Andrzej Szulkin

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HAMILTONIAN SYSTEMS WITH PERIODIC NONLINEARITIES

SZULKIN A., STOCKHOLM, Sweden

1. Abstract result

Let $M = E \times T^k$, where E is a real Hilbert space with an inner product $\langle , \rangle$ and T^k is the k-torus. We will be concerned with a class of functionals $\Phi \in C^1(M, \mathbf{R})$ of the form $\Phi(x, v) = \frac{1}{2}\langle Lx, x\rangle - \psi(x, v)$, where L and ψ satisfy the following hypotheses:

(i) $L : E \to E$ is a bounded, linear and selfadjoint operator to which there corresponds an orthogonal decomposition $E = E^+ \oplus E^-$ into L-invariant subspaces such that $\langle Lx, x\rangle$ is positive definite on E^+ and negative definite on E^-.

(ii) The gradient of ψ, denoted $\nabla\psi$, is compact (in the sense that it maps bounded sets into compact ones).

(iii) $\nabla\psi(M)$ is bounded, i.e., there exists a constant C such that $\|\nabla\psi(x, v)\| \leq C$ $\forall(x, v) \in M$.

Recall that a differentiable functional Φ is said to satisfy *the Palais-Smale condition* (PS) if each sequence (z_n) such that $\Phi(z_n)$ is bounded and $\nabla\Phi(z_n) \to 0$ possesses a convergent subsequence. Note that it is an easy consequence of hypotheses (i)-(iii) that our functional Φ satisfies (PS). Recall also that a set $A \subset M$ is said to be *of category k in M* (denoted $\mathrm{cat}_M(A) = k$) if k is the smallest integer such that A can be covered by k closed sets which are contractible to a point in M. If $A = M$, we will write $\mathrm{cat}\,(M) = \mathrm{cat}_M(M)$. Properties of the category may be found e.g. in [4, 9, 11, 12].

Theorem 1. *Suppose that the functional Φ satisfies hypotheses (i)-(iii). Then Φ possesses at least $k + 1$ distinct critical points.*

If $E^- = \{0\}$, Φ is bounded below. So in this case Theorem 1 follows from a result by Palais [9] (because $\mathrm{cat}\,(M) = \mathrm{cat}\,(T^k) = k + 1$ [11]). For finite dimensional E^- Theorem 1 was first proved by Chang [1, 2]. A different argument, which we sketch below, has been proposed by Fournier and Willem [8]. The proof for $\dim E^- \leq \infty$ is due to the author [13].

Remark 1. (i) The conclusion of Theorem 1 remains valid if T^k is replaced by a compact manifold V^d such that $\mathrm{cuplength}\,(V^d) = k$ [1, 2, 8, 13].

(ii) The conclusion remains valid if L has a finite dimensional kernel E^0 and $\psi(x^0, v) \to -\infty$ (or $\psi(x^0, v) \to +\infty$) as $\|x^0\| \to \infty$, $x^0 \in E^0$ [1, 2, 13].

If $E^- = \{0\}$, the proof of Theorem 1 is easy to obtain directly (without invoking Palais' theorem [9]) by using a standard argument [4, 12] based on the minimax characterization of critical values of Φ as

$$b_j = \inf_{\mathrm{cat}_M(A) \geq j} \sup_{(x,v)\in A} \Phi(x, v), \qquad 1 \leq j \leq k + 1$$

and the deformation lemma. Note that $\text{cat}_M(\{0\} \times T^k) = \text{cat}\,(T^k) = k+1$, so all b_j are well defined and finite. If $\dim E^- > 0$, the functional Φ becomes unbounded below and the values b_j defined above are equal to $-\infty$. It is therefore necessary to employ a different argument.

We will introduce two notions of relative category and indicate how they enter into the proof of Theorem 1. A continuous mapping $\eta : [0,1] \times M \to M$ such that $\eta(0,z) = z \; \forall z \in M$ will be called *a deformation of* M. Let A, N be two closed subsets of M. The set A is said to be *of category* k *in* M *relative to* N, denoted $\text{cat}_{M,N}(A) = k$, if k is the smallest integer such that $A = A_0 \cup A_1 \cup ... \cup A_k$, where all A_j are closed, all A_j with $j \geq 1$ are contractible in M, and there exists a deformation η_0 of M satisfying $\eta_0(1, A_0) \subset N$ and $\eta_0(t, N) \subset N$ $\forall t \in [0,1]$. If such k does not exist, $\text{cat}_{M,N}(A) = \infty$.

The above notion of relative category is due to Fournier and Willem [7, 8]. Our definition is a slight modification of theirs and may be found in [13].

Let $\mathcal{D}$ be a given class of deformations of M such that the trivial deformation $\eta(t,z) = z$ $\forall (t,z)$ is in $\mathcal{D}$, and whenever η_1, η_2 are in $\mathcal{D}$, so is the deformation obtained by letting η_1 be followed by η_2. The set A is said to be *of category* k *in* M *relative to* N *and* $\mathcal{D}$, denoted $\text{cat}^{\mathcal{D}}_{M,N}(A) = k$, if k is the smallest integer such that $A = A_0 \cup A_1 \cup ... \cup A_k$, where A_j and η_0 are as in the preceding definition and $\eta_0 \in \mathcal{D}$. If such k does not exist, $\text{cat}^{\mathcal{D}}_{M,N}(A) = \infty$.

The above definition may be found in [13].

Suppose $\dim E^- < \infty$. Denote $\Phi_a = \{z \in M : \Phi(z) \leq a\}$, $\Gamma_j = \{A \subset M :$ *A is closed and* $\text{cat}_{M,\Phi_a}(A) \geq j\}$ and

$$c_j = \inf_{A \in \Gamma_j} \sup_{z \in A} \Phi(z), \qquad 1 \leq j \leq k+1.$$

Then $c_j \geq a$ (because $\text{cat}_{M,\Phi_a}(A) = 0$ whenever $A \subset \Phi_a$). Denote $M_R = \{(x,v) \in M : \|x\| \leq R\}$. It can be shown that if a is sufficiently small and R sufficiently large, then

$$(*) \qquad \text{cat}_{M,\Phi_a}(M_R) \geq \text{cat}_{B \times T^k, S \times T^k}(B \times T^k) \geq k+1,$$

where $B = \{x \in E^- : \|x\| \leq R\}$ and $S = \partial B$ (more precisely, $(*)$ holds for a certain modified functional $\widetilde{\Phi}$ which has the same critical points as Φ, cf. [8]). So $\Gamma_j \neq \emptyset$ for $1 \leq j \leq k+1$ and the numbers c_j are well defined and finite. Now a standard argument shows that Φ has at least $k+1$ critical points.

If $\dim E^- = \infty$, then B is contractible to S, so $\text{cat}_{B \times T^k, S \times T^k}(B \times T^k) = 0$ and the above argument fails (because all Γ_j may be empty). Therefore a further modification is needed in order to prove Theorem 1 in the most general case. Let $\mathcal{D}$ be the class of deformations which consists, roughly speaking, of solutions for $0 \leq t \leq 1$ of initial value problems of the form

$$\frac{d\eta}{dt} = -\omega(\eta)V(\eta), \qquad \eta(0,x,v) = (x,v),$$

where V is a certain pseudo-gradient vector field for Φ, $\omega : M \to [0,1]$ is locally Lipschitz continuous, and $\omega = 0$ in a neighbourhood of the set of critical points of Φ (the class $\mathcal{D}$ is in fact somewhat larger, see [13]). Then it can be shown that for R sufficiently large and a sufficiently small,

$$\text{cat}^{\mathcal{D}}_{M,\Phi_a}(M_R) \geq \text{cat}_{B \times T^k, S \times T^k}(B \times T^k) \geq k+1,$$

where $B = \{x \in \widetilde{E}^- : \|x\| \leq R\}$ is a ball in a *finite dimensional* subspace $\widetilde{E}^-$ of E^-. So replacing cat by $\text{cat}^{\mathcal{D}}$ in the definition of c_j we obtain the conclusion.

2. Hamiltonian systems

Consider the Hamiltonian system of differential equations

$$(HS) \qquad \dot{z} = JH_z(z,t),$$

where $z = (p,q) \in \mathbf{R}^N \times \mathbf{R}^N$, $H \in C^1(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ and

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

is the standard symplectic matrix. Assume H is 2π-periodic in t. Let $H^{1/2}(S^1, \mathbf{R}^{2N}) \equiv H^{1/2}$ be the Sobolev space of 2π-periodic $\mathbf{R}^{2N}$-valued functions

$$z = \sum_{k \in \mathbf{Z}} c_k e^{ikt}, \qquad \textit{where } c_k \in \mathbf{C}^{2N} \textit{ and } c_{-k} = \bar{c}_k,$$

such that

$$\sum_{k \in \mathbf{Z}} (1 + |k|)|c_k|^2 < \infty.$$

Define

$$\widetilde{\Phi}(z) = \frac{1}{2} \int_0^{2\pi} (-J\dot{z} \cdot z)\, dt - \int_0^{2\pi} H(z,t)\, dt.$$

It is known [10] that $\widetilde{\Phi} \in C^1(H^{1/2}, \mathbf{R})$, critical points of $\widetilde{\Phi}$ correspond to 2π-periodic solutions of (HS) and the gradient of $\int_0^{2\pi} H(z,t)\, dt$ is compact.

Suppose H is 2π-periodic in all variables. Note that if z is a 2π-periodic solution of (HS), so are all functions $\hat{z}$ such that $\hat{z} - z \in 2\pi\, \mathbf{Z}^{2N}$ (by periodicity of H). Two solutions $\hat{z}$ and z will be called *geometrically distinct* if $\hat{z} - z \notin 2\pi\, \mathbf{Z}^{2N}$. Let $H^{1/2} = H^+ \oplus H^0 \oplus H^-$ be the decomposition corresponding to the positive, zero and negative part of the spectrum of $-J\dot{z}$. Denote $E = H^+ \oplus H^-$. Then to each $z \in H^{1/2}$ there corresponds a unique pair (x, v) such that $x \in E$ and v is the mean value of z modulo 2π. Clearly, v may be considered as an element of the torus T^{2N}, so $(x,v) \in E \times T^{2N} \equiv M$. Define

$$\Phi(x,v) = \frac{1}{2} \int_0^{2\pi} (-J\dot{x} \cdot x)\, dt - \int_0^{2\pi} H(z,t)\, dt.$$

It is easy to see that $\Phi \in C^1(M, \mathbf{R})$, critical points of Φ correspond to 2π-periodic solutions of (HS) and, unlike for $\widetilde{\Phi}$, distinct critical points correspond to solutions which are geometrically distinct. Since Φ satisfies hypotheses (i)-(iii) of Section 1, we have the following

Theorem 2 [13]. *Suppose that H is 2π-periodic in all variables. Then (HS) possesses at least $2N+1$ geometrically distinct 2π-periodic solutions.*

This result, which is known to imply an affirmative answer to one of Arnold's conjectures, has been first time proved by Conley and Zehnder for $H \in C^2(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ [5]. A different proof, also for $H \in C^2$, may be found in Chang [1, 2].

Using Remark 1 and a variant of the above argument, one can easily prove

Theorem 3 [13]. *Suppose that $H \in C^1(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ is 2π-periodic in q and t, $H(z,t) = \frac{1}{2}Bp\cdot p + G(z,t)$, where B is a symmetric $N \times N$-matrix and the derivative G_z is bounded. If the null space of B, $N(B)$, is nontrivial, suppose also that $G(p,q,t) \to +\infty$ (or $G(p,q,t) \to -\infty$) uniformly in t as $|p| \to \infty$, $p \in N(B)$. Then (HS) possesses at least $N+1$ geometrically distinct 2π-periodic solutions.*

Similar results, for nonlinearities of class C^2, may be found in [3, 6].

Remark 2. The proofs in [1-3, 5, 6] employ a finite dimensional reduction which requires that $H \in C^2$ (and H_{zz} be bounded). On the other hand, our Theorem 1 allows one to avoid this reduction and therefore have $H \in C^1$.

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