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TRANSFER OF BOUNDARY CONDITIONS FOR TWO-DIMENSIONAL PROBLEMS

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1. Formulation of the problem and main results

Let us consider a boundary value problem for a differential equation in a domain Ω. Very often such a situation occurs that the solution of this problem is physically interesting only on a part of Ω and the original domain is in fact introduced only for the reason of simple formulation of boundary conditions. It is, therefore, interesting and maybe also effective to reformulate the original problem only on the domain which we are interested in, in other words, to find the equivalent boundary conditions on the reduced domain. Since the attempt to generalize the idea of the one-dimensional transfer of boundary conditions (see, e.g. [1]) leads to the same problems we will speak also about transferring boundary condition.

Thus, let the domain Ω and its subdomain Ω_2 be given. Put $\Omega_1 = \Omega \setminus \Omega_2$ and suppose that Ω_1 is also a domain. Suppose, further, that the boundaries of Ω, Ω_1, and Ω_2 are all Lipschitzian. Let the original problem be the problem of solving the differential equation

$$Lu = f \quad \text{in} \quad \Omega ,$$

where L is an elliptic differential operator of the second order, with the boundary condition

$$u = g \quad \text{on} \quad \partial\Omega .$$

Let us recall now the weak formulation of this problem. Let there be given a function $w \in H^1(\Omega)$ such that $w = g$ on $\partial\Omega$ and a bilinear form $a(u,v)$ defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ which is continuous and V-elliptic. We have the well-known result: There exists one function $v \in H_0^1(\Omega)$ such that

$$a(v,\varphi) = -a(w,\varphi) + (f,\varphi) \quad \text{for any} \quad \varphi \in H_0^1(\Omega);$$

the weak solution of the original problem is then defined as the function $u = v + w$.

Let us introduce now a problem on Ω_2: Let V be the space of such functions u of $H^1(\Omega_2)$ for which $u = 0$ on $\partial\Omega \cap \partial\Omega_2$. Let us define first the operator of continuation $\Pi : V \to H_0^1(\Omega)$ in such a way that for $\varphi \in V$ the function $\Pi\varphi$ is the solution of the homogeneous problem on Ω_1; thus, it holds

$$(1) \qquad a(\Pi\varphi, \psi) = 0 \quad \text{for any} \quad \psi \in H_0^1(\Omega).$$

The mapping Π is injective and continuous and it holds $\|\varphi\|_V \cong \|\Pi\varphi\|_{H^1(\Omega)}$. Further, let us define a bilinear form A on V by the equation

$$A(u,v) = a(\Pi u, \Pi v).$$

It is obvious that A is continuous and V-elliptic. Again, we can assert that there exists one and only one function $\tilde{v}$ such that

$$A(\tilde{v}, \varphi) = -a(w, \Pi\varphi) + (f, \Pi\varphi) \quad \text{for any} \quad \varphi \in V.$$

The function $\tilde{u} = \tilde{v} + w|_{\Omega_2}$ is then considered to be the weak solution of the boundary value problem defined on Ω_2.

Now we can formulate the following theorem.

Theorem 1. The restriction of v on Ω_2 equals to $\tilde{v}$, i.e., $v|_{\Omega_2} = \tilde{v}$ and, consequently, also $u|_{\Omega_2} = \tilde{u}$.

2. Example

It is very important in actual situation to be able to divide the restricted bilinear form on a differential operator and boundary conditions. This is, generally, a rather difficult problem. In order to indicate the form of equivalent (or transferred) conditions, we will investigate a simple example.

Let there be given the Poisson equation

$$(2) \qquad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = f(r,\varphi)$$

on a circle $K = \{(r,\varphi);\ 0 \leqq r < R,\ 0 \leqq \varphi < 2\pi\}$ with the boundary condition

$$(3) \qquad u(R,\varphi) = 0.$$

Let us try to restrict this problem on the annulus $K_{r_1,R} = \{(r,\varphi);\ r_1 \leqq r \leqq R\}$ and to find on its inner boundary an equivalent boundary condition. To avoid some difficulties accompanying the straightforward approach indicated above we use here the results about the one-dimensional transfer of boundary conditions.

For this reason, let us write the solution of the problem (2),

(3) in the form of a Fourier series

$$(4) \qquad u(r,\varphi) = \frac{1}{2} a_0(r) + \sum_{k=1}^{\infty} \left[a_k(r)\cos k\varphi + b_k(r)\sin k\varphi \right].$$

If we denote the Fourier coefficients of the right-hand term of (2) by c_k and d_k we can write for the coefficients a_k and b_k of u the differential equations

$$(5) \qquad -(ra_k'(r))' + \frac{k^2}{r} a_k(r) = -r\, c_k(r), \quad k = 0,1,\ldots$$

$$-(rb_k'(r))' + \frac{k^2}{r} b_k(r) = -r\, d_k(r), \quad k = 1,2,\ldots$$

with appropriate boundary conditions at $r = 0$ and $r = R$.

Since the classical theory of transfer of boundary conditions is elaborated in detail for a differential equation of the form

$$(6) \qquad -(py')' + qy = f$$

with $1/p$, q, and f being Lebesgue integrable or having a singularity in one of the coefficients $1/p$ and q (see [1], [2]), we must first introduce a theorem covering the case of equations (5), i.e., the case admitting the singularities in both $1/p$ and q.

<u>Theorem 2.</u> Let $p, q\colon [a,b] \to \mathbb{R}$ are such that $p > 0$, $q \geqq 0$ almost everywhere in $[a,b]$, $1/p, q \in \mathcal{L}(a+\varepsilon, b)$ for any $0 < \varepsilon < b-a$ and $1/p, q \notin \mathcal{L}(a,b)$. Further, let η be a nonpositive (absolutely continuous) solution of the differential equation

$$(7) \qquad \eta' = \frac{1}{p}\eta^2 - q.$$

Then any solution of (6) with the Lebesgue integrable right-hand term satisfies the differential equation

$$(8) \qquad \eta(t)y(t) + p(t)y'(t) = \xi(t)$$

for any $t \in [a,b]$ and ξ is defined by the differential equation

$$(9) \qquad \xi' = \frac{1}{p}\eta\xi - f$$

with the initial condition

$$(10) \qquad \xi(a) = 0.$$

Let us note that under the hypothesis of Theorem 2 any solution of (6) satisfies $p(a)y'(a) = 0$ (and, also, $y(a) = 0$). Thus, the equation (8) may be looked at as the result of transferring this condition.

Let us turn back to the problem (2), (3). In our special situation the equation (7) has the form

$$\eta' = \frac{1}{r}\eta^2 - \frac{k^2}{r}$$

and a function satisfying the assumptions of Theorem 2 is, obviously, the function $\eta = -k$. Computing the corresponding function ξ by solving (9) with the initial condition (10) we obtain from (5)

$$a_k(r) - \frac{r}{k} a_k'(r) = -\frac{1}{k}\int_0^r \left(\frac{\varrho}{r}\right)^k \varrho\, c_k(\varrho)\, d\varrho ,$$

$$b_k(r) - \frac{r}{k} b_k'(r) = -\frac{1}{k}\int_0^r \left(\frac{\varrho}{r}\right)^k \varrho\, d_k(\varrho)\, d\varrho , \quad k = 1,2,\dots$$

Substituting these results into (4) we have (after some manipulation) the relation

(11)
$$u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\psi)\, d\psi + $$
$$+ \frac{r}{2\pi}\int_0^{2\pi} \ln 2\left[1-\cos k(\psi-\varphi)\right] u_r(r,\psi)\,d\psi = \gamma(r,\varphi),$$
$$0 \leqq r \leqq R,$$

where

$$\gamma(r,\varphi) =$$
$$= \frac{1}{2\pi}\int_0^r \int_0^{2\pi} \varrho f(\varrho,\psi) \ln\left[1-2\frac{\varrho}{r}\cos(\varphi-\psi) + \left(\frac{\varrho}{r}\right)^2\right] d\varrho\, d\psi .$$

Thus, in equation (11) we have found a form of the equivalent boundary condition on the inner circular boundary of the annulus.

Bibliography

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