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NEW FUNCTIONAL SPACES AND LINEAR NONSTATIONARY PROBLEMS OF MATHEMATICAL PHYSICS

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1. Introduction

The range of physical problems in which the time dimension has to be considered is great. Very often it is usual to consider the time dependent problems separately, classifying them according to the mathematical structure as parabolic and hyperbolic. We shall deal with equations the structure of which is more complicated and analyse common approach to their theoretical and numerical analysis.

Consider differential equations of the form

$$\sum_{k=0}^{n} A_k D_t^k u + \sum_{k=0}^{s} b_k D_t^k u = f^*, \qquad \text{in } \Omega \times R^+ \tag{1.1}$$

where A_k are time independent elliptic operators, b_k are non - negative constants, $D_t^k = \partial^k / \partial t^k$, $s = n + 1$ or $s = n + 2$.

According to n and s the equation (1.1) can be parabolic or hyperbolic and includes the governing equations of quasistatic and dynamic problems of viscoelastic plates and shallow shells.

We assume a bounded domain Ω in R^n with a smooth boundary $\partial\Omega$ and consider Dirichlet boundary conditions and non-homogeneous initial conditions.

Simultaneously we can consider integro-differential equations

$$\int_0^t A (t - \tau) u (\tau) d\tau + \rho D_t^2 u = f^* \qquad \text{in } \Omega \times R^+ \tag{1.2}$$

with corresponding boundary and initial conditions, where A is an elliptic operator with respect to space variables.

According to the form of the elliptic operator A the equation (1.2) can be the governing equation of quasistatic problems for viscoelastic plates and shallow shells of materials of the integral type for $\rho = 1$ and of dynamic problems for $\rho \neq 1$.

2. Variational formulation

We assume that the right-hand sides of (1.1) and (1.2) belong to the weighted Hilbert space of function valued in Sobolev spaces $L_2(R_t^+, H^m, \sigma)$ endowed with the norm

$$\| f \|_{L_2(R^+, H^m, \sigma)} = \left(\int_0^\infty \| f(t) \|_{H^m}^2 e^{-2\sigma t} dt\right)^{1/2} . \tag{2.1}$$

Then applying Laplace transform to (1.1) and (1.2) we arrive at

$$\sum_{k=0}^{n} p^k A_k \tilde{u} + \sum_{k=0}^{s} p^k b_k \tilde{u} = \tilde{f} \tag{2.2}$$

and

$$\tilde{A}(p) \tilde{u} + \rho p^2 \tilde{u} = \tilde{f} , \tag{2.3}$$

where tildes denote Laplace transforms and $\tilde{f}$ includes also initial conditions.

In order to arrive at a variational formulation we introduce complex Sobolev spaces of functions $H^m(\Omega, p_\sigma^+)$ which are parametrically dependent on the transform parameter p and analytic in the right-hand halfplane $p_\sigma^+ = \{p \mid \mathrm{Re}\, p \geq \sigma \geq 0\}$ with the scalar product

$$(\tilde{u}, \tilde{v})_m = \int_\Omega \sum_{|\alpha| \leq m} D^\alpha \tilde{u} \, \overline{D^\alpha \tilde{v}} \, d\Omega \tag{2.4}$$

and the norm

$$\| \tilde{u} \|_m = \left(\int_\Omega \sum_{|\alpha| \leq m} |D^\alpha \tilde{u}|^2 \, d\Omega\right)^{1/2} \tag{2.5}$$

α beeing a multiindex.

Now we define the bilinear form for $\tilde{u}, \tilde{v} \in H_0^m(\Omega, p_\sigma^+)$,

$$B[\tilde{u}, \tilde{v}] = \sum_{k=0}^{n} p^k (A_k \tilde{u}, \tilde{v}) + \sum_{k=0}^{s} b_k p^k (\tilde{u}, \tilde{v}). \tag{2.6}$$

Since A_k are positive definite operators and b_k non-negative constants, for positive real values of p $B[\tilde{u}, \tilde{v}]$ is a positive definite bilinear form. Then for real values of p Laplace transforms of $\tilde{u}$ and $\tilde{f}$ assume real values and one can construct the functional of generalized (in the sense of Laplace transform) potential energy

$$2\,\tilde{V}\,(\tilde{u}) = B\,[u,u] - 2\,(\tilde{f},\tilde{u}). \tag{2.7}$$

Now we can formulate the following variational theorem:

Theorem 1. For $\tilde{f} \in L_2(\Omega,p_\sigma^+)$ there exists a unique weak (generalized) solution $\tilde{u}$ of (2.2) which for positive values of p minimizes the functional of the generalized potential energy (2.7) and conversely the element, which for positive values of p minimizes (2.7), fulfills (2.2). The solution for other values of p is given by the analytic continuation and the solution of (1.1) by inverse Laplace transform.

Further one can prove that (2.6) fulfills the conditions of the Lax - Milgram theorem for $p \in p_\sigma^+ - \{\infty\}$.

3.New functional spaces

When we want to derive a priori estimates it is necessary to introduce new functional spaces. We shall consider the spaces $K^{m,r}(\Omega,\sigma)$ of functions parametrically dependent on the transform parameter p and analytic in p_σ^+ with the norm

$$\|\tilde{f}\|_{K^{m,r}(\Omega,\sigma)} = \Big(\sup_{p_1>\sigma}\int_{-\infty}^{\infty}\{\|\tilde{f}\,(p_1+ip_2)\|^2_{H^m} + (1+|p|^2)^r\,\|\tilde{f}\,(p_1+ip_2)\|^2_{H^0}\}\,dp_2\Big)^{1/2} \tag{3.1}$$

where $p = p_1 + ip_2$. For $r = 0$ we arrive at Hardy spaces of functions valued in Sobolev spaces.

Simultaneously we introduce weigthed anisotropic Sobolev spaces endowed with the norm

$$\|f\|_{H^{m,r}_{,0}(\Omega,R^+,\sigma)} = \{\int_0^{\infty}\|f\|^2_{H^m}\,e^{-2\sigma t}\,dt + \|f\,e^{-\sigma t}\|^2_{H^r(R^+,\,H^0(\Omega))}\}^{1/2} \tag{3.2}$$

where we assume that

$$\frac{\partial f^k\,(x,0)}{\partial t^k} = 0, \quad k = 0, 1, \ldots, r-1. \tag{3.3}$$

Then we can prove the following theorem.

Theorem 2. Laplace transform is an isomorphic mapping of

$H^{m,r}_{,0}\ (\Omega, R^+, \sigma)$ on $K^{m,r}\ (\Omega,\sigma)$.

When we want to decide what spaces are convenient for an analysis of a time dependent problem, it is necessary to analyse properties of eigenvalues. In dealing with this analysis we restrict ourselves to the differential equation

$$D_t^2 u + A_1 D_t u + A_0 u = f^* \tag{3.4}$$

with Dirichlet boundary conditions and following initial conditions

$$u(x,0) = u_0 , \qquad D_t u(x,0) = u_1. \tag{3.5}$$

Then the Laplace transform yields

$$(p^2 + A_1 p + A_0)\ \tilde{u} = \tilde{f}^* + A_1 u_0 + p\ u_0 + u_1 . \tag{3.6}$$

Firstly we shall consider the case

$$A_1 = \kappa A_0. \tag{3.7}$$

Denoting λ_n eigenvalues of A_0 and X_n eigenfunctions of A_0 and multiplying scalarly (3.6) by X_n we arrive at

$$[\ p^2 + (\kappa p + 1)\lambda_n\ ]\ \tilde{u}_n = \tilde{f}_n + (\kappa\lambda_n + p)\ u_{0n} + u_{1n}. \tag{3.8}$$

Then

$$\tilde{u} = \sum_{n=1}^{\infty} \frac{1}{p^2 + (\kappa p+1)\lambda_n}\ [\tilde{f}_n + (\kappa\ \lambda_n + p)u_{0n} + u_{1n}]. \tag{3.9}$$

Now eigenvalues of the considered non-selfadjoint problem are the roots of the equations

$$p^2 + \kappa\lambda_n p + \lambda_n = 0, \tag{3.10}$$

what gives

$$p^*_{n1,2} = -\frac{1}{2}\lambda_n\ \ [1 \pm (1 - \frac{4}{\lambda_n \kappa^2})^{1/2}]. \tag{3.11}$$

From the point of view of space and convergence analysis it is important to find limit points of the spectrum (3.11). We have

$$\lim_{n\to\infty} p^*_{n1} = -\infty = -0(\lambda_n) \tag{3.12}$$

and

$$\lim_{n\to\infty} p_{n2} = -\frac{1}{\kappa}. \tag{3.12}$$

Thus the spectrum of the considered non-selfadjoint problem has a finite limit point and an infinite one.

The similar result can be proved in the case of general symmetric elliptic operators A_0 and A_1 when we take into account their asymptotic properties.

Then from an analysis of (3.9) it is obvious that the choice of spaces $K^{m,r}(\Omega,\sigma)$ and $H^{m,r}_{,0}(\Omega x R^+_,\sigma)$ depends on the asymptotic properties of eigenvalues of the problem for $n=\infty$ and not on orders of derivatives. [1-2].

Now we can formulate and prove:

Theorem 3. Let $\tilde{f} \in K^{2(r-1)m,r-1}(\Omega,\sigma)$ $r\geq 0$ and $u_0=u_1=0$. Then the solution of (3.6) satisfies $u \in K^{2rm,r}(\Omega,\sigma)$ and there exists a constant C such that

$$\|\tilde{u}\|_{K^{2m,r}} \leq C \|\tilde{f}\|_{K^{2(r-1)m,r-1}}. \tag{3.13}$$

Then using the isomorphism of spaces $K^{m,r}(\Omega,\sigma)$ and $H^{m,r}_{,0}(\Omega x R^+_,\sigma)$ we have.

Theorem 4. Let $f \in H^{2(r-1)m,r-1}_{,0}(\Omega\times R^+_,\sigma), r\geq 0, u_0=u_1=0$. Then the solution of (3.4) satisfies $u \in H^{2(r-1)m,r-1}_{,0}(\Omega x R^+_,\sigma)$ and there exists a constant C such that

$$\|u\|_{H^{2rm,r}_{,0}} \leq C \|f\|_{H^{2(r-1)m,r-1}_{,0}} \tag{3.14}$$

When $\tilde{f}$ does not belong to $K^{m,r}(\Omega,\sigma)$, we apply Hardy spaces $\mathcal{H}_2(H^m(\Omega,p^+_\sigma))$ of $H^m(\Omega,p^+_\sigma)$ valued functions parametrically dependent on p and analytic in p^+_σ with the norm [3-4]

$$\|\tilde{f}\|_{\mathcal{H}_2(H^m(\Omega,p^+_\sigma))} = \left(\sup_{p_1>\sigma} \int_{-\infty}^{\infty} \|\tilde{f}(p_1+ip_2)\|^2_{H^m}\, dp_2\right)^{1/2}. \tag{3.15}$$

Then we have:

Theorem 5. Laplace transform is an isometric isomorphic mapping of $L_2(R^+_,H^m,\sigma)$ on $\mathcal{H}_2(H^m(\Omega,p^+_\sigma))$.

Now analysing (3.9) we can formulate and prove.

Theorem 6. Let $\tilde{f} \in H_2(H^{-m}(\Omega,p_\sigma^+))$, $u_0 \in H^m(\Omega)$ and $u_1 \in H^{-m}(\Omega)$. Then the solution $\tilde{u}$ of (3.6) satisfies $\tilde{u} \in H_2(H^m(\Omega,p_\sigma^+))$ and there exists a constant C such that

$$\|\tilde{u}\|_{H_2(H^m(\Omega,p_\sigma^+))} \le C\{\|\tilde{f}\|_{H_2(H^{-m}(\Omega,p_\sigma^+))} + \|u_0\|_{H^m(\Omega)} + \|u_1\|_{H^{-m}(\Omega)}\}. \tag{3.16}$$

Then applying the theorem 5 we arrive at.

Theorem 7. Let $f \in L_2(R^+,H^{-m}(\Omega))$, $u_0 \in H^m(\Omega)$ and $u_1 \in H^{-m}(\Omega)$ Then the solution u of (3.4) satisfies $u \in L_2(R^+,H^m(\Omega))$ and there exists a constant C such that

$$\|u\|_{L_2(R^+,H^m(\Omega))} \le C\{\|f\|_{L_2(R^+,H^{-m}(\Omega))} + \|u_0\|_{H^m(\Omega)} + \|u_1\|_{H^{-m}(\Omega)}\}. \tag{3.17}$$

4. Finite element solution

We look for the solution in the form

$$\tilde{u}_n = \sum_{\alpha=1}^{n} \tilde{a}_\alpha(\ )\, \Phi_\alpha(x), \tag{4.1}$$

where $\{\Phi_\alpha\}_1^n$ are global basis functions depending only on space variables and the coefficients $\tilde{a}_\alpha(p)$ are functions of the transform parameter. We consider global finite element basis functions, which are piece wise polynomials in space variables. Since the set $\{\Phi_\alpha\}_1^n$ is linearly independent, it forms over the fields of functions of p analytic in p_σ^+ the basis of a n-dimensional subspace $S_n(\Omega,p_\sigma^+)$ of $H^m(\Omega,p_\sigma^+)$. In the same we may introduce $S_n^{k,m}(\Omega,p_\sigma^+)$ families of finite element subspaces [5,6] in which basis functions are polynomials in space variables of degree $k+1>m$ and form a dosed subspaces of $H^m(\Omega,p_\sigma^+)$.

Inserting (4.1) into (2.7) and applying the usual variational procedure we arrive at

$$\frac{\partial\tilde{V}}{\partial\tilde{a}_\beta} = \sum_{\alpha=1}^{n} \tilde{a}_\alpha\, B[\Phi_\alpha,\Phi_\beta] - (\tilde{f},\Phi_\beta) = 0. \tag{4.2}$$

Thus

$$\tilde{a}_\alpha(p) = \sum_{\beta=1}^{n} \frac{\tilde{F}_{\alpha\beta}(p)(\tilde{f},\Phi_\beta)}{\Delta(p)} \tag{4.3}$$

where $\tilde{F}_{\alpha\beta}$ is the adjoint matrix and

$$\Delta(p) = | B[\Phi_\alpha,\Phi_\beta)| \tag{4.4}$$

is the determinant of the matrix of the system. It is obvious that $\tilde{F}_{\alpha\beta}(p) / \Delta(p)$ is a rational function of p. Thus the inverse transform can be achieved by the method of decomposition into partial fractions.

When dealing with the error estimation of this generalization of the finite element method applying the approach and results of [3-4] we can prove.

Theorem 7. Let $S_n(\Omega,p_\sigma^+)$ belongs to $S_n^{k,m}(\Omega,p_\sigma^+)$ where $k + 1 > m$ and let $\tilde{e} = \tilde{u} - \tilde{u}^x$, where $\tilde{u}$ is the exact solution of (3.8) with $u_0 = u_1 = 0$ and $\tilde{u}^x$ is its finite element approximation, is the aproximation error. Then there exists a constant C independent of $\tilde{u}$, h and p, such that for sufficiently small h, a regular refinement and for $\tilde{f} \in K^{2sm,s}(\Omega,\sigma)$, $s \geq 1$

$$\|\tilde{e}\|_{K^{m,1/2}(\Omega,\sigma)} \leq C\, h^{\nu}\, \|f\|_{K^{2(s-1)m,s-1}(\Omega,\sigma)} \tag{4.5}$$

where $\nu = \min\,[k + 1 - m,\ (2s-1)\, m]$.

Similar results hold for $e = u - u^x$.

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