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PRESSURE JUMPS IN THE DAM PROBLEMS

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The phenomenon modelled is the flow of water through earth dams. Let the region occupied by the dam be denoted by Ω, p the water pressure, Θ the porosity of the dam, s the (relative) water saturation, J the water flux, ρ the water density and e_z the unit vector in the opposite direction of gravity. Then the flow is governed by the continuity equation.

$$\Theta\, \partial_t s = -\nabla \cdot J$$

and Darcy's law

$$J \sim (\nabla p + \rho e_z)$$

where one often assumes the constant to depend on s and x, e.g.

$$J = -a(x)\, K(s)\, (\nabla p + \rho e_z)$$

with a bounded away from zero, K monotone and continuous with $K(1) = 1$. For the relationship between s and p there are two models,
a small scale model (saturated, unsaturated flow) where $s = s(p)$ is a continuous monotone function and a large scale (free boundary) model where

$$0 \le s \le 1,\ p \ge 0,\ (1 - s)\cdot p \equiv 0\ .$$

The boundary conditions on the impervious part of the boundary are (ν denoting the outer normal)

$$J \cdot \nu = 0 \quad \text{in } \Gamma_N$$

and on the pervious part Γ_D

$$p = p_D > 0 \quad \text{on the boundary to water}$$
$$p \le p_D = 0 \quad \text{and } J \cdot \nu \ge 0 \text{ on the boundary to air}$$

(and for the small scale model $p < p_D \Rightarrow J \cdot \nu = 0$) the so called outflow condition.

Taken together these equations have the weak formulation (for an initial water saturation s_o)

$$(1)\qquad -\int_o^T \int_\Omega \Theta\, s\, \partial_t \zeta - \int_\Omega \Theta\, s_o \rho(0) + \int_o^T \int_\Omega (a\, K(s)\, \nabla p\, \nabla\zeta + a\rho\, K(s)\, \partial_z \zeta) \le 0$$

for all $\zeta \in H_2^1((0,T) \times \Omega)$ with $\xi(T,\cdot) \equiv 0$, $\zeta \geq 0$ on Γ_D and $\zeta\ p_D \equiv 0$ on Γ_D where $(p_+ - p_D) \in \{u \in L_2(0,T,H_2^1(\Omega)) \mid u_{|\Gamma_0} \equiv 0\}$ and for the large scale model

(2) $$0 \leq s \leq 1\ ,\ p \geq 0\ ,\ (1-s)\ p \equiv 0\ .$$

Since in this free boundary problem there is no time derivative for p, it is natural to ask whether p is a continuous function in time. That this is not the case follows from the following counter example:

Let Ω be a cylindrical domain

$$\Omega = \Omega' \times (z_o,0) \quad \text{with} \quad z_o < 0$$
$$\Gamma_N = \Omega' \times \{z_o\} \cup \partial\Omega' \times (z_o,0)$$
$$p = p_o = \text{const} > 0 \quad \text{on} \quad \Gamma_D = \{0\} \times \Omega'$$
$$s_o \equiv 0\ ,\ \rho = 0\ ,\ \Theta = a = 1\ .$$

Then the solution can be given explicitely

$$s = \begin{cases} 1 & \text{for } z\ \sqrt{2p_o t} \\ 0 & \text{otherwise} \end{cases} \qquad p = \begin{cases} p_o\left(1 + \frac{z}{\sqrt{2p_o t}}\right)_+ & \text{f. } z_o < -\sqrt{2p_o t} \\ p_o & \text{f. } z_o > -\sqrt{2p_o t} \end{cases} .$$

So there occurs a jump in pressure allover the domain at the time when the free boundary hits the impervious boundary.

So the possible conjecture left is that pressure jumps up but not down. And this can be proved under additional assumptions. I shall state the theorem with more regularity assumptions than are actually necessary to save some writing. The result has been achieved in collaboration with G. Gilardi, Pavia.

Theorem (Gilardi-L.)
Suppose $\partial\Omega \in C^2$; $\Gamma_D \subset \partial\Omega$; $\Gamma_N = \partial\Omega \setminus \Gamma_D$; $\Gamma_N, \Gamma_D \in C^2$; $a \in C^2(\bar\Omega)$, $\Theta \in C^1(\bar\Omega)$, $p_D \in C^{1,\alpha}((0,T) \times \Omega$,
and suppose

$$\partial_z a \geq 0\ ,\ \nu_z \leq \Omega \quad \text{on} \quad \Gamma_N\ , \text{ where } \nu \text{ is the outer normal then:}$$

$$\partial_+ p \in \mathcal{M}((0,T) \times \Omega)\ ,\ \partial_+ p_- \in L_\infty((0,T) \times \Omega)\ .$$

Remark the assumptions on $\partial_z a$ and $\nu_z|_{\Gamma_N}$ are crucial for the method of proof. If $\partial_z a \geq 0$ it is a conjecture that s is a characteristic function.

Heuristically then the argument goes as follows (for simplicity take $a = \Theta = \rho \equiv 1$) $\partial_t p$ is harmonic in $s \equiv 1$, with Neumann data on Γ_N and C^α Dirichlet data on Γ_D. On the free boundary $\Gamma_f = \partial\{s = 1\} \cap \Omega$ one has

$$\partial_t p = |\nabla p| \cdot \partial_t \Gamma_f = |\nabla p|(|\nabla p| + \nu_f \cdot e_z) \geq -\frac{1}{4}(\nu_f \cdot e_z)^2 \geq -\frac{1}{4}.$$

So there is a bound on $\partial_t p$.

Sketch of proof.

To make this precise let $\partial_t^h p = (p(t) - \frac{1}{h}\int_{t-h}^{t} p(\tau)d\tau)\frac{1}{2h}$.

One has to show that

$\partial_t^h p < C$, C large negative implies

$\partial_t^h p$ superharmonic,

It is sufficient to show, since p is subharmonic and harmonic whenever $s \equiv 1$ in an open set,

$$\frac{1}{h^2}\int_{t-h}^{t} p(\tau,x)\, d\tau > |C| \quad \text{implies} \quad s(t,\cdot) \equiv 1 \text{ in } B_h(x).$$

Now take a ball falling with characteristic speed

$B = B_{2h}(x + (t - \tau)e_z)$, one calculates

$$\partial_\tau \int_B (1 - s) = - \int_B \Delta p.$$

Now for subharmonic nonnegative p on has

Lemma: Let $u \geq 0$, $\Delta u \geq 0$ in B_ρ than there exists a constant $C(n)$ with

$$\int \Delta u > \frac{C(n)}{\rho^{n+1}} \int_{B_\rho} u[\text{meas}(\{u = 0\} \cap B_\rho)]^{1-1/n}.$$

The Lemma implies

$$\begin{aligned}\partial_\tau \int_B (1 - s) &\leq -\frac{C(n)}{h^{n+1}} \int_B p\, [\text{meas}\{p = 0\}]^{1-1/n} \\ &\leq -\frac{C(n)}{h^{n+1}} \int_B p\, [\int_B (1 - s)]^{1-1/n} \\ &\leq -\frac{\bar{c}}{h} p(\tau,x)\, [\int_B (1 - s)]^{1-1/n}\end{aligned}$$

since $[\int_B (1 - s)(t - h)]^{\frac{1}{n}} < \omega_n^{\frac{1}{n}} 2h$

one sees that for $\int_{t-h}^{t} p(\tau,x) \geq \frac{2}{n}\omega_n^{\frac{1}{n}} \bar{c}^{-1} h^2$

$\int_B (1 - s)(t) = 0$. Which implies that $\max\left(\partial_t^h p, -\frac{\omega_n^{1/n}}{nc}\right)$ is superharmonic in $\Omega_h = \{x | d(x, (\Omega) > 2h\}$. Dealing in a similar way with the boundary strip one gets the result.

Appendix, proof of the lemma (w.l.og. $\rho = 1$):

Let u_h be the harmonic continuation of the boundary value of u. One has for any $\beta > 0$ the following capacity estimate

$$\int \Delta u = \int \Delta(u - u_h) \geq c\beta[\text{meas}(\{u - u_h > \beta\})]^{1-2/n}$$

So if $⨍_{B_{\frac{1}{2}}} u < \frac{1}{2} ⨍_{B_1} u$, nothing has to be proved. Otherwise define

$\Gamma := [\text{meas}(\{u = 0\})]^{\frac{1}{n}}$, $\Gamma < \frac{1}{2}$ w.l.o.g. 1^{st} case : $\text{meas}(\{u=0\} \cap B_{1-\Gamma}) > \frac{1}{2}\Gamma^n$

$u_h(x) > c ⨍ u\,(1 - |x|)$ by the Hopf maximum principle. et $\beta = c ⨍ u\,\Gamma$ in the capacity estimate to prove the lemma. 2^{nd} case : $\text{meas}(\{u=0\} \cap B_{1-\Gamma}) < \frac{1}{2}\Gamma^n$, then there is $\Gamma' > 1-\Gamma$ with $H^{n-1}(\partial B_{\Gamma'} \cap \{u = 0\}) > \frac{1}{2}\Gamma^{n-1}$.

Define $\bar{u}_h$ by, $\bar{u}_h = u$ in $\partial B_1 \cup \partial B_{\Gamma'}$, $\Delta \bar{u}_h = u$ in $B_1 \setminus (\partial B_{\Gamma'})$, $\Delta \bar{u}_h \geq 0$ in B, and $\int \Delta \bar{u}_h < \int \Delta u$, as $\bar{u}_h > u$. But by Hopf's maximum principle

$$\int \Delta \bar{u}_h \geq \int_{\{u=0\} \cap \partial B_{\Gamma'}} \partial_\nu \bar{u}_h \geq \frac{1}{2}\Gamma^{n-1} c ⨍_{B_{\frac{1}{2}}} u \geq c \int u\,\Gamma^{n-1}.$$