M)M < 1$ then any weak solution u, $|u| \leq M$, to (3.3) is a quasi-minimum for the functional (3.5). Therefore the L^p-estimate holds *).

Finally, we want to mention two further examples of quasi-minima.

3. Weak solutions of the obstacle problem, i.e. for example

$$u \geq \psi \quad \text{in} \quad \Omega : \quad \int_\Omega DuD(u - v)dx \leq 0 \quad \forall v \,, \quad v \geq \psi \,, \quad \operatorname{supp}(u - v) \subset\subset \Omega$$

are quasi-minima for

$$\int_\Omega (|Du|^2 + |D\psi|^2)dx \,.$$

4. Quasiconformal (or quasi-regular) mappings are quasi-minima for

$$\int_\Omega |Du|^n dx$$

and therefore in $H^{1,n+\varepsilon}(\Omega,\mathbb{R}^n)$, in particular Hölder-continuous.

The definition of quasi-minima appears in [39] and the results of this section have been developed by M. Giaquinta - E. Giusti and have not been published.

We conclude this section with an example which shows that no Hölder regularity theory (even partial) can be developed for quasi--minima, in the vector valued case.

Let us start with the following remarks. Set

$$a_{ij}^{kh}(x) = \delta_{ij}\delta^{kh} + \frac{d_i^k d_j^h}{u_{x_\ell}^s d_\ell^s}$$

where

$$d_i^k = b_i^k - u_{x_i}^k \,, \quad b \in L^2(\Omega) \,, \quad \int_\Omega b_i^k \phi_{x_i}^k dx = 0 \quad \forall \phi \in C_0^\infty(\Omega,\mathbb{R}^n) \,.$$

Then

$$\int_\Omega a_{ij}^{kh}(x) u_{x_i}^k \phi_{x_j}^h dx = 0 \quad \forall \phi \in C_0^\infty(\Omega,\mathbb{R}^n) \,. \tag{3.6}$$

---

*) It is an open question whether we can get the same result under the weaker assumption $a(M)M < 1$.

The ellipticity and boundedness of the coefficients a_{ij}^{kh} corresponds respectively to,

$$u_x \cdot d > 0 \;,$$

$$\frac{b \cdot d}{u_x \cdot d} \leqq M \;.$$

It is on the basis of this simple remark that the examples of E. De Giorgi [18] and Giusti - Miranda [55] can be regarded. Actually the following choice for $n \geqq 3$:

$$u(x) = |x|^{-1} x \;,$$

$$b_i^k = |x|^{-1}\left(n\delta_{ik} + \frac{n}{n-2}\,\frac{x_k x_i}{|x|^2}\right)$$

permits to construct a discontinuous weak solution to the elliptic system (3.6). Let y_α be a sequence of points in Ω and let us set

$$U^k(x) = \sum_\alpha u^k(x - y_\alpha)\varepsilon_\alpha \;,$$

$$B_i^k(x) = \sum_\alpha b_i^k(x - y_\alpha)\varepsilon_\alpha$$

and

$$\mathcal{a}_{ij}^{kh} = \delta_{ij}\delta^{kh} + \frac{D_i^k D_j^h}{U_{x_\ell}^\ell D_\ell^s}$$

where

$$D_i^k = B_i^k - U_{x_i}^k \;.$$

Since

$$\int_\Omega B_i^k(x)\phi_{x_i}^k\,dx = 0 \quad \forall\phi \in C_0^\infty(\Omega,\mathbb{R}^n) \;,$$

it is a simple matter of calculation to show that after a suitable choice of the ε_α the vector U belongs to $H^{1,2}_{loc}(\Omega,\mathbb{R}^N)$ and is a solution of the elliptic system

$$- D_j(\mathcal{a}_{ij}^{kh}\, D_i U^k) = 0 \;, \quad h = 1,\dots,n \;.$$

Remark that U may be singular in a dense subset.

The above construction was shown to the author by J. Souček in April 1980.

From the point of view of quasi-minima, the example described shows that there exists a vector-valued quasi-minimum for

$$\int_{B_1(0)} |Du|^2 dx \;, \quad B_1(0) \subset \mathbb{R}^3$$

singular at all points $x \in B_{1/2}(0)$ with rational coordinates.

The Hölder regularity results of this section do not permit to gill the gap b) in Sec. 2 Chap. I; the step $C^{0,\gamma} \to C^{1,\gamma}$ is missing. This step needs some work, we refer to [73] and to [36] for a different approach.

4. Quasi-minima and quasi-convexity

In this section we want to show how the notion of quasi-minimum can be used together with the semicontinuity Theorem 1.3 in Chap. I and a variational principle in order to prove the existence of minimum points for a class of quasi-convex functionals (in the sense of Morrey).

Consider the multiple integral

$$\mathcal{F}[u;\Omega] = \int_\Omega F(x,u,Du)\,dx \tag{4.1}$$

where, for the sake of simplicity,

$$|p|^m \leqq F(x,u,p) \leqq a|p|^m$$

and assume that F is quasi-convex, i.e. for a.e. $x_0 \in \Omega$, for every $u_0 \in \mathbb{R}^N$, $\xi_0 \in \mathbb{R}^{nN}$ and for all $\phi \in C^1_0(\Omega,\mathbb{R}^N)$

$$\frac{1}{|\Omega|}\int_\Omega F\big(x_0,u_0,\xi_0 + D\phi(x)\big)\,dx \geqq F(x_0,u_0,\xi_0) \ .$$

We shall prove

THEOREM 4.1. Let $\tilde{u} \in H^{1,m}(\Omega,\mathbb{R}^N)$ *. Then there exists a minimum point* u *of* $\mathcal{F}$ *on* $\tilde{u} + H^{1,m}_0(\Omega,\mathbb{R}^N)$ *. Moreover,* $u \in H^{1,q}_{loc}(\Omega,\mathbb{R}^N)$ *for some* $q > m$.

We need the following variational principle in I. Ekeland [22]:

THEOREM 4.2. Let (V,d) *be a complete metric space,* $F : V \to [0,+\infty]$ *a lower semicontinuous functional, not identically* $+\infty$ *. Let* $\eta > 0$ *and* $w \in V$ *verify*

$$F(w) \leqq \inf_V F + \eta \ .$$

Then there exists $v \in V$ *such that* $F(v) \leqq F(w)$, $d(v,w) \leqq 1$ *and* v *is the (only) minimum point of the functional*

$$F(u) + \eta d(u,v) \ .$$

The functional in (4.1) is lower semicontinuous in the complete metric space $\{u \in H^{1,1}(\Omega,\mathbb{R}^N) : u = \tilde{u} \text{ on } \partial\Omega\}$. Hence we can apply

Theorem 4.2, and the function v we obtain is obviously a quasi-minimum for a functional of the same type, which is independent of η for η small. In particular, there exists a minimizing sequence of quasi-minima with a uniform constant A.

Theorem 3.1 (ii) implies then that there exists a minimizing sequence $\{u_k\}$ in $\tilde{u} + H_0^{1,m}(\Omega,\mathbb{R}^N)$ such that for $\tilde{\Omega} \subset\subset \Omega$

$$||u_k||_{H^{1,q}(\Omega,\mathbb{R}^N)} \leqq \text{const. independent of } k$$

where q is larger than m (and independent of $\tilde{\Omega}$).

We can then conclude the proof of Theorem 4.1, simply by means of (semicontinuity) Theorem 1.3 *) in Chap. I, as for all $\tilde{\Omega} \subset\subset \Omega$ we have

$$\mathcal{F}[u;\tilde{\Omega}] \leqq \liminf_{k\to\infty} \mathcal{F}[u_k;\tilde{\Omega}] \leqq \min_{\tilde{u}+H_0^{1,m}(\Omega,\mathbb{R}^N)} \mathcal{F} .$$

The p r o o f above is a re-reading of the proof in [76].

III. Partial regularity

As we have had occasion to mention, the study of the "partial regularity" of extremals or, more generally, of weak solutions of elliptic nonlinear systems starts with the work of C. B. Morrey [96] and E. Giusti, M. Miranda [56] in 1968. Nowadays we have two different methods for getting such type of results:

a) The one in [96], [56]. It is an indirect argument, i.e. a reduction to absurd argument; and it works very well for studying weak solutions of systems (of the type of systems) in variation for general multiple integrals, essentially when no explicit dependence on u appears.

b) The methods introduced in M. Giaquinta - E. Giusti [38] and developed and improved in [43], [44], [45], [39]. It is of direct type and relies on a perturbation argument **) which uses as an essential tool the L^p-estimate for the gradient. It allows to handle some classes of quasilinear and nonlinear systems (as well as of multiple integrals with explicit dependence on u), too.

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*) Here we use the quasi-convexity

**) of the type of the one which appears in [89], [13]: Korn device.

Anyway, the two methods seem not to be completely interchangeable; we refer to [36] for a discussion. In these lectures we shall not talk about the first one and we simply refer to [96], [56], [52], [112] and to [36]. Moreover, we shall confine ourselves to describing the main ideas of proving a few results obtained. Therefore this chapter, which should be the central one, has to be understood as an introduction. For more information we refer to the papers quoted and to [36]. In particular, much space should have been dedicated to diagonal systems, the methods developed for proving everywhere regularity, and its connections with harmonic mappings; instead, even reluctantly, we simply refer to [59], [60], [61], [42].

1. Quasilinear elliptic systems

In Chapter I we have seen that any extremal of functionals of the type

$$\int_\Omega F(Du)\,dx \tag{1.1}$$

with

$$|\xi|^2 \leq F_{p^i_\alpha p^j_\beta}(p)\,\xi^i_\alpha \xi^j_\beta \leq L|\xi|^2 \qquad \forall \xi$$

or, more generally, any weak solution $u \in H^{1,2}_{loc}(\Omega,\mathbb{R}^N)$ to systems of the type

$$-\,D_\alpha A^\alpha_i(Du) = 0\ , \quad i = 1,\dots,N\ ,$$

with

$$|A^\alpha_i(p)| \leq c|p|\ , \quad |A^\alpha_{ip^j_\beta}(p)| \leq L\ ,$$

$$A^\alpha_{ip^j_\beta}(p)\,\xi^i_\alpha\xi^j_\beta \geq \lambda|\xi|^2 \qquad \forall\xi \quad (\lambda > 0)$$

has first derivatives in $H^{1,2}_{loc}$ *) satisfying the quasilinear elliptic system

$$\int_\Omega \delta_{\ell s} A^\alpha_{ip^j_\beta}(U)\,D_\beta U^j_s D_\alpha \phi^i_\ell\,dx = 0 \qquad \forall \phi \in H^1_0(\Omega,\mathbb{R}^{nN})$$

where

$$U = (U^j_s) = (D_s u^j)\ .$$

Therefore the question of $C^{1,\alpha}$-regularity for extremals of the functionals in (1.1) can be reduced to the question of the Hölder regularity of weak solutions to quasilinear elliptic systems of the type

*) We assume $A^\alpha_i(p)$ of class C^1.

(1.2) $$\int_\Omega A_{ij}^{\alpha\beta}(u) D_\alpha u^i D_\beta \phi^j dx = 0 \quad \forall \phi \in H_0^1(\Omega, \mathbb{R}^N) ,$$

where $A_{ij}^{\alpha\beta}(u)$ are continuous functions in u, satisfying the ellipticity condition

(1.3) $$A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j \geqq \lambda|\xi|^2 \quad \forall \xi \quad (\lambda > 0)$$

and

(1.4) $$|A_{ij}^{\alpha\beta}| \leqq L .$$

We have (compare with [96], [56], [38], [43])

THEOREM 1.1. *Suppose that the coefficients* $A_{ij}^{\alpha\beta}(u)$ *are continuous and verify* (1.3), (1.4). *Let* u *be a weak solution to system* (1.2). *Then there exists an open set* $\Omega_0 \subset \Omega$ *such that* u *is locally Hölder-continuous with any exponent less than* 1 *in* Ω_0. *Moreover,* $\mathcal{H}^{n-s}(\Omega \setminus \Omega_0) = 0$ *for some* $s > 2$; *here* $\mathcal{H}^k$ *denotes the* k*-dimensional Hausdorff measure.*

We now want to sketch the proof of this theorem following the method of [38] and assuming, moreover, that the coefficients $A_{ij}^{\alpha\beta}$ are bounded and uniformly continuous. This implies in particular that there exists a continuous, bounded, increasing, concave function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

(1.5) $$|A_{ij}^{\alpha\beta}(u) - A_{ij}^{\alpha\beta}(v)| \leqq \omega(|u - v|^2) .$$

P r o o f of Theorem 1.1. Let $x_0 \in \Omega$ and $R < \min\{\mathrm{dist}(x_0,\partial\Omega),1\}$. Let $A_{ij0}^{\alpha\beta} = A_{ij}^{\alpha\beta}(u_{x_0,R})$ and let v be the solution to the Dirichlet problem

$$\begin{cases} - D_\beta(A_{ij}^{\alpha\beta}\, D_\alpha v^i) = 0 , \quad j = 1,\dots,N , \text{ in } B_{R/2}(x_0) , \\ v - u \in H_0^1(B_{R/2}(x_0), \mathbb{R}^N) . \end{cases}$$

Then we have, see Sec. 3, Chap. I, for all $\rho < R/2$

$$\int_{B_\rho(x_0)} |Dv|^2 dx \leqq c\left(\frac{\rho}{R}\right)^n \int_{B_{R/2}(x_0)} |Dv|^2 dx ,$$

hence

(1.6) $$\int_{B_\rho(x_0)} |Du|^2 dx \leqq c\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |Du|^2 dx + c\int_{B_{R/2}} |D(u - v)|^2 dx .$$

If we set $w = u - v$, we have $w = 0$ on $\partial B_{R/2}$ and

$$\int_{B_{R/2}(x_0)} A^{\alpha\beta}_{ij0} D_\alpha w^i D_\beta \phi^j dx = \int_{B_{R/2}(x_0)} [A^{\alpha\beta}_{ij0} - A^{\alpha\beta}_{ij}(u)] D_\alpha u^i D_\beta \phi^j dx$$

$$\forall \phi \in H^1_0(B_{R/2}, \mathbb{R}^N) .$$

In particular, we may take $\phi = w$, so that using the ellipticity in (1.3), Hölder regularity and (1.5) we get

$$(1.7) \qquad \int_{B_{R/2}(x_0)} |D(u - v)|^2 dx \leq c \int_{B_{R/2}(x_0)} \omega^2(|u - u_{x_0,R}|^2)|Du|^2 dx.$$

On the other hand, using the L^p-estimate for the gradient, compare Sec. 3 Chap. II, and the boundedness of ω, we have (for some $\sigma > 2$)

$$(1.8) \qquad \int_{B_R(x_0)} \omega^2 |Du|^2 dx \leq \left(\int_{B_{R/2}} |Du|^\sigma dx\right)^{2/\sigma} \left(\int_{B_{R/2}} \omega^{\frac{2\sigma}{\sigma-2}} dx\right)^{\frac{\sigma-2}{\sigma}} \leq$$

$$\leq c \int_{B_R} |Du|^2 dx \left(\fint_{R_R} \omega(|u - u_{x_0,R}|^2) dx\right)^{\frac{\sigma-2}{\sigma}}$$

and, as ω is a concave function,

$$(1.9) \qquad \fint_{B_R} \omega dx \leq \omega\left(\fint_{B_R} |u - u_{x_0,R}|^2 dx\right) .$$

Putting together (1.6), (1.7), (1.8), (1.9), with a simple use of Poincaré inequality, we get

$$(1.10) \qquad \int_{B_\rho} |Du|^2 dx \leq c\left[\left(\frac{\rho}{R}\right)^n + \lambda(x_0,R)\right] \int_{B_R} |Du|^2 dx$$

where

$$\chi(x_0,R) = \omega\left(\frac{c}{R^{n-2}} \int_{B_R} |Du|^2 dx\right)^{\frac{\sigma-2}{\sigma}}$$

for all $\rho < R/2$. Since (1.10) is obvious for $\frac{R}{2} \leq \rho < R$, we get (1.10) for all $\rho < R$. Set

$$\Phi(x_0,R) = R^{2-n} \int_{B_R(x_0)} |Du|^2 dx .$$

From (1.10) we deduce for $0 < \tau < 1$

$$(1.11) \qquad \Phi(x_0,\tau R) \leq K[1 + \chi(x_0,R)\tau^{-n}]\tau^2 \Phi(x_0,R) .$$

Let now $0 < \gamma < 1$ and choose τ in such a way that $2K\tau^{2-2\gamma} = 1$.

Let $x_0 \in \Omega$ and let R be such that

(1.12) $$\chi(x_0,R) < \tau^n .$$

Then we have from (1.11)

$$\Phi(x_0,\tau R) \leqq \tau^{2\gamma}\Phi(x_0,R)$$

and hence

$$\chi(x_0,\tau R) \leqq \chi(x_0,R) \leqq \tau^n .$$

By induction we get for every k :

$$\Phi(x_0,\tau^k R) \leqq \tau^{2\gamma k}\Phi(x_0,R)$$

and hence for every $\rho < R$

(1.13) $$\Phi(x_0,\rho) \leqq \tau^{-n+2+2\gamma}\left(\frac{\rho}{R}\right)^{2\gamma}\Phi(x_0,R) .$$

Now, since χ is a continuous function of x_0 , if (1.12) holds for a point $x_0 \in \Omega$, then there exists a ball $B(x_0,r)$ such that for every $x \in B(x_0,r)$ we have

$$\chi(x,R) < \tau^n .$$

We conclude then that (1.13) holds uniformly for all $x \in B(x_0,r)$. It follows, compare Sec. 3, Chap. I, that u is Hölder-continuous in $B(x_0,r)$ with the exponent γ . In conclusion, there exists an open set $\Omega_0 \subset \Omega$ such that the solution u is locally Hölder-continuous, with the exponent γ , in Ω_0 . Since we have

$$\Omega_0 = \{x : \liminf_{R\to 0} R^{2-n} \int_{B_R(x)} |Du|^2 dy = 0\} \quad {}^{*)}$$

$$\{x : \liminf_{R\to 0} R^{-n} \int_{B_R(x)} |u - u_{x,R}|^2 dy = 0\} ,$$

we see that Ω_0 is nonvoid, $\mathrm{meas}(\Omega\setminus\Omega_0) = 0$, and independent of γ . On the other hand, Ω_0 depends on u and not only on the data of the system.

The second part of the theorem has to do with the problem of the pointwise definition of $H^{1,p}_{loc}$ functions. It is a consequence of the following result in [51]:

for $v \in L^1_{loc}(\Omega)$ and $0 \leqq \alpha < n$, set

$$E_\alpha = \{x \in \Omega : \limsup_{\rho\to 0+} \rho^{-n} \int_{B_\rho(x)} |v(y)|dy > 0\} .$$

---

*) Because of the Caccioppoli inequality.

Then we have

$$\mathcal{H}^{\alpha}(E_{\alpha}) = 0$$

simply noting that $Du \in L^q$ for some $q > 2$ and

$$\Big(R^{2-n}\int_{B_R} |Du|^2 dx\Big)^{1/2} \leq \Big(R^{q-n}\int_{B_R} |Du|^q\Big)^{1/q} .$$

Q.E.D.

REMARK. The proof of Theorem 1.1 shows that there exists an ε_0 depending on the data of the system such that x_0 is a regular point, i.e. $x_0 \in \Omega_0$, if and only if

$$R^{2-n}\int_{B_R(x_0)} |Du|^2 dx < \varepsilon_0$$

for some R , or equivalently (maybe for a different ε_0)

$$R^{-n}\int_{B_R(x_0)} |u - u_{x_0,R}|^2 dx < \varepsilon_0$$

for some R .

The case of (non-uniformly) continuous coefficients needs some technical adjustments. We shall not discuss the details, see [56], [52], [43] and we limit ourselves to remark that now $\Omega - \Omega_0$ would be

$$\Omega - \Omega_0 = \{x \in \Omega : \liminf_{\rho\to 0+} \rho^{2-n}\int_{B_\rho(x)} |Du|^2 dy > \varepsilon_0\} \cup$$

$$\cup \{x \in \Omega : \limsup_{\rho\to 0+} |u_{x,\rho}| = +\infty\} .$$

The technique described above permits to study general quasilinear elliptic systems of the type

$$(1.14) \qquad \int_\Omega A^{\alpha\beta}_{ij}(x,u)D_\beta u^j D_\alpha \phi^i dx = \int_\Omega a^\alpha_i(x,u)D_\alpha\phi^i dx + \int_\Omega b_i(x,u,Du)\phi^i dx$$

$$\forall\phi \in C^\infty_0(\Omega,\mathbb{R}^N)$$

(and even higher order systems) and obtain "optimal" partial (or everywhere) regularity results for weak solutions according to the growth conditions verified by the functions $a^\alpha_i(x,u)$ and $b_i(x,u,p)$ on the right hand side and the assumptions we make on the leading part $A^{\alpha\beta}_{ij}$. It would be very lengthy and technically complicated to describe these results, therefore we simply refer for example to [43], [44], [45], and [36].

Here we confine ourselves to discuss rapidly a "limit case" of (1.14) and more precisely the regularity of weak solutions u, i.e. $u \in H^1 \cap L^\infty(\Omega,\mathbb{R}^N)$, to systems with quadratic right-hand side,

$$\int_\Omega A_{ij}^{\alpha\beta}(x,u) D_\beta u^j D_\alpha \phi^i dx = \int_\Omega b_i(x,u,Du)\phi^i dx \ , \tag{1.15}$$

where we assume that (1.3), (1.4) still hold and

$$|b(x,u,p)| \leq a|p|^2 \ . \tag{1.16}$$

Following the lines of the proof of Theorem 1.1 one can prove, see [38], the following: *suppose* $A_{ij}^{\alpha\beta}$ *to be continuous and let* u, $|u| \leq M$, *be a solution to* (1.5). *Assume moreover that the ellipticity constant* λ *and the constant* a *in* (1.16) *satisfy the relation*

$$2Ma < \lambda \ . \tag{1.17}$$

Then u *is Hölder-continuous with any exponent* $\gamma < 1$ *in* Ω, *except possibly for a closed singular set* Σ, *whose Hausdorff dimension does not exceed* $n-q$, *for some* $q > 2$. *)

We have therefore the same result as in Theorem 1.1, but with the additional condition (1.17). However, as we have already stated, (1.17) is a natural condition, apart possibly for the factor 2. In fact the conclusion above does not hold without the assumption $Ma \leq \lambda$ even if $n = 2$ and the system is diagonal, i.e. $A_{ij}^{\alpha\beta} = \delta_{ij}A^{\alpha\beta}$.

Diagonal systems have been studied extensively, compare for example the survey papers [58], [60], because of their importance in differential geometry. The Euler system of the energy of harmonic mappings between Riemannian manifolds or the system satisfied by surface with prescribed mean curvature in isotermal parameters have exactly this structure.

We have, see [63], [130] and for a simpler proof [41]:

<u>*THEOREM 1.2.*</u> *Suppose* $A_{ij}^{\alpha\beta} = \delta_{ij}A^{\alpha\beta}(x)$ *with* $A^{\alpha\beta} \in L^\infty$; *let* (1.3), (1.4), (1.16) *hold and let* u, $|u| \leq M$, *be a weak solution to* (1.15) *and* $Ma < \lambda$. *Then* u *is locally Hölder-continuous.*

The literature on harmonic mappings is so large that we have not any possibility even to hint at it. We simply refer to the report [21] by J. Eells and L. Lemaire, and, for results in dimension $n \geq 3$, to

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*) We remark that the only point where (1.17) is used is in order to obtain the L^p-estimate for the gradient.

[61], [60], [59] and [42].

In the case of Theorem 1.1 when the coefficients are more than merely continuous, the solution $u(x)$ will show higher regularity in Ω_0. This is a simple consequence of the linear theory. However, in the more general case (1.14), as the right-hand side shows dependence on Du, in order to use the linear theory one first has to prove that u is in $C^{1,\gamma}_{loc}(\Omega_0, \mathbb{R}^N)$. In fact this can be done, still in the spirit of the proof of Theorem 1.1, and we refer for a very simple proof to [38], [42], see also [36].

We conclude this section with a discussion of a few problems that appear naturally.

There is a general <u>problem of studying the singular set</u>. In particular: is the singular set analytic or semianalytic? Are there different characterizations of the singular set? (See for example [57].) Are the singularities isolated in dimension 3 or more generally in the first dimension they appear? (In the next section we shall see one case with a positive answer.)

Connected with these problems is the problem of <u>giving reasonable condition for the solutions to be everywhere regular</u>. We mention some results in [46], [47] and the very interesting result in [127], see also [23], [64], that says that extremals of elliptic integrals of the type

$$\int_\Omega F(|Du|^2)dx$$

are everywhere regular. Other structural conditions in the case of diagonal systems with quadratic right-hand sides can be found in [63], [39]. But the problem is still open.

There are, finally, <u>topological problems</u> like: Is the regularity a generic property? Which are the topological properties of the class of systems with smooth solutions or with non-smooth solutions? In particular, there is a problem of the stability (or non-stability) of the singularities.

Finally, one could look for analogous results for parabolic systems and to the (specific) problem of the evolution of singularities (we refer for some basic results to [37], [48], [49], [15]).

2. Minima of quadratic multiple integrals

The results in Section 1 do not cover the case of minima of regular multiple integrals of the Calculus of Variations. And almost no results on the partial regularity of the minima of integrals like

$$(2.1)\qquad \int_\Omega F(x,u,Du)dx$$

under "natural conditions" are known. Let us try again to point out some difficulties. Assume F smooth and

$$(2.2)\qquad |p|^2 - k \leqq F(x,u,p) \leqq a|p|^2 + k\ .$$

We are not allowed to think of u as a solution of the Euler equation; and if we want to use the Euler equation with natural conditions we need

(i) growth assumptions on F_u and F_p , for example

$$|F_p(x,u,p)| \leqq L|p|\ ,$$
$$|F_u(x,u,p)| \leqq L|p|^2\ ;$$

(ii) to assume that u is bounded.

Of course we can assume (i), (ii), but we do not know when (ii) holds. Under the assumptions (i) and (ii), u is a solution of a system of the type

$$(2.3)\qquad \int_\Omega [A_i^\alpha(x,u,Du)D_\alpha\phi^i + B_i(x,u,Du)\phi^i]dx = 0 \qquad \forall\phi \in H_0^1 \cap L^\infty$$

and it does not seem easy to get some partial regularity result in that case simply requiring the natural growth, i.e.

$$(2.4)\qquad \begin{cases} |A_i^\alpha(x,u,p)| \leqq L|p|\ , \\ |A^\alpha_{ip^j_\beta}(x,u,p)| \leqq L\ , \\ |A^\alpha_{iu^\ell}(x,u,p)| \leqq L|p|\ , \end{cases}$$

$$(2.5)\qquad |B(x,u,p)| \leqq a|p|^2 + b\ ,$$

and the strong ellipticity

$$(2.6)\qquad A^\alpha_{ip^j_\beta}\xi^i_\alpha\xi^j_\beta \geqq \lambda|\xi|^2 \quad \forall\xi \quad (\lambda > 0)\ .$$

We note that in general (2.3), ..., (2.6) are not sufficient for proving that $u \in H^{2,2}_{loc}$ and satisfies the system in variation, i.e. are not enough for linearizing the system even in the case $B \equiv 0$. We should need to know that u is not only bounded but also continuous.

Anyway, for systems of the type (2.3) we have the following result, [44]:

THEOREM 2.1. a) *Let* $u \in H^{1,2}(\Omega,\mathbb{R}^N)$ *be a weak solution to system*

$$-D_\alpha A_i^\alpha(x,u,Du) = 0 \quad i = 1,\dots,N .$$

Suppose (2.4), (2.6) *hold. Then the first derivatives of* u *are Hölder-continuous in an open set* Ω_0 *and* $\text{meas}(\Omega \setminus \Omega_0) = 0$.

b) *Let* $u \in H^{1,2} \cap L^\infty(\Omega,\mathbb{R}^N)$ *be a weak solution to* (2.3). *Suppose that* (2.4), (2.5), (2.6) *are satisfied and that* $2aM < \lambda$ *where* $|u| \leq M$. *Then the first derivatives of* u *are Hölder-continuous in an open set* Ω_0 *and* $\text{meas}(\Omega \setminus \Omega_0) = 0$.

We refer to [44] for the proof in principle uses the technique of [38], [43] plus some sharper L^p-estimates for the gradient, and to [44], [36] for a discussion of this result.

Theorem 2.1, however, leaves the problem of the regularity of the minima of the functional (2.1) under "natural" conditions open, except in dimension 2, compare also C. B. Morrey [95]. In fact, if $n = 2$, under the assumption of Theorem 2.1, we have $u \in H^{1,p}_{loc}$ for some $p > 2$; then by the Sobolev imbedding theorem, u is Hölder-continuous. Therefore $u \in H^{2,2}_{loc}$ and, moreover, one can show, see [44], [14], that $u \in H^{2,p}_{loc}$. Hence we can conclude:

a) *Under the assumptions of Theorem 2.1, if* $n = 2$, *then the derivatives of weak solutions are Hölder-continuous everywhere.*

Moreover, since the minimum points $u \in H^{1,2}(\Omega,\mathbb{R}^N)$ of the functional (2.1) are Hölder-continuous in dimension 2, compare Sec. 3, Chap. II, we get:

b) *The minimum points* u *of the functional* (2.1) *are, if* $n = 2$, C^1*-Hölder-continuous* *). *Therefore they are as regular as* F *permits.*

In the rest of this section we want to describe two contributions to the problem of the partial regularity of minima in dimension $n \geq 3$, due to M. Giaquinta, E. Giusti [39], [40] and refering to the special case of quadratic functionals, i.e. multiple integrals of the type

$$(2.7) \qquad \mathcal{F}[u;\Omega] = \int_\Omega A_{ij}^{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^j dx ,$$

*) Of course, provided that the analogues of (2.4), (2.5), (2.6) hold.

where $A^{\alpha\beta}_{ij}(x,u)$ are continuous (for the sake of simplicity we shall assume uniformly continuous), bounded:

$$|A^{\alpha\beta}_{ij}| \leq L \tag{2.8}$$

and satisfy the ellipticity condition

$$A^{\alpha\beta}_{ij}\xi^i_\alpha\xi^j_\beta \geq \lambda|\xi|^2 \quad \forall\xi \quad (\lambda > 0) . \tag{2.9}$$

We have, see [39],

THEOREM 2.2. *Let* $u \in H^{1,2}(\Omega,\mathbb{R}^N)$ *be a minimum point for the functional* $\mathcal{F}$ *in* (2.7); *let* $A^{\alpha\beta}_{ij}$ *be (uniformly) continuous and let* (2.8), (2.9) *hold. Then there exists an open set* $\Omega_0 \subset \Omega$ *such that* $u \in C^{0,\gamma}(\Omega_0,\mathbb{R}^N)$ $\forall\gamma < 1$. *Moreover* $\mathcal{H}^{n-q}(\Omega \setminus \Omega_0) = 0$ *for some* $q > 2$.

P r o o f . Let us sketch the proof. Let $x_0 \in \Omega$, $R < \frac{1}{2}\operatorname{dist}(x_0,\partial\Omega)$ and let v be the solution of the variational problem

$$\begin{cases} \displaystyle\int_\Omega A^{\alpha\beta}_{ij}(x_0,u_{x_0,R})D_\alpha v^i D_\beta v^j dx \to \min , \\ v - u \in H^1_0(B_R(x_0),\mathbb{R}^N) . \end{cases}$$

Then we have, compare Sec. 3, Chap. I,

$$\int_{B_\rho(x_0)} |Dv|^2 dx \leq c_1\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |Dv|^2 dx \quad \forall\rho < R \tag{2.10}$$

and moreover

$$\int_{B_R(x_0)} |Dv|^p dx \leq c_2 \int_{B_R(x_0)} |Du|^p dx \quad (2 < p < \sigma) . \tag{2.11}$$

Let $w = u - v$; we have $w \in H^1_0(B_R,\mathbb{R}^N)$ and

$$c_3 \int_{B_R(x_0)} |Dw|^2 dx \leq \int_{B_R(x_0)} A^{\alpha\beta}_{ij}(x_0,u_{x_0,R})D_\alpha w^i D_\beta w^j dx .$$

On the other hand,

$$\int_{B_R} A^{\alpha\beta}_{ij}(x_0,u_{x_0,R})D_\alpha v^i D_\beta w^j dx = 0$$

and therefore

$$\int_{B_R} A^{\alpha\beta}_{ij}(x_0,u_{x_0,R})D_\alpha w^i D_\beta w^j dx = \int_{B_R} A^{\alpha\beta}_{ij}(x_0,u_R)D_\alpha u^i D_\beta w^j dx =$$

$$= \int_{B_R} [A_{ij}^{\alpha\beta}(x_0,u_R) - A_{ij}^{\alpha\beta}(x,u)] D_\alpha(u^i + v^i) D_\beta w^j dx +$$

$$+ \int_{B_R} [A_{ij}^{\alpha\beta}(x,v) - A_{ij}^{\alpha\beta}(x,u)] D_\alpha v^i D_\beta v^j dx +$$

$$+ \int_{B_R} A_{ij}^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^j dx - \int_{B_R} A_{ij}^{\alpha\beta}(x,v) D_\alpha v^i D_\beta v^j dx .$$

Since u minimizes $\mathcal{F}$ and $u = v$ on ∂B_R, the sum of the last two terms is nonpositive. Therefore

$$\int_{B_R} |Dw|^2 dx \leq c_4 \int_{B_R} (|Du|^2 + |Dv|^2) [\omega^2(R^2 + |u - u_R|^2) + \\ + \omega^2(R^2 + |u - v|^2)] dx$$

where ω is a continuous, bounded, increasing, concave function with $\omega(0) = 0$ and

$$|A_{ij}^{\alpha\beta}(x,u) - A_{ij}^{\alpha\beta}(y,v)| \leq \omega(|x - y|^2 + |u - v|^2) .$$

Now using the L^q-estimate for Du, (2.4), the boundedness of ω, the Poincaré inequality (in the same way as we did in Section 1) and combining (2.10) and (2.12), it is not difficult to deduce the following inequality:

$$\int_{B_\rho(x_0)} (1 + |Du|^2) dx \leq$$

$$\leq c\left[\left(\frac{\rho}{R}\right)^n + \omega\left(R^2 + cR^{2-n} \int_{B_R(x_0)} |Du|^2 dx\right)^{1-\frac{2}{q}}\right] \int_{B_{2R}(x_0)} (1 + |Du|^2) dx$$

for every $\rho < R < \frac{1}{2}\operatorname{dist}(x_0,\partial\Omega)$ and for some $q > 2$. The result then follows as in the proof of Theorem 1.1 Sec. 1.

Q.E.D.

Under the assumption that the coefficients split as

$$(2.13) \qquad A_{ij}^{\alpha\beta}(x,u) = g_{ij}(x,u) G^{\alpha\beta}(x)$$

we have now, see [40], more information on the singular set, and, more precisely,

THEOREM 2.3. The singular set of a bounded minimum u *has Hausdorff dimension not greater than* $n - 3$. *In dimension* $n = 3$ *it consists at most of isolated points.*

We note that, although of particular type, functionals in (2.7) with coefficients given by (2.13) are of interest in the theory of harmonic mappings of Riemannian manifolds. In fact if u is a mapping from a Riemannian manifold M into a Riemannian manifold N (with metric tensors $G_{\alpha\beta}(x)$, $g_{ij}(x)$, respectively) the energy is given in local coordinates by

$$\int g_{ij}(u)G^{\alpha\beta}(x)D_\alpha u^i D_\beta u^j \sqrt{G(x)}\, dx$$

where $(G^{\alpha\beta})=(G_{\alpha\beta})^{-1}$ and $G = \det(G_{\alpha\beta})$.

The method of proof follows closely the one developed in the theory of minimal surfaces and uses the following two lemmas:

LEMMA 2.4. *Let* $A^{(\nu)}(x,z) = A_{ij}^{\alpha\beta(\nu)}(x,z)$ *be a sequence of continuous functions in* $B \times \mathbb{R}^N$ (B *is the unit ball in* $\mathbb{R}^n$) *converging to* $A(x,z)$ *and satisfying the inequalities*

$$(2.14)\qquad \begin{cases} |A^{(\nu)}(x,z)| \leq M, \\ A^{(\nu)}\xi\cdot\xi \geq |\xi|^2 \quad \forall\xi, \\ |A^{(\nu)}(x,z) - A^{(\nu)}(x',z')| \leq \omega\left(|x - x'|^2 + |z - z'|^2\right) \end{cases}$$

where $\omega(t)$ *is a bounded continuous concave function with* $\omega(0) = 0$. *For each* $\nu = 1,2,\dots$ *let* $u^{(\nu)}$ *be a minimum on* B *for the functional*

$$\mathcal{F}^{(\nu)}(u^\nu) = \int A^{(\nu)}(x,u^{(\nu)})Du^{(\nu)}Du^{(\nu)}dx$$

and suppose that $u^{(\nu)} \rightharpoonup v$ *weakly in* $L^2(B,\mathbb{R}^N)$.
Then v *is a minimum in* B *for the functional*

$$\int A(x,u)DuDudx .$$

Moreover, if x_ν *is a singular point for* $u^{(\nu)}$ *and* $x_\nu \to x_0$, *then* x_0 *is a singular point for* v.

The second lemma is a monotonicity result (very similar to the one which appears in the theory of minimal surfaces). And it is for this lemma that the special form (2.13) of the coefficients is needed *). We may assume

$$(2.15)\qquad G^{\alpha\beta}(0) = \delta^{\alpha\beta};$$

moreover, we assume that

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*) Any extension of this lemma to a more general class of coefficients would imply an immediate extension of Theorem 2.3.

$$(2.16) \qquad \int_0^1 \frac{\omega(t^2)}{t}\, dt < +\infty \, .$$

Then the monotonicity result is

LEMMA 2.5. Let u *be a local minimum in* B *for* $\mathcal{F}$ *in* (2.7) *with coefficients* A *given by* (2.13) *and satisfying* (2.14), (2.15), (2.16). *Then for every* ρ , R , $0 < \rho < R < 1$ *we have*

$$(2.17) \qquad \int_{\partial B} |u(Rx) - u(\rho x)|^2 d\, H^{n-1} \leqq \gamma_1 \log \frac{R}{\rho} \,[\Phi(R) - \Phi(\rho)]$$

where

$$(2.18) \qquad \Phi(t) = t^{2-n} \exp\Big(\gamma_2 \int_0^t \frac{\omega(s^2)}{s}\, ds\Big) \int_{B_t} A(x,u)DuDudx \, .$$

We confine ourselves here to proving the second part of Theorem 2.3 and we refer to [40] for the first part and for the proof of Lemmas 2.4, 2.5.

P r o o f . We first observe that the function $\Phi(t)$ in (2.18) is increasing, and therefore tends to a finite limit when $t \to 0$ (since it is also bounded). Suppose now that u has a sequence of singular points x_0 , converging to $x_0 = 0$ and let $R_\nu = 2|x_\nu| <$ < 1 . The function $u^{(\nu)}(x) = u(R_\nu x)$ is a local minimum in B for the functional

$$\mathcal{F}^{(\nu)}[u^\nu;B] = \int_B A^{(\nu)}(x,u^{(\nu)})Du^{(\nu)}Du^{(\nu)}dx \, , \quad A^{(\nu)}(x,z) = A(R_\nu x,z) \, .$$

Moreover, each $u^{(\nu)}$ has a singular point y_ν with $|y_0| = \frac{1}{2}$. Since the $u^{(\nu)}$ are uniformly bounded, we can suppose (passing possibly to a subsequence) that $u^{(\nu)}$ converge weakly in $L^2(B)$ to some function v and that $y_\nu \to y_0$. The coefficients $A^{(\nu)}(x,u)$ are bounded and uniformly continuous in $\bar{B} \times B_L$ (L being a bound for $|u|$) and hence we may apply Lemma 2.4 and conclude that v is a minimum for

$$\mathcal{F}_0[v;B] = \int_B A(0,v)DvDvdx \, .$$

Also from Lemma 2.4 it follows that v has a singular point at y_0 . Let now $0 < \lambda < \mu < 1$, and let us apply inequality (2.17) to $\rho =$ $= \lambda R_\nu$ and $R = \mu R_\nu$. We have

$$\int_{\partial B} |u^{\nu}(\lambda x) - u^{(\nu)}(\mu x)|^2 d\mathcal{H}^{n-1} \leqq \gamma_1 \log \frac{\mu}{\lambda} \left[\Phi(\mu R_{\nu}) - \Phi(\lambda R_{\nu})\right] .$$

If we let $\nu \to \infty$, the right-hand side converges to zero and hence for almost every value of λ and μ we have

$$\int_{\partial B} |v(\lambda x) - v(\mu x)|^2 d\mathcal{H}^{n-1} = 0$$

so that v is homogeneous of degree zero.

We may therefore conclude that the whole segment joining 0 with y_0 is formed by singular points for v. This contradicts Theorem 2.2 and in particular the conclusion that the set of singular points has dimension strictly less than $3 - 2 = 1$.

Q.E.D.

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