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SOME FEATURES OF THE NODAL RECOVERY OF GRADIENTS FROM FINITE ELEMENT APPROXIMATIONS WHICH PRODUCES SUPERCONVERGENCE

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1. Introduction

In the solving of elliptic boundary value problems it is often the case that the quantity of major interest is not the primary unknown in terms of which the problem is formulated, but some secondary quantity which is derived from the primary unknown. Examples of this are flux in potential problems and stress in problems of linear elasticity, which depend on derivatives of the respective primary unknowns of potential and displacement.

In the context of finite element methods for the numerical solution of elliptic problems of this type, these considerations have motivated techniques of **recovery** of gradients from finite element approximations to the (primary) solutions of the problems to be much studied, and for the topic of **superconvergence** to come to prominence in the analysis of errors.

The idea of superconvergence is of course not new and was originally exploited by engineers who noted that for certain elements there exist specific points, **stress points**, at which the gradients of the approximations were superior to those found generally, see e.g. Barlow [3]. For meshes of triangular elements these special points occur at the Gauss points on the element sides and involve automatically tangential derivatives. However, if a recovery procedure is used to produce from these tangential derivatives **recovered gradients** of the finite element approximation at the **element nodes**, then the rate of convergence of the recovered gradients is higher than that found normally. This is the phenomenon of **gradient superconvergence.** It is possible, by interpolating to these recovered nodal gradients with vectors, each component of which is a function from the original finite element space, to obtain a **recovered gradient function,** which is itself superconvergent. A vast literature on gradient superconvergence now exists, and many of the publications in this field are listed in Křížek and Neittaanmäki [8].

There is, in the treatment of (nodal) recovered gradient functions based on piecewise linear and piecewise quadratic finite element approximations, a common thread of analysis which produces the superconvergent error analysis. In this present short paper we seek to describe the main features of this thread.

2. Boundary Value Problems and Finite Element Discretisation

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with boundary $\partial\Omega$. We consider the mixed boundary value problem in which the solution $u(\mathbf{x})$ satisfies

$$\mathcal{L}u \equiv - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} \left(a_{ij}(\mathbf{x}) \frac{\partial u}{\partial x_j} \right) = f(\mathbf{x}) \ , \ \mathbf{x} \in \Omega \ , \tag{2.1}$$

together with Dirichlet boundary conditions $g^D(\mathbf{x})$, $\mathbf{x} \in \partial\Omega^D$ and Neumann boundary conditions $g^N(\mathbf{x})$, $\mathbf{x} \in \partial\Omega^N$, where $\partial\Omega \equiv \partial\Omega^D \cup \partial\Omega^N$ and

$$a_{ij}(\mathbf{x}) = a_{ji}(\mathbf{x}) \ , \ \sum_{i,j=1}^{2} a_{ij}(\mathbf{x})\xi_i\xi_j \geq c \sum_{i=1}^{2} \xi_i^2 \ , \ c > 0 \ . \tag{2.2}$$

This formulation is written for the scalar case with a single differential equation in (2.1), but it can also cover the case where $\mathbf{u}(\mathbf{x})$ is a vector and (2.1) is a system of equations.

The weak form of (2.1) together with the boundary conditions is that in which we seek $u \in H_E^1$ such that

$$a(u,v) = (f,v) + (g^N,v)_{\partial\Omega^N} \quad \forall \ v \in H \ , \tag{2.3}$$

where

$$a(u,v) \equiv \sum_{i,j=1}^{2} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, d\mathbf{x} \ ,$$

$$H_E^1 \equiv \left\{ v \ : \ v \in H^1(\Omega), \ v = g^D \text{ on } \partial\Omega^D \right\} \ , \tag{2.4}$$

$$H \equiv \left\{ v \ : \ v \in H^1(\Omega), \ v = 0 \text{ on } \partial\Omega^D \right\} \ . \tag{2.5}$$

When Ω is covered by a quasiuniform triangular mesh, a Galerkin technique is applied to approximate (2.3) using globally continuous piecewise polynomial functions over the mesh. Thus we define the finite dimensional subspace $S^h(\Omega) \subset H^1(\Omega)$ and the Galerkin problem, find $u_h \in S_E^h$ such that

$$a(u_h,v_h) = (f,v_h) + (g^N,v_h)_{\partial\Omega^N} \quad \forall \ v_h \in H^h \ , \tag{2.6}$$

where

$$S_E^h \equiv \left\{ v_h \ : \ v_h \in S^h(\Omega), \ v_h = g_I^D \text{ on } \partial\Omega^D \right\} \ , \tag{2.7}$$

$$H^h \equiv \left\{ v_h \ : \ v_h \in S^h(\Omega), \ v_h = 0 \text{ on } \partial\Omega^D \right\} \ , \tag{2.8}$$

where g_I^D is the interpolant from the Galerkin space to g^D.

3. Superconvergence of Recovered Gradients

The piecewise polynomial functions in $S^h(\Omega)$ have discontinuous gradients across element edges. However, using **averaging schemes** which utilise the fact that the tangential derivatives at the element edge Gauss points of the interpolant $u_I \in S_E^h$ to u are exact for u a polynomial of one order higher than the polynomials in $S^h(\Omega)$, gradients can be **recovered** at the nodes and to each component of these can be fitted a polynomial from $S^h(\Omega)$, thus defining the recovered gradient function $\nabla^R v_h \in (S^h(\Omega))^2$ for any $v_h \in S^h(\Omega)$. This type of recovery has the property that $\nabla^R u_I$ is exact for a regular triangular mesh, when again u is a polynomial of one order higher than that of the polynomials in $S^h(\Omega)$.

In deriving the error estimates for u_h, the global regularity of u is paramount as u_h is a **global** approximation, the calculation of which is done over the whole of Ω. As most problems of type (2.1) which are encountered in practice have solutions with low global regularity, $u \notin H^2(\Omega)$, in the derivation of error estimates for the recovered gradient functions it is necessary to work on subdomains where the **local regularity** is higher than that found globally. We thus define subdomains $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset \Omega$ such that u has suitable regularity in Ω_2; Ω_2 is also constructed so that it can be meshed uniformly.

In this case using the Bramble-Hilbert Lemma, [4], and the Sobolev embedding theorem, see Adams [1], it can be shown that

$$\left|\nabla u - \nabla^R u_I\right|_{0,p,\Omega_0} \le Ch^{\alpha+1}\left|u\right|_{\alpha+2,p,\Omega_0} \tag{3.1}$$

for piecewise polynomials of degree α, $\alpha = 1$ or 2, and $p = 2$ or ∞, see [5]-[7]. It follows immediately from (3.1), using the triangle inequality that

$$\left|\nabla u - \nabla^R u_h\right|_{0,p,\Omega_0} \le C\left\{\left|u_I - u_h\right|_{1,p,\Omega_0} + h^{\alpha+1}\left|u\right|_{\alpha+2,p,\Omega_0}\right\} \tag{3.2}$$

so that in seeking estimates on $\left|\nabla u-\nabla^R u_h\right|_{0,p,\Omega_0}$ we have to estimate $\left|u_I-u_h\right|_{1,p,\Omega_0}$.

This is achieved via the use of the property that $a(u-u_I,v_h) = 0$, $v_h \in S_0^h(\Omega_1)$, for u a polynomial of one order higher than that of the space $S^h(\Omega)$. Again using the Bramble-Hilbert Lemma and the Sobolev embedding theorem, it can be shown that

$$\left|a(u - u_I,v_h)\right| \le Ch^{\alpha+1}\left\|u\right\|_{\alpha+2,p,\Omega_1}\left|v_h\right|_{1,p/(p-1),\Omega_1} \tag{3.3}$$

which is the fundamental inequality in the derivation of the L_2 estimate

$$\left\|u_I - u_h\right\|_{1,\Omega_0} \le C\left\{\left|u_I - u_h\right|_{0,\Omega_1} + h^{\alpha+1}\left\|u\right\|_{\alpha+2,\Omega_1}\right\} \tag{3.4}$$

see [10].

For the L_∞ case, we find that

$$|u_I - u_h|_{1,\infty,\Omega_0} \le Ch^{-\varepsilon}\left\{\|u_I - u_h\|_{1,\Omega_1} + h^{\alpha+1}\|u\|_{\alpha+2,\infty,\Omega_1}\right\} \tag{3.5}$$

see [6], [7] and [10] and using the result (3.4) on Ω_1 we obtain the estimate

$$|u_I - u_h|_{1,\infty,\Omega_0} \le Ch^{-\varepsilon}\left\{|u_I - u_h|_{0,\Omega_2} + h^{\alpha+1}\|u\|_{\alpha+2,\infty,\Omega_2}\right\}. \tag{3.6}$$

In (3.4) and (3.6) the controlling term is $|u_I-u_h|_{0,\Omega_i}$, i = 1,2 which is itself controlled by $\|u-u_h\|_{-q,\Omega}$, for any $q \ge 0$, see Nitsche and Schatz [9]. In the present case optimal estimates are obtained by taking $q = \alpha-1$ and the bound on $\|u-u_h\|_{-q,\Omega}$ naturally depends on the global regularity of u. Following this approach the best estimate for $|u_I-u_h|_{0,\Omega_i}$ is $O(h^{\alpha+1})$ when $u \in H^{\alpha+1}(\Omega)$, thus giving in (3.2) a **superconvergent** $(O(h^{\alpha+1}))$ estimate.

It has to be said that the estimates for the cases when $u \notin H^{\alpha+1}(\Omega)$ are of lower order than those above but can still be of higher order than the corresponding estimates obtained without recovery, see [7] and [10].

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