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ON THE APPROXIMATION OF SOLENOIDAL AND POTENTIAL VECTOR FIELDS

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1. Introduction

In the present paper we proceed with further treatment of the questions considered previously in [1-4]. We state here some of our recent results which turn out to be essentially more general than those stated in our papers [1-4].

The key question underlying considerations of [1-4] is just as simple as is complicated the answer to it. And the question is whether the two functional spaces $\overset{\circ}{J}{}^{1}_{p}(\Omega)$ and $\hat{J}^{1}_{p}(\Omega)$ of solenoidal vector fields coincide or not. Here $\overset{\circ}{J}{}^{1}_{p}(\Omega)$ is a closure of the subspace $\overset{\circ}{J}{}^{\infty}(\Omega) = \{\vec{v}(x): \vec{v}(x) \in \overset{\circ}{C}{}^{\infty}(\Omega),\ \operatorname{div}\vec{v}(x) = 0\}$ in Sobolev space $\overset{\circ}{W}{}^{1}_{p}(\Omega)$ of vector fields $\vec{v}(x) = (v_1,\dots,v_n)$, $x = (x_1,\dots,x_n) \in \Omega$ and Ω is a domain (open connected set) in R^n, $n \geq 2$. The norm in $\overset{\circ}{W}{}^{1}_{p}(\Omega)$ is introduced here by the equality

$$\|\vec{v}\|_{\overset{\circ}{W}{}^{1}_{p}(\Omega)} = \sum_{|\alpha| \leq l} \|D^{\alpha}_{x}\vec{v}(x)\|_{L_p(\Omega)},$$

where $l \geq 1$ is integer and $1 < p < \infty$. Finally $\hat{J}^{1}_{p}(\Omega) = \{\vec{v}(x): \vec{v}(x) \in \overset{\circ}{W}{}^{1}_{p}(\Omega),\ \operatorname{div}\vec{v}(x) = 0\}$.

It is clear that $\overset{\circ}{J}{}^{1}_{p}(\Omega) \subset \hat{J}^{1}_{p}(\Omega)$, both being the subspaces of $\overset{\circ}{W}{}^{1}_{p}(\Omega)$. The problem is whether the inverse inclusion takes place or not. In other words, this is the problem of approximation of $\overset{\circ}{W}{}^1_p(\Omega)$ solenoidal vector fields in the norm of $\overset{\circ}{W}{}^1_p(\Omega)$ by $\overset{\circ}{J}{}^{\infty}(\Omega)$ vector fields.

The question of coincidence of $\overset{\circ}{J}{}^1_p(\Omega)$ and $\hat{J}^1_p(\Omega)$ was posed by J. G. Heywood in [5] . Before the appearance of [5] in 1976, the coincidence of the two spaces seemed to be out of question. And the only attempt to prove the coincidence, namely the identity $\overset{\circ}{J}{}^1_2(\Omega) = \hat{J}^1_2(\Omega)$, was due to J. - L. Lions (see book [6] , p. 67 and p. 100). In [6] the question of coincidence of $\overset{\circ}{J}{}^1_2(\Omega)$ and $\hat{J}^1_2(\Omega)$ was reduced to that of their annihilators, which consist of $\overset{=}{W}{}^1_2(\Omega)$ potential vector fields. However, at least for unbounded domains Ω , identity of the annihilatora was left, in fact, without a proof, since the only argument was just a reference to the footnote on p. 320 of [7] , which for unbounded Ω is inadequate (for details and complete treatment, using such an approach, see [8]) .

The first counter example for $\overset{\circ}{J}{}^1_2(\Omega)$ and $\hat{J}^1_2(\Omega)$, showing that the spaces may not coincide if Ω is unbounded, was constructedby J.G. Heywood in [5] . Further counter examples, this time for $\overset{\circ}{J}{}^1_p(\Omega)$ and $\hat{J}^1_p(\Omega)$ with unbounded $\Omega = x=(x',x_n):|x'|<f(x_n)$, $x_n \in R^1$, $f(x_n) \geq \delta > 0$, were constructed by the authors in [1] (see also [3]), where $1 \geq 1$ and $n \geq 2$ could be any integer numbers and, what is important, $1<p<\infty$.

There are reasons to believe that regularity of $\partial\Omega$ has nothing to do with the question of coincidence of $\overset{\circ}{J}{}^1_p(\Omega)$ and $\hat{J}^1_p(\Omega)$ if only one defines $\overset{\circ}{W}{}^1_p(\Omega)$ as closure in Sobolev space $W^1_p(\Omega)$ of its subspace $\overset{\circ}{C}{}^{\infty}(\Omega)$. One of the reasons is the result obtained by C. J. Amick, who proved the identity $\overset{\circ}{J}{}^1_2(\Omega) = \hat{J}^1_2(\Omega)$ for bounded domains $\Omega \subset R^2$ requiring only finite connectedness of $\partial\Omega$ (private communication by C. J. Amick).

The main difficulty one encounters in consideration of the

$\overset{\circ}{W}{}^1_p(\Omega)$ subspaces $J^1_p(\Omega)$ and $\hat{J}^1_p(\Omega)$ is the classification of domains, according to which one could determine dim $\hat{J}^1_p(\Omega)$ / / $\overset{\circ}{J}{}^1_p(\Omega)$, i. e. the dimension of a quotient space $\hat{J}^1_p(\Omega)$ / $J^1_p(\Omega)$. So far there has been introduced no such general classification.

In publications that followed [5] , O. A. Ladyzhenskaya, V. A. Solonnikov, K. J. Piletskas and the authors considered domains with N outlets at infinity, i. e. domains Ω that can be represented in a form $\Omega = \bigcup_{j=0}^{N} \Omega_j$, where Ω_0 is a bounded domain and each of the domains Ω_j , $j=1,\dots,N$, is unbounded and called an outlet at infinity if $\Omega_j \cap \Omega_k = \emptyset$ for $j \neq k$. Domains with N outlets at infinity were introduced in [9,10] by O. A. Ladyzhenskaya and V. A. Solonnikov, who treated the case $p=2$, $l=1$ and $n=2$ or 3 . Further treatment, including the cases $1<p<\infty$, $l\geq 1$ and $n\geq 2$, is due to the authors and is contained in [1-4,8,12] (see also [17] that followed [2]).

At present time it seems that a natural way to approach the more or less general classification of domains is a way of introducing the more and more general definitions of an outlet at infinity. In [3,12] we introduced the following definition. A domain $\omega \subset R^n$ is called an outlet at infinity if ω is congruent to the domain

$$\omega_1 = \left\{ x=(x',x_n): \ x' \quad F\left(\frac{x'}{|x'|},x_n\right), \ \ x_n>0 \right\} ,$$

where $F\left(\frac{x'}{|x'|},x_n\right) \geq \delta > 0$ satisfies local Lipschitz condition and there exist positive constants C_j, $j = 1,2,3$, such that for $x_n>0$

$$\sup_{|x'|=1} F(x',x_n) \quad C_1 \inf_{|x'|=1} F(x',x_n) \quad C_2 x_n + C_3 \ ;$$

otherwise a domain ω is called an outlet at infinity if it is congruent to some special Lipschitz domain ω_2 (i. e., there exists open cone $\Gamma = \{ x=(x',x_n) : |x'|<\alpha x_n \ , \ x_n>0, \ \alpha>0 \}$

such that $\omega_2 + \Gamma \subset \omega_2)$. Accordingly, we shall call ω an outlet of the type ω_1 , or of the type ω_2 .

Our definition is not the most general one, since the most general definition is the following one. A domain ω is called an outlet at infinity if ω is an unbounded domain. But no one can operate with such definition as yet, except for the case $1 < p \leq n/(n-1)$, where geometrical structure of outlets is of no importance.

It should be noted that there are two sides of the problem. On the one hand, it is desirable to obtain conditions on Ω , sufficient for the coincidence of $\overset{\bullet 1}{J}_p(\Omega)$ and $\overset{\hat{}1}{J}_p(\Omega)$, which is easier; on the other hand, it is desirable to obtain conditions on Ω , necessary for the coincidence of the two spaces, which is much more difficult, since in the latter case one has to construct explicitly counter examples. That explains the restrictions we impose on outlets. And that is why our definitiom of an outlet turns out to be more or less general as yet.

We shall assume that an outlet of the type ω_1 satisfies or does not satisfy the following condition

$$\int_0^\infty \frac{dx_n}{[\text{mes } S(x_n)]^{p-1}} = +\infty , \tag{1}$$

where $S(\xi)$ is the cross-section of ω_1 with the plane $x_n = \xi, \xi > 0$, and mes $S(\xi)$ is a $(n-1)$-dimensional mesure of $S(\xi)$.

The condition (1) was introduced in [11] by V. A. Solonnikov and K. J. Piletskas for the case $p=2$, $l=1$ and $n=2$ or 3 . At about the same time, the conditiom (1) was introduced independently by the authors in [1,2] for the cases $1 < p < \infty$, $l \geq 1$ and $n \geq 2$. The same cases were also considered by K. J. Piletskas, who used the condition (1) later in [17] .

One of the most interesting results of the present paper is that $\overset{\circ 1}{J}_p(\Omega) = \overset{\circ 1}{J}_p(\Omega)$ for any domain $\Omega \subset R^n$ having the uniform C^1 - regularity property (see [18]) for $1 < p \leq n/(n-1)$, $n \geq 2$ and $l \geq 1$. The case $1 < p \leq n/(n-1)$ is the only one, where the answer to the question whether $\overset{\circ 1}{J}_p(\Omega)$ and $\overset{\circ 1}{J}_p(\Omega)$ coincide or not is independent of geometrical structure of Ω . And one cannot leave out the geometrical structure of Ω trying to answer all the questions concerning $\overset{\circ 1}{J}_p(\Omega)$ and $\hat{\overset{\circ 1}{J}}_p(\Omega)$ (e. g., one should assume that Ω is a domain with N outlets at infinity in the sense defined above) if only $n/(n-1) < p < \infty$.

Another interesting result of the present paper concerns the approximation of potential vector fields in $L_p(\Omega)$ by $C^\infty(\overline{\Omega})$ potential vector fields having bounded supports. More precisely, S. L. Sobolev in [14] and later O. V. Besov in [15] (see also [16]) posed the question of coincidence of the two $L_p(\Omega)$ subspaces $G_p(\Omega)$ and $\hat{G}_p(\Omega)$, where $G_p(\Omega)$ is a closure in $L_p(\Omega)$ of its subspace consisting of restrictions on Ω of $\overset{\circ}{G}{}^\infty(R^n) = \{\vec{v}(x) : \vec{v}(x) = \nabla\psi(x) \ , \psi \in \overset{\circ}{C}{}^\infty(R^n)\}$ vector fields, and $\hat{G}_p(\Omega) = \{\vec{v}(x) : \vec{v}(x) = \nabla\psi(x) \in L_p(\Omega)\}$. It is clear that $G_p(\Omega) \subset \hat{G}_p(\Omega)$ both consisting of $L_p(\Omega)$ potential vector fields.

In [14] S. L. Sobolev proved that $G_p(R^n) = \hat{G}_p(R^n)$. Later, in [15] O. V. Besov proved that $G_p(\Omega) = \hat{G}_p(\Omega)$ for a domain $\Omega \subset R^n$ with a unique outlet at infinity of the type ω_2 .

In the present paper we state that the question posed by J. G. Heywood in 1976 and the question posed by S. L. Sobolev in 1963 and O. V. Besov in 1969 are, in fact, one and the same question. Namely, $\dim \hat{\overset{\circ 1}{J}}_p(\Omega)/\overset{\circ 1}{J}_p(\Omega) = \dim \hat{G}_{p'}(\Omega)/G_{p'}(\Omega)$ for any domain $\Omega \subset R^n$ having the uniform C^1 - regularity property [18] , where $p' = p/(p-1)$, $1 < p < \infty$, $n \geq 2$ and $l \geq 1$. Geometrical structure of Ω is here of no importance.

We also state in the present paper some of the properties pf $\overset{\circ}{J}{}^{1}_{p}(\Omega)$ and $\hat{\overset{\circ}{J}}{}^{1}_{p}(\Omega)$ which are independent of the geometrical structure of a domain Ω .

2. Statement of the results

Instead of $\overset{\circ}{J}{}^{1}_{p}(\Omega)$ and $\hat{\overset{\circ}{J}}{}^{1}_{p}(\Omega)$ it is more convenient to consider the subspaces $\overset{\circ}{J}_p(\Omega)$ and $\hat{\overset{\circ}{J}}_p(\Omega)$ of $L_p(\Omega)$. Here $\overset{\circ}{J}_p(\Omega)$ is defined as a closure in $L_p(\Omega)$ of its subspace $\overset{\circ}{J}{}^{\infty}(\Omega)$, and the other subspace $\hat{\overset{\circ}{J}}_p(\Omega) = \{\vec{v}(x) : \vec{v}(x) \in L_p(\Omega),\ E\vec{v}(x) \in \overset{\circ}{J}_p(R^n)\}$, where E is the following zero extension operator

$$E\vec{v}(x) = \begin{cases} \vec{v}(x) & \text{for } x \in \Omega , \\ 0 & \text{for } x \in R^n \setminus \Omega . \end{cases}$$

It is clear that $\overset{\circ}{J}_p(\Omega) \subset \hat{\overset{\circ}{J}}_p(\Omega)$. The inverse inclusion may not happen.

Introduction of $\overset{\circ}{J}_p(\Omega)$ and $\hat{\overset{\circ}{J}}_p(\Omega)$ is justified by the easily verifiable fact that

$$\begin{aligned} \overset{\circ}{J}_p(\Omega) &= \hat{G}_{p'}(\Omega)^{\perp} , \\ \hat{\overset{\circ}{J}}_p(\Omega) &= G_{p'}(\Omega)^{\perp} , \end{aligned} \tag{2}$$

where $p' = p/(p-1)$. Thus the $L_p(\Omega)$ subspaces $\overset{\circ}{J}_p(\Omega)$ and $\hat{\overset{\circ}{J}}_p(\Omega)$ are annihilators of the $L_{p'}(\Omega)$ subspaces $\hat{G}_{p'}(\Omega)$ and $G_{p'}(\Omega)$, accordingly.

Note that one can rewrite (2) in the form

$$\begin{aligned} \hat{G}_{p'}(\Omega) &= \overset{\circ}{J}_p(\Omega)^{\perp} , \\ G_{p'}(\Omega) &= \hat{\overset{\circ}{J}}_p(\Omega)^{\perp} . \end{aligned} \tag{3}$$

Note also that since in [14] S. L. Sobolev proved the identity $G_p(R^n) = \hat{G}_p(R^n)$ for $1 < p < \infty$, there is no need to introduce the $L_p(R^n)$ subspace $\hat{\overset{\circ}{J}}_p(R^n)$ (for details see [1]) .

It turns out that some problems concerning $G_p(\Omega)$ and $\hat{G}_p(\Omega)$ can be solved easier than those concerning $\overset{\circ}{J}{}^1_p(\Omega)$ and $\hat{\overset{\circ}{J}}{}^1_p(\Omega)$, which made it possible to establish many of the results stated below.

To be short, we hereafter let $\overset{\circ}{J}{}^0_p(\Omega) = \overset{\circ}{J}_p(\Omega)$, $\hat{\overset{\circ}{J}}{}^0_p(\Omega) = \hat{\overset{\circ}{J}}_p(\Omega)$ and $\overset{\circ}{W}{}^0_p(\Omega) = L_p(\Omega)$ in the case where $l=0$. Only for the sake of shortness, number l in the present paper is considered to be integer.

Trying to construct a sequence of $\overset{\circ}{J}{}^\infty(\Omega)$ vector fields approximating in $\overset{\circ}{W}{}^1_p(\Omega)$ some given $\hat{\overset{\circ}{J}}{}^1_p(\Omega)$ vector field in the case where Ω is unbounded, one can find out that it is the boundedness of supports of approximating sequence elements which constitutes the essence of the approximation problem. If one abandons the boundedness of supports, the approximation problem will have then the unambiguous solution. More precisely, denoting $\hat{\overset{\circ}{J}}{}^\infty_p(\Omega) = \left\{\vec{v}(x) : \vec{v}(x) \in \hat{\overset{\circ}{J}}{}^l_p(\Omega) \quad \forall l \geq 0 \quad , \ \mathrm{dist}(\mathrm{supp}|\vec{v}|, \partial\Omega) > 0\right\}$ we have the following result.

<u>Theorem 1.</u> Let $\Omega \subset R^n$ be any domain with noncompact boundary having the uniform C^1 - regularity property, $n \geq 2$, $l \geq 0$ and $1 < p < \infty$. Then $\hat{\overset{\circ}{J}}{}^l_p(\Omega)$ coincides with the closure in $\overset{\circ}{W}{}^l_p(\Omega)$ of its subspace $\hat{\overset{\circ}{J}}{}^\infty_p(\Omega)$.

The answer to the question whether $\overset{\circ}{J}{}^l_p(\Omega)$ and $\hat{\overset{\circ}{J}}{}^l_p(\Omega)$ coincide or not may be ambiguous if $\partial\Omega$ is noncompact (i. e., unbounded) . As to the case where $\partial\Omega$ is compact (i. e., bounded), the following theorem holds.

<u>Theorem 2.</u> Let $\Omega \subset R^n$ be any domain with compact boundary having the cone property, $l \geq 0$, $n \geq 2$ and $1 < p < \infty$. Then $\overset{\circ}{J}{}^l_p(\Omega) = \hat{\overset{\circ}{J}}{}^l_p(\Omega)$ and $G_p(\Omega) = \hat{G}_p(\Omega)$.

Note that a domain Ω is said to have the cone property if there exists a fixed finite open cone C such that each point x is the vertex of some cone contained in Ω and congruent to C (see [18]) .

Thus domains Ω with compact boundaries are now out of question, and we proceed with treatment of domains having noncompact boundaries.

It is clear that the closure in $L_p(\Omega)$ of its subspace $\overset{\circ l}{J}_p(\Omega)$ with $l \geq 1$ coincides with $\overset{\circ}{J}_p(\Omega)$. Similarly, due to Theorem 1, the closure in $L_p(\Omega)$ of its subspace $\hat{\overset{\circ}{J}}{}^l_p(\Omega)$ with $l \geq 1$ coincides with $\hat{\overset{\circ}{J}}_p(\Omega)$. The following result makes it possible to reduce by means of (2),(3) the problems concerning $\overset{\circ l}{J}_p(\Omega)$ and $\hat{\overset{\circ}{J}}{}^l_p(\Omega)$ to the problems concerning $G_p(\Omega)$ and $\hat{G}_p(\Omega)$.

<u>Theorem 3.</u> Let $\Omega \subset R^n$ be any domain with noncompact boundary having the uniform C^1 - regularity property, $n \geq 2$, $l \geq 1$ and $1 < p < \infty$. Then $\overset{\circ l}{J}_p(\Omega) = \hat{\overset{\circ}{J}}{}^l_p(\Omega)$ if and only if $\overset{\circ}{J}_p(\Omega) = \hat{\overset{\circ}{J}}_p(\Omega)$, and $\dim \hat{\overset{\circ}{J}}{}^l_p(\Omega)/\overset{\circ l}{J}_p(\Omega) = \dim \hat{\overset{\circ}{J}}_p(\Omega)/\overset{\circ}{J}_p(\Omega)$ in the case where $\hat{\overset{\circ}{J}}_p(\Omega) \neq \overset{\circ}{J}_p(\Omega)$.

The complete proof of all stated here results is to be published in Vol. 23 of the Siberean Mathematical Journal.

Due to Theorem 3 and (2),(3) the following three identities are equivalent to each other:

$$\text{(a)} \quad \overset{\circ l}{J}_p(\Omega) = \hat{\overset{\circ}{J}}{}^l_p(\Omega) \quad \text{with} \quad l \geq 1;$$

$$\text{(b)} \quad \overset{\circ}{J}_p(\Omega) = \hat{\overset{\circ}{J}}_p(\Omega); \tag{4}$$

$$\text{(c)} \quad G_{p'}(\Omega) = \hat{G}_{p'}(\Omega) \quad \text{with} \quad p' = p/(p-1);$$

where $\Omega \subset R^n$ is any domain with noncompact boundary having the uniform C^1 - regularity property, $n \geq 2$ and $1 < p < \infty$. Accordingly, if for instance $\overset{\circ}{J}_p(\Omega) \neq \hat{\overset{\circ}{J}}_p(\Omega)$, then we have

$$\dim \hat{\overset{\circ}{J}}{}^l_p(\Omega)/\overset{\circ l}{J}_p(\Omega) = \dim \hat{\overset{\circ}{J}}_p(\Omega)/\overset{\circ}{J}_p(\Omega) = \dim \hat{G}_{p'}(\Omega)/G_{p'}(\Omega).$$

Thus the value of l has nothing to do with the question whether $\mathring{J}{}^{l}_{p}(\Omega)$ and $\hat{\mathring{J}}{}^{l}_{p}(\Omega)$ coincide or not. Meanwhile, the values of p and n are of great importance. For any given domain $\Omega \subset R^n$ with noncompact boundary and any given $n \geq 2$ the answer to the question whether $\mathring{J}{}^{l}_{p}(\Omega)$ and $\hat{\mathring{J}}{}^{l}_{p}(\Omega)$ coincide or not, if it happens to ambiguous, depends only on the value of p ,except for the case where $1 < p \leq n/(n-1)$. In the latter case the following theorem holds.

<u>Theorem 4.</u> Let $\Omega \subset R^n$ be any domain with noncompact boundary having the uniform C^1 - regularity property, $n \geq 2$ and $1 < p \leq n/(n-1)$. Then $\mathring{J}_p(\Omega) = \hat{\mathring{J}}_p(\Omega)$, which implies that $\mathring{J}{}^{l}_{p}(\Omega) = \hat{\mathring{J}}{}^{l}_{p}(\Omega)$ with $l \geq 1$ and $G_{p'}(\Omega) = \hat{G}_{p'}(\Omega)$ with $p' = p(p-1)^{-1}$, i. e. $n \leq p' < \infty$.

As to dependance of the approximation problem on the value of p in the case where $n/(n-1) < p < \infty$, it may be described in terms of the following statement.

<u>Theorem 5.</u> Let $\Omega \subset R^n$ be any domain with noncompact boundary having the uniform C^1 - regularity property, $n \geq 2$ and $n/(n-1) < p < \infty$. Then $\mathring{J}_p(\Omega) = \hat{\mathring{J}}_p(\Omega)$ implies that $\mathring{J}_q(\Omega) = \hat{\mathring{J}}_q(\Omega)$ in the case where $n/(n-1) < q < p$.

One can easily deduce from Theorem 5 all conceivable corollaries concerning $\mathring{J}{}^{l}_{p}(\Omega)$ and $\hat{\mathring{J}}{}^{l}_{p}(\Omega)$ with $l \geq 1$ and $G_{p'}(\Omega)$, $\hat{G}_{p'}(\Omega)$ with $p' = p/(p-1)$, e. g. $G_p(\Omega) = \hat{G}_p(\Omega)$ with $1 < p < n$ implies that $G_q(\Omega) = \hat{G}_q(\Omega)$ in the case where $p < q < n$. Note that due to Theorem 4 we have $G_p(\Omega) = \hat{G}_p(\Omega)$ in the case where $n \leq p < \infty$ if Ω satisfies the requirements of Theorem 4.

By virtue of Theorem 5 in the case where $\mathring{J}_p(\Omega) = \hat{\mathring{J}}_p(\Omega)$ to answer to the question whether $\mathring{J}_q(\Omega)$ and $\hat{\mathring{J}}_q(\Omega)$ coincide or not is unambiguous if $q < p$. Contrary to that, there can be no unambiguous answer to the question if $q > p$, which is confirmed by counter examples constructed by the authors in [1,3] .

It should be noted that J. G. Heywood was the first to suggest in [5] that $\overset{\circ}{J}{}^1_2(\Omega) = \hat{\overset{\circ}{J}}{}^1_2(\Omega)$ for any domain $\Omega \subset R^2$ (cf. with our Theorem 4 and note that $n/(n-1) = 2$ if $n = 2$) .

Thus solution of the approximation problem for any given $n \geq 2$ depends only on two variables: p and Ω . The dependence on Ω is much more complicated then that on p. It is the dependence on geometrical structure of Ω . The following theorem contains a condition on Ω which is sufficient for the identity $\overset{\circ}{J}_p(\Omega) = \hat{\overset{\circ}{J}}_p(\Omega)$ to take place.

<u>Theorem 6.</u> Let $\Omega \subset R^n$ be any domain with noncompact boundary having the uniform C^1- regularity property, $n \geq 2$ and $1 < p < \infty$. Then $\overset{\circ}{J}_p(\Omega) = \hat{\overset{\circ}{J}}_p(\Omega)$ if the following condition is satisfied

$$\underline{\lim}_{R\to\infty} \frac{\operatorname{mes}\Omega_R}{R^{p'}} < +\infty \qquad \text{with } p' = p/(p-1),$$

where $\Omega_R = \{x:\ x \in \Omega,\ |x| < R\}$ with $R > 0$ and $\operatorname{mes}\Omega_R$ is the measure of set Ω_R ($\underline{\lim}$ stands for the limit inferior).

One can easily verify that Theorem 4 is just a consequence of Theorem 6, since $\operatorname{mes}\Omega_R$ is bounded by the measure of the ball $\{x:\ x \in R^n\ ,\ |x| < R\}$, and $p' \geq n$ in the case where $1 < p \leq n/(n-1)$.

As it was mentioned in the Introduction, the conditions on Ω necessary for coincidence of $\overset{\circ}{J}{}^1_p(\Omega)$ and $\hat{\overset{\circ}{J}}{}^1_p(\Omega)$ need explicit counter examples to be constructed, which is not an easy problem. Nevertheless, we have the following result.

<u>Theorem 7.</u> Let $\Omega_1, \Omega_2 \subset R^n$ be any two domains with noncompact boundaries having the uniform C^1- regularity property,

$n \geq 2$ and $1 < p < \infty$. Let also $\Phi : \Omega_1 \to \Omega_2$ be a differentiable mapping such that $\Omega_2 = \Phi(\Omega_1)$ with Jacobian J_Φ. If there exist positive constants C_1 and C_2 such that for $x \in \Omega_1$

$$C_1 < |J_\Phi| < C_2 \quad ,$$

then we have $\overset{\circ}{J}_p(\Omega_1) = \hat{\overset{\circ}{J}}_p(\Omega_1)$ if and only if $\overset{\circ}{J}_p(\Omega_2) = \hat{\overset{\circ}{J}}_p(\Omega_2)$, and $\dim \hat{\overset{\circ}{J}}_p(\Omega_1)/\overset{\circ}{J}_p(\Omega_1) = \dim \hat{\overset{\circ}{J}}_p(\Omega_2)/\overset{\circ}{J}_p(\Omega_2)$ in the case $\hat{\overset{\circ}{J}}_p(\Omega_j) \neq \overset{\circ}{J}_p(\Omega_j)$ with $j=1$ or 2.

From Theorem 7 one can deduce all corollaries concerning $\overset{\circ}{J}{}^l_p(\Omega_j)$, $\hat{\overset{\circ}{J}}{}^l_p(\Omega_j)$ with $l \geq 1$, and $G_{p'}(\Omega_j)$, $\hat{G}_{p'}(\Omega_j)$ with $p' = p/(p-1)$, $j=1,2$.

As to the dependence on geometrical construction of Ω, we have the following results.

<u>Theorem 8.</u> Let $\Omega \subset R^n$ be a domain with unique outlet at infinity having the cone property, $n \geq 2$ and $n/(n-1) < p < \infty$. Then $\overset{\circ}{J}_p(\Omega) = \hat{\overset{\circ}{J}}_p(\Omega)$, accordingly $\overset{\circ}{J}{}^l_p(\Omega) = \hat{\overset{\circ}{J}}{}^l_p(\Omega)$ with $l \geq 1$ and $G_{p'}(\Omega) = \hat{G}_{p'}(\Omega)$ with $p' = p/(p-1)$.

<u>Theorem 9.</u> Let $\Omega \subset R^n$ be a domain with $N \geq 2$ outlets at infinity having the cone property, $n \geq 2$ and $n/(n-1) < p < \infty$. Let also k of the outlets be of the type ω_1 satisfying condition (1), where $0 \leq k \leq N-2$; and let the $N-k$ of the outlets be of the type ω_2, or of the type ω_1, in the latter case not satisfying condition (1). Then

$$\dim \hat{\overset{\circ}{J}}_p(\Omega)/\overset{\circ}{J}_p(\Omega) = N-k-1 \quad .$$

Note that in the case where $N=k+1$ or k we have $\overset{\circ}{J}_p(\Omega) = \hat{\overset{\circ}{J}}_p(\Omega)$. Note also that Theorems 8, 9 concern only domains with outlets of the type ω_1 or of the type ω_2 and only the case where $n/(n-1) < p$. The case $p \leq n/(n-1)$ is contained in Theorem 4.

From Theorems 8, 9 one can easily deduce all conceivable corollaries concerning $\overset{\circ}{J}{}^{1}_{p}(\Omega)$, $\hat{J}^{1}_{p}(\Omega)$ with $l \geq 1$ and $G_{p'}(\Omega)$, $\hat{G}_{p'}(\Omega)$ with $p' = p/(p-1)$. Note that in the special case where $l=1$, Theorems 8, 9 were stated in [12] and proved in [3]. For domains with only two outlets at infinity Theorem 9 was previously stated in [2]. A theorem similar to our Theorem 9 was also stated by K. J. Piletskas in [17], who used another definition of outlets.

It should be noted that the proof of almost all stated here results is based on the explicitly constructed in [13] solution $\vec{v}(x) \in \overset{\circ}{W}{}^{1}_{p}(\Omega)$ of the equation $\operatorname{div} \vec{v} = f(x)$ for bounded domains $\Omega \subset R^n$, $n \geq 2$, having the cone property, with any given $f(x) \in L_p(\Omega)$ satisfying the necessary compatibility condition

$$\int_{\Omega} f(x)\, dx = 0.$$

This solution satisfies the inequality

$$\|\vec{v}\|_{\overset{\circ}{W}{}^{1}_{p}(\Omega)} \leqq C \|f\|_{L_p(\Omega)}$$

with positive constant C depending only on n, p and Ω (for proof and further details see [8]).

Thus the approximation problem under consideration has been solved in the following three cases;

(a) Ω is a domain with compact boundary having the cone property and $1 < p < \infty$, $n \geq 2$;

(b) Ω is a domain with noncompact boundary having the uniform C^1- regularity property and $1 < p \leq n/(n-1)$, $n \geq 2$,

(c) Ω is a domain with $N \geq 1$ outlets at infinity (in the sense defined above) and $n/(n-1) < p < \infty$, $n \geq 2$.

All other cases represent problems which are open as yet.

The ways of mathematical progress are inscrutable. Indeed, the approximation problem concerning $\overset{\circ}{J}{}^1_p(\Omega)$ and $\hat{J}{}^1_p(\Omega)$ was posed in 1976 and seemed to be a new problem. But the problem turned out, in fact, to be an old one, posed already in 1963 for $G_p(R^n)$ and $\hat{G}_p(R^n)$, and in 1969 for $G_p(\Omega)$ and $\hat{G}_p(\Omega)$. Notwithstanding that it was posed later, the new approximation problem concerning solenoidal vector fields has added a variety of new facts and particular examples to the old approximation problem concerning potential vector fields.

The two problems are now to be treated as the only one.

References

[1] V.N. Maslennikova and M.E. Bogovsky. Denseness of finite solenoidal vector fields. Sibir. Mat. Zhurnal, Vol. 19, No. 5 (1978), 1092-1108 (Russian). English translation: Sib. Math. J., Vol. 19(1979), 771-783.

[2] M. E. Bogovsky and V. N. Maslennikova. On denseness of finite solenoidal vector fields. Uspekhi Mat. Nauk, Vol 33, No. 3 (1978), p. 152 (Russian).

[3] V. N. Maslennikova and M. E. Bogovsky. Sobolev spaces of solenoidal vector fields. Sibir. Mat. Zhurnal, Vol. 22, No. 3 (1981), 91-118 (Russian).

[4] V. N. Maslennikova and M. E. Bogovsky. On the approximation of solenoidal and potential vector fields. Uspekhi Mat. Nauk, Vol. 36, No. 4 (1981), 239-240 (Russian).

[5] J. G. Heywood. On uniqueness questions in the theory of viscous flow. Acta Math., Vol. 106, No. 1-2(1976), 61-102.

[6] J.-L. Lions. Quelques méthodes de résolution des problémes aux limite non linéaires. Dunod Gauthier-Villars,Paris, 1969.

[7] E. Magenes and G. Stampachia. I problemi al contorno per le equazioni differenziali di tipo ellitico. Ann Scuola Norm. Sup. Pisa, Vol. 12, f. 3 (1958), 247-358.

[8] M. E. Bogovsky. Solution of some vector analysis problems connected with operators div and grad. Trudy Seminar. S. L. Sobolev. (in Novosibirsk), No 1 (1980), 5 - 40 (Russian).

[9] O. A. Ladyzhenskaya and V. A. Solnnikov. Some problems ov vector analysis and generalized formulations of boundary value problems for the Navier-Stokes equations. J. Soviet Math. Vol. 10, No 2 (1978), 257-286.

[10] O. A. Ladyzhenskaya and V. A. Solonnikov. On the solution of boundary and initial boundary value problems for the Navier--Stokes equations in domains with noncompact boundaries. Leningrad. Universitet Vestnik, No 13(1977), 39-47 (Russian).

[11] V. A. Solonnikov and K. J. Piletskas. Some spaces of solenoidal vectors and the solvability of a boundry value problem for a system of Navier-Stokes equations in domains with noncompact boundaries. Zap. Nauchn. Sem Leningrad. Otdel. Mat. Inst. Steklov (LOMI), Vol. 73(1977), 136-151 (Russian).

[12] M. E. Bogovsky. On the L_p-theory of the Navier-Stokes system for unbounded domains with noncompact boundaries. Dokl. Akad. Nauk SSSR, Vol. 255, No. 6 (1980), 1296-1301 (Russian).

[13] M. E. Bogovsky. Solution of the first boundary value problem for the equation of continuity of an incompressible fluid. Soviet Math. Dokl., Vol. 20, No 5 (1979). 1094-1098.

[14] S. L. Sobolev. Denseness of finite functions in the space $L_p^{(m)}(E_n)$. Sibir. Mat. Zhurnal, Vol. 4, No 3 (1963), 673-682 (Russian).

[15] O. V. Besov. Behaviour of differentiable functions at infinity

and denseness of finite functions. Trudy Inst. Steklov.(MIAN), Vol. 105 (1969), 3-14 (Russian).

[16] O. V. Besov, V. P. Ilyin, S. M. Nikolsky. Integral representation of functions and imbedding theorems. Nauka, Moscow, 1975 (Russian).

[17] K. J. Piletskas. On spaces of solenoidal vectors. Zap. Nauchn. Sem. Leningrad. Otdel Mat. Inst. Steklov (LOMI), Vol. 96 (1980), 237-239 (Russian).

[18] R. A. Adams. Sobolev spaces. Academic Press, New York, 1975.

[19] J. G. Heywood. Auxiliary flux and pressure conditions for Navier-Stokes problems. Proc. IUTAM Symp., Paderborn, 1979.

[20] J. G. Heywood. Classical solutions of the Navier-Stokes equations. Proc. IUTAM Symp., Paderborn, 1979.

[21] J. G. Heywood. On classical solutions of the nonstationary Navier-Stokes equations in two and three dimensions. Fluid Dynamics Transactions, Vol. 10.

[22] J. G. Heywood. The Navier-Stokes equations: On the existance, regularity and decay of solutions. Indiana Univ. Math. J., Vol. 29, No 5(1980), 639-682.