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BOUNDARY VALUE PROBLEMS AT RESONANCE FOR VECTOR SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

The aim of this paper is to use a version of the Leray-Schauder continuation theorem (see section 2) to prove the existence of solutions for second order vector ordinary differential equations with linear boundary conditions. The type of sharp sufficient conditions given in section 3 are motivated by the recent work of Kazdan and Warner [4], de Figueiredo and Gossez [2], and Brézis and Nirenberg [1] for semilinear scalar elliptic equations of the form

$$Lu = f(x, u),$$

and dealing with resonance problems at the first eigenvalue of L. We consider linear boundary value problems for systems of ordinary nonlinear differential of the form

$$x'' + f(t, x, x') = 0$$

with f verifying the Carathéodory conditions, allowing for f a dependence in x'. Moreover the technique of proof differs from the ones used in the papers quoted above and could be successfully applied to some of the problems considered there, as illustrated in [5]. In the special case of a system

$$x'' + f(t, x) = 0 \tag{1.1}$$

with boundary conditions

$$x(o) = x(\pi) = 0$$

(1.2) $\quad$ (resp. $x(o) - x(2\pi) = x'(o) - x'(2\pi) = 0$

the obtained conditions for existence of a solution are

(1.3) $\quad F(t) := \overline{\lim}_{|x|\to\infty} |x|^{-2}(x|f(t, x) \leqslant 1 \;, \int_o^{\pi} F(t)\sin^2 t dt < \frac{\pi}{2}$

(1.4) $\quad$ (resp. $F(t) \leqslant 0 \neg \int_o^{2\pi} F(t)dt < 0$)

(with $(|)$ the inner product in $\mathbb{R}^n$) so that F(t) can be equal to the first eigenvalue 1 (resp. 0) of the associated linear problem

$$x'' + \mu x = 0$$

$$x(o) = x(\pi) = 0 \quad (\text{resp. } x(o) - x(2\pi) = x'(o) - x'(2\pi) = 0)$$

on any subset of $[o, \pi]$ (resp. $[o, 2\pi]$) of positive measure such that the second condition in (1.3) (resp. (1.4)) is verified. In the case of a first order vector equation

$$x' + f(t, x) = 0$$

it has been shown in [5] that the first condition in (1.4) could be replaced by the mere existence of F(t) and the second one replaced by

$$\int_o^{2\pi} F(t)dt \neq 0$$

It is an open question to know if (1.4) for (1.1) - (1.2) could be generalized by

$$F(t) \leqslant a \;, \quad \int_o^{2\pi} F(t)dt < 0$$

for same $a > 0$. Let us finally notice that Theorem 2 gives existence conditions distinct of the ones of Hartman's type described for example in [3] , Chapter V.

2. A USEFUL VERSION OF THE LERAY-SCHAUDER CONTINUATION THEOREM

Let X, Z be real normed spaces, L : dom $L \subset X \to Z$ a linear Fredholm mapping of index zero and $N : X \to Z$ a (not necessary linear) mapping which is L-compact on bounded subsets of X (shortly L-completely continuous on X). See [3] p. 12-13 for the corresponding definitions. We shall be interested in proving the existence of a solution for the abstract problem

(2.1) $$Lx = Nx$$

Theorem 1. Assume that there exist an open bounded set $\Omega \subset X$ and a linear L-completely continuous mapping $A : X \to Z$ such that :

(i) $0 \in \Omega$

(ii) $\ker (L - A) = \{0\}$

(iii) $Lx \neq (1 - \lambda) Ax + \lambda Nx$ for all $x \in$ dom $L \cap \partial\Omega$ and all $\lambda \in]0,1[$.

Then (2.1) has at least one solution in dom $L \cap \overline{\Omega}$.

For a proof of this theorem, special cases of which have been used by many authors, see [5]. If L is invertible, then a natural choice for A is $A = 0$ and one gets the usual Leray-Schauder condition for mappings of the form $I - L^{-1}N$ and $L^{-1}N$ completely continuous. If L is not invertible, it is shown in [5] how to use this theorem to give more simple proofs of some results of Cesari and Kannan, De Figueiredo, Brézis and Nirenberg for abstract equations. One could get similarly the results of De Figueiredo and Gossez on elliptic problems.

3. AN EXISTENCE THEOREM FOR VECTOR SECOND ORDER DIFFERENTIAL EQUATIONS WITH LINEAR BOUNDARY CONDITIONS

Let $I = [a,b]$ be a compact internal of $\mathbb{R}$, $f : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ a mapping satisfying the Caratheodory conditions.

Let (1.1) denote the Euclidian norm in $\mathbb{R}^n$ and $(u|v)$ the inner product of u and v.

Let now

$$\Lambda : C^1(I, \mathbb{R}^n) \to \mathbb{R}^n$$

be a linear continuous mapping such that

(A) For each $x \in \ker \Lambda$, $(x'(b) \mid x(b)) - (x'(a) \mid x(a)) = 0$

(B) The problem

$$x'' + \mu x = 0 , \quad \Lambda(x) = 0$$

has a nontrivial solution if and only if $\mu = \mu_i$ $(i = 1, 2, \ldots)$ with

$$0 \leq \mu_1 < \mu_2 < \ldots .$$

of finite multiplicity and such that the corresponding eigenfunctions form a complete orthogonal set in $L^2(I, \mathbb{R}^n)$ with the inner product

$$< y,z > = \int_I (y(t) \mid z(t)) \, dt$$

(C) There exists $\gamma > 0$ such that for each $x \in \ker\Lambda$ such that

$$< x,u > = 0$$

for all u belonging to the eigenspace relative to μ_1, one has

$$|x|_\infty \leq \gamma |x'|_2$$

where

$$|x|_\infty = \max_{t \in [a,b]} |x(t)| , \quad x' = dx/dt \quad \text{and}$$

$$|y|_2 = \left(\int_I |y(t)|^2 dt \right)^{1/2}$$

Important special cases of mappings Λ satisfying (A) - (B) - (C) are given by

$\Lambda x = (x(a), x(b))$ (Dirichlet or Picard boundary conditions)
$\Lambda x = (x'(a), x'(b))$ (Neumann boundary conditions)
$\Lambda x = (x(a) - x(b), x'(a) - x'(b))$ (periodic boundary conditions)
$\Lambda x = (x(a), x'(b)), \ldots$

We shall be interested in proving the existence of solutions for the boundary value problem

$$x'' + f(t, x, x') = 0 \tag{3.1}$$

$$\Lambda(x) = 0, \tag{3.2}$$

i.e. the existence of mapping $x : I \to \mathbb{R}^n$ having an absolutely continuous first derivative and verifying (3.1) a.e. in I and (3.2).

By taking $X = C^1(I, \mathbb{R}^n)$ with the norm

$$\|x\|_\infty = \max(|x|_\infty, |x'|_\infty),$$

$Z = L^1(I, \mathbb{R}^n)$ with the norm $|x|_1 = \int_I |x(t)|\, dt$, dom $L = \{x \in X : x'$ is absolutely continuous on I and $\Lambda(x) = 0\}$, $L : \operatorname{dom} L \subset X \to Z$, $x \mapsto -x''$, $N : X \to Z$, $x \mapsto f(\cdot, x, x')$, it is routine to check that L is Fredholm of index zero and closed and that N is L-completely continuous. The problem is then equivalent to an abstract one of type (2.1) and we shall prove the following existence theorem.

<u>Theorem 2.</u> <u>Let</u> f <u>be like above and let</u> Λ <u>satisfy conditions (A), (B) and (C).</u> <u>Assume that the following conditions hold</u> :

$$F(t) := \overline{\lim_{|x| \to \infty}} \left(\sup_{u \in \mathbb{R}^n} \frac{(x \mid f(t, x, u))}{|x|^2} \right) \leq \mu_1 \tag{D}$$

<u>uniformly in</u> $t \in I$, <u>i.e. for each</u> $\varepsilon > 0$, <u>there exists</u> $\rho_\varepsilon > 0$ <u>and</u> $\delta_\varepsilon \in L^1(I, \mathbb{R}_+)$ <u>such that for a.e.</u> $t \in I$, <u>all</u> x <u>with</u> $|x| \geq \rho_\varepsilon$ <u>and all</u> $u \in \mathbb{R}^n$, <u>one has</u>

$$(x \mid f(t, x, u)) \leq (\mu_1 + \varepsilon)|x|^2 + \delta_\varepsilon |x|$$

(E) For each φ belonging to the eigenspace relative to μ_1, one has

$$\int_I F(t)|\varphi(t)|^2 dt < \mu_1 \int_I |\varphi(t)|^2 dt$$

(F) For each $R > 0$ there exists $\varphi_R \in C^0(\mathbb{R}_+, \mathbb{R}_+^*)$ nondecreasing with $\lim_{s \to \infty} s^{-2}\varphi_R(s) = 0$ and such that for a.e. $t \in I$ and all x with $|x| \leqslant R$, one has

$$|f(t, x, u) \leqslant \varphi_R(|u|).$$

Then problem (3.1-3.2) has at least one solution.

Proof

We shall apply Theorem 1 with $Ax = ax$ and

$$a < \mu_1 \tag{3.3}$$

which clearly satisfies the L-complete continuity requirement and condition (ii). The inequality in (iii) becomes here

$$x'' + (1 - \lambda) ax + \lambda f(t, x, x') \neq 0 \tag{3.4}$$

and we shall first show that there exists $R > 0$ such that for any $\lambda \in]0, 1[$ and any x verifying $\Lambda(x) = 0$ and

$$|x|_\infty \geqslant R$$

(3.4) is satisfied. If it is not the case, there will exist a sequence (λ_n) with $\lambda_n \in]0, 1[$ $(n \in \mathbb{N})$ and a sequence (x_n) of elements of dom L such that

$$|x_n|_\infty \geqslant n$$

and

$$x_n'' + (1 - \lambda_n) ax_n + \lambda_n f((t, x_n, x'_n) = 0 \tag{3.5}$$

Using the assumption (A) (3.5) implies

$$|x'_n|_2^2 = (1 - \lambda_n)a|x_n|_2^2 + \lambda_n < x_n, f(., x_n, x'_n)> . \tag{3.6}$$

Let now $\varepsilon > 0$; Caratheodory conditions on f and assumption (D) insure the existence of $\tilde{\delta}_\varepsilon \in L^1(I, \mathbb{R}_+)$ such that for a.e. $t \in I$

$$(x_n(t) \mid f(t, x_n(t), x'_n(t)) \leqslant (\mu_1 + \varepsilon)|x_n(t)|^2 + \tilde{\delta}_\varepsilon(t)|x_n(t)|$$

which together with (3.3) and (3.6) gives

$$(3.7) \qquad |x'_n|_2^2 \leqslant (\mu_1 + \varepsilon)\,|x_n|_2^2 + |\tilde{\delta}_\varepsilon|_1 |x_n|_\infty$$

Now let us write

$$(3.8) \qquad x_n = u_n + v_n$$

where u_n (resp. v_n) belongs (resp. is orthogonal) to the eigenspace associated to μ_1. Then

$$|x'_n|_2^2 - \mu_1 |x_n|_2^2 = |v'_n|_2^2 - \mu_1 |v_n|_2^2$$

and we have the Wirtinger's inequality (which follows from assumption (B))

$$\mu_2 |v_n|_2^2 \leqslant |v'_n|_2^2 \quad .$$

Those results together with (3.7) and assumption (C) imply that

$$|v_n|_\infty \leqslant \gamma\, \mu_2\, (\mu_2 - \mu_1)^{-1} \,[\, \varepsilon(b-a)^2 |x_n|_\infty^2 + |\tilde{\delta}_\varepsilon|_1 |x_n|_\infty \,]$$

and hence that

$$(3.9) \qquad |x_n|_\infty^{-1} \, |v_n|_\infty \to 0 \qquad \text{if} \quad n \to \infty.$$

Let now

$$y_n = x_n / |x_n|_\infty \quad , \quad n = 1, 2, \ldots .$$

Using (3.9), the finite dimension of the eigenspace associated to μ_1 and going to subsequences, one can assume that (for the norm $|\cdot|_\infty$),

$$y_n \to y \quad , \quad \lambda_n \to \lambda_0$$

with y in the eigenspace associated to μ_1, $|y|_\infty = 1$ and $\lambda_0 \in [0,1]$.

From (3.6) and Wirtinger's inequality we then obtain, if X_n denotes the characteristic function of the set

$$I_n = \{t \in I : |x_n(t)| \neq 0\} = \{t \in I : |y_n(t)| \neq 0\},$$

$$\lambda_n \int_I X_n(t)|y_n(t)|^2 \frac{(x_n(t)|f(t, x_n(t), x'_n(t))}{|x_n(t)|^2} dt$$

$$\geq [\mu_1 - (1-\lambda_n)a]\,|y_n|_2^2$$

Hence, by assumption (D) and by Fatou's lemma,

$$\lambda_o \int_I \overline{\lim_{n\to\infty}} \left[X_n(t)\, |y_n(t)|^2 \frac{(x_n(t) \mid f(t, x_n(t), x'_n(t))}{|x_n(t)|^2} \right] dt + (1-\lambda_o)a|y|_2^2$$

(3.10)

$$\geq \mu_1 |y|_2^2$$

But, on I_n, $|x_n(t)| = |y_n(t)|\,|x_n|_\infty \to \infty$ if $n \to \infty$ and, as $y_n \to y$, $X_n \to 1$ a.e. in I, which implies together with (3.10) and assumption (D) that

$$\lambda_o \int_I F(t)\, |y(t)|^2\, dt + (1-\lambda_o)\, a \int_I |y(t)|^2\, dt$$

$$\geq \mu_1 \int_I |y(t)|^2\, dt ,$$

a contradiction with assumption (E). Thus, if $x \in \mathrm{dom} L$ satisfies

(3.11) $$x'' + (1-\lambda)\, a\, x + \lambda f(t, x, x') = 0$$

for some $\lambda \in]0,1[$, one has

$$|x|_\infty < R$$

and hence, for a.e. $t \in I$, using assumption (F),

$$|x''(t)| \leq |a|\, R + \varphi_R(|x'(t)|) = \tilde{\varphi}_R(|x'(t)|)$$

with $\tilde{\varphi}_R \in C^0(\mathbb{R}_+, \mathbb{R}_+^*)$ nondecreasing and $\lim_{s\to\infty} s^{-2}\, \tilde{\varphi}_R(s) = 0$.

By a result of Schmitt (see e.g. [3] p. 69) this implies the existence of $S = S(R) > 0$ such that any solution $x \in \text{dom } L$ of (3.11) verifies

$$|x'|_\infty < S.$$

Therefore, if

$$\Omega = \{x \in C^1(I, \mathbb{R}^n) : |x|_\infty < R \, , \; |x'|_\infty < S\}$$

conditions (i) and (iii) of Theorem 1 are satisfied and the proof is complete.

REFERENCES

1. H. BREZIS and L. NIRENBERG, Characterizations of the ranges of some nonlinear operators and applications to boundary value problems, Annali Scuola Norm. Sup., to appear.

2. D.G. de FIGUEIREDO and J.P. GOSSEZ, Perturbation non linéaire d'un problème elliptique linéaire près de sa première valeur propre, C.R. Acad. Sci. Paris (A) 284 (1977) 163-166.

3. R.E. GAINES and J. MAWHIN, "Coincidence Degree and Nonlinear Differential Equations", Lecture Notes in Math. vol. 568, Springer, Berlin, 1977.

4. J.L. KAZDAN and F.W. WARNER, Remarks on some quasilinear elliptic equations, Comm. Pure Applied Math. 28 (1975) 567-597.

5. J. MAWHIN, Landesman - Lazer's type problems for nonlinear equations, Conf. Semin. Mat. Univ. Bari n° 147, 1977.

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