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ON THE REGULARITY FOR 2nd ORDER NONLINEAR ELLIPTIC SYSTEMS

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Let an elliptic system of m equations in $\Omega \subset \mathbb{R}_n$ be considered:

$$-D_\alpha(a_\alpha^i(x,u,\nabla u)) + \bar{a}^i(x,u,\nabla u) = -D_\alpha f_\alpha^i(x) + \bar{f}^i(x), \tag{1}$$
$$\alpha = 1,\dots,n \quad i=1,\dots,m \quad x=(x_1,\dots,x_n) \quad u=(u^1,\dots,u^m).$$

It is known that for the couples (m,n) with m=1 or n=2 the regularity holds, i.e. the weak solutions of (1) belong to $C^{1,\alpha}$. In case of (m,n) general it does not take place, not even if the functions $a, \bar{a}, f$ and $\bar{f}$ are analytic. The attempts at the deeper analysis led to the formulation of the following problems:

Firstly, to obtain the regularity for all (m,n) by modification of the concept of regularity itself. Acting in this direction the mathematicians developed the theory of partial regularity. In case of the system (1) we get, roughly speaking, the following: For each weak solution u of (1) there exists closed "very small" set Σ such that $u \in C^{1,\alpha}(\Omega \setminus \Sigma)$. First theorems of this type are due to C.B.Morrey, later the italian mathematicians brought this theory to its today s form. (For the references see [1]).

Secondly, to finish the investigation of the set of couples (m,n). The history of these efforts is sufficiently known. (Cf. [1],[2]). In our paper [3] we dealt with the obstinate cases (m,3), (m,4) which had resisted for long. We constructed the systems

$$D_\alpha a_\alpha^i(\nabla u)=0 \text{ on } \Omega = B(0,1), \quad \alpha=1,\dots,n \quad i=1,\dots,m \;(= (n+1)n/2) \quad n=3,4,\dots$$

in such a way that their coefficients are analytic, the uniqueness for Dirichlet BVP holds and for suitably chosen analytic boundary function the Dirichlet BVP has the solution with bounded and discontinuous gradient.

Thirdly, to characterize regular subclasses of the set of all systems (1). Being interested in the regularity of weak solutions with bounded gradient (which is reasonable from the point of view of applications) we can use as a good characterization the condition of the Liouville type.

In the first place we consider the interior regularity (cf. [4]). Let in (1) a^i_α, $\bar{a}^i$ be in $C^1(\bar{\Omega} \times \mathbb{R}_m \times \mathbb{R}_{mn})$, $f^i_\alpha \in W_{1,p}(\Omega)$, $\bar{f}^i \in W_{1,p/2}(\Omega)$ where $p > n$. Let us assume further

$$(2) \qquad (\partial a^i_\alpha / \partial \xi^j_\beta)(x,u,\xi)\, \eta^i_\alpha \eta^j_\beta > 0, \qquad (x,u,\xi) \in \Omega \times \mathbb{R}_m \times \mathbb{R}_{mn}.$$

Definitions. The system (1) has the property (R) if its each weak solution $u \in W_{1,2,loc}(\Omega; \mathbb{R}_m)$ for which $\nabla u \in L_\infty(\Omega; \mathbb{R}_{mn})$ belongs to $C^{1,\alpha}(\Omega; \mathbb{R}_m)$. The system (1) has the property (L)(Liouville) if for each couple $(x_0,u_0) \in \Omega \times \mathbb{R}_m$ and for each weak solution $U \in W_{1,2,loc}(\mathbb{R}_n)$ of the system

$$(3) \qquad D_\alpha a^i_\alpha (x_0, u_0, \nabla U) = 0, \quad \alpha = 1,\dots,n \quad i = 1,\dots,m$$

the following assertion is valid: If $\nabla U \in L_\infty(\mathbb{R}_n; \mathbb{R}_{mn})$ then U is a polynomial of at most the first degree.

THEOREM 1. (L)$\Rightarrow$(R) .

Remark 1. In case of n=2 (L) can be proved directly (see [4]). For m=1 and for the systems (1) with $a^i_\alpha(x,u,\xi) = A^{ij}_{\alpha\beta}(x,u)\, \xi^j_\beta$ we can establish (L) passing in (3) to the equations in variations and using the results of HILDEBRANDT and WIDMAN [5] and CAMPANATO [6] respectively. It follows from THEOREM 1 now that all these special cases have the property (R) .

Remark 2. (Sketch of the proof) . Let (1) have the property (L) and let u be its weak solution with bounded gradient. For $B(x,R)$ put

$$V(x,R) = R^{-n} \int_{B(x,R)} [\, v(y) - v_{x,R}]^2 dy, \qquad v_{x,R} = [\mu B(x,R)]^{-1} \int_{B(x,R)} u(y)$$

where $v = \nabla u$. It follows from (L) that

(4) For each $x_0 \in \Omega$ there exists $R_k \to 0_+$ such that $\lim_{k\to\infty} V(x_0,R_k)=0$.

The assumptions on (1) enable us to pass from (1) to the quasilinear system for v. For each bounded weak solution of this system there exists $\varepsilon > 0$ such that if $V(x_0,R) < \varepsilon$ with some $R>0$ then $v \in C^{0,\alpha}(\overline{B(x_0,\rho)})$ with some $\rho \in (0,R)$ and $\alpha = \min\{1/2, 1-n/p\}$. This together with (4) gives the result.

As to the regularity up to the boundary, the situation is much more complicated. (For the details see [7]). Denote $\Omega = \{x \in R_n^+;\ |x_i| < 1 \text{ for } i=1,\dots,n\}$, $\Gamma = \{x \in R_n;\ |x_i| < 1,\ i=1,\dots,n-1,\ x_n=0\}$. We consider again the system (1) with $f_\alpha^i=0$. The weak formulation of the boundary value problem (BVP) is

(5) $\quad u - u_0 \in \mathcal{V}$,

(6) $\quad$ For each $\varphi \in \mathcal{V}$:

$$\int_\Omega [a_\alpha^i D_\alpha \varphi^i + \bar{a}^i \varphi^i - \bar{f}^i \varphi^i]\,dx = \int_\Gamma [h^i(x,u) - g^i(x)]\,\varphi^i dS,$$

where $u_0 \in W_{2,p}(\Omega; \mathbb{R}_m)$, $g \in L_\infty(\Gamma; \mathbb{R}_m)$, $h \in C^1(\Gamma \times \mathbb{R}_m; \mathbb{R}_m)$ are given functions, $\mathcal{V} = \{v \in W_{1,2}(\Omega; \mathbb{R}_m);\ \mathbb{C}v=0 \text{ on } \Gamma,\ v=0 \text{ on } \partial\Omega \setminus \Gamma\}$, and $\mathbb{C}$ is a constant matrix of the type $\begin{pmatrix} I, \mathbb{B} \\ 0, 0 \end{pmatrix}$.

<u>Definitions.</u> The BVP has <u>the property (R_B)</u> if its each solution u for which $\nabla u \in L_\infty(\Omega; \mathbb{R}_{mn})$ belongs to $C^{1,\alpha}(\Omega \cup \Gamma; \mathbb{R}_m)$. The BVP has <u>the property L_B</u> if for each triplet $(x_0,u_0,d) \in \Gamma \times \mathbb{R}_m \times \mathbb{R}_{mn}$ and for every function $U \in W_{1,2,loc}(\overline{\mathbb{R}}_n^+; \mathbb{R}_m)$ for which

(7)
$$\int_{\mathbb{R}_n^+} a_\alpha^i(x_0,u_0,\nabla U)\, D_\alpha \varphi^i(x)\,dx = \int_{\{x;\ x_n=0\}} d^i \varphi^i(x)\,dS$$

$$\text{for all } \varphi \in C_0^\infty(\mathbb{R}_n; \mathbb{R}_m),\quad \mathbb{C}\varphi = 0 \text{ on } \{x;\ x_n=0\},$$

(8) $\mathbb{C}U$ is the polynomial of at most the first degree on $\{x;\ x_n=0\}$ the following assertion is valid: If $\nabla U \in L_\infty(\mathbb{R}_n^+; \mathbb{R}_{mn})$, then U is the polynomial of at most the first degree on $\mathbb{R}_n^+$.

<u>THEOREM 2.</u> $(L) \wedge (L_B) \Rightarrow (R_B)$.

<u>Remark 3.</u> We say that <u>the system (1) has the property (R^*)</u> if

for each "frozen" system (3) there exists $T>0$ such that each weak solution u of (3) on R_n with the bounded gradient ∇u is of the space $C^{1,\alpha}(\overline{B(0,T)})$ and the Hölderian norm of the gradient depends only on $\|\nabla u\|_{\infty}$. The following result converse to THEOREM 1 holds: $(R^*) \Rightarrow (L)$. For BVP the analogical assertion is valid (see [4], [7]).

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