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Algebraic methods in the theory of global properties of the oscillatory equations $Y'' = Q(t)Y$

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ALGEBRAIC METHODS IN THE THEORY OF GLOBAL PROPERTIES OF THE OSCILLATORY EQUATIONS $Y'' = Q(t)Y$

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CONTENTS.

I. INTRODUCTION

The theory of global properties of the ordinary 2nd order linear differential equations in the real domain has, in the last twenty years, made a remarkable progress. It originated from the problem of global equivalence of the nth order linear equations ($n \geqslant 2$) as the first step towards its solution. The question was, first, to solve the problem of global equivalence in the simplest case $n = 2$ and then, with acquired experience, get to the core of the problem in the general case. The result of this first step was quite satisfactory; in particular, the problem of global equivalence for 2nd order equations had been completely solved ([1], [2]).

The theory of global properties of the 2nd order equations in question consists of several partial theories concentrated about its most important notions. Among the latter is the theory of global properties of the oscillatory equations in the interval $\underline{R} = (-\infty, \infty)$ that deserves, for its algebraic character, a particular attention. It is exactly this theory, simply: the algebraic theory of the oscillatory equations, that forms the object of my lecture.

The basic notion of the theory in question is that of the group of phases, $\mathfrak{G}$. By the latter we understand the set composed of all phase functions together with the binary operation defined by composing functions. A phase function is any real function of class $C^3_{\underline{R}}$, unbounded on both sides, whose derivative of the first order is always different from zero. Various objects connected with oscillatory equations in $\underline{R}$ give rise to an algebraic structure of the group $\mathfrak{G}$. The latter is richly articulated and is, of course, in certain relations with the properties of the equations in question. The above algebraic objects are, in particular, the groups of dispersions of the single oscillatory equations, namely - let us note - groups formed by all phase functions transforming these equations into themselves and, furthermore, decompositions of the group $\mathfrak{G}$ into left and right cosets generated by groups of dispersions. The algebraic theory of the oscillatory equations is a study of the structure of the group $\mathfrak{G}$ in connection with the properties of the equations in question. We shall consider, in particular, equations with periodic coefficients. That opens a new way to Floquet's classical theory for 2nd order equations, leading to many new results in this field. This is a brief outline of the following lecture.

II. GENERAL THEORY

1. <u>Starting situation.</u> Consider equations of the Jacobian form

$$(Q) \qquad Y'' = Q(t)Y, \quad t \in \underline{R}$$

and, from the beginning, suppose that we deal always with oscillatory equations. The coefficient Q, also called the carrier of (Q), is supposed to be of class $C^0_{\underline{R}}$: $Q \in C^0_{\underline{R}}$. The equation (Q) is called oscillatory if each of its integrals has an infinite number of zeros towards $-\infty$ and ∞ . By an integral of (Q) we understand a solution of (Q) defined in the entire $\underline{R}$. The integral identically equal to zero is generally omitted.

As to the way of notation and terminology let us add the following remark: We often write Q instead of (Q); sometimes we speak, e.g., about integrals of the carrier Q instead of integrals of the equation (Q). The symbol $\underline{M}$ denotes the set of all carriers Q or all equations (Q).

2. <u>Kummerian transformations.</u> The general theorem about global transformations of 2nd order equations states that, in case of oscillatory equations in $\underline{R}$, every equation Q may be globally transformed into any equation R. That means: there exist phase functions X(t) such that, for every integral Y of Q, the function

$$Z(t) = \frac{c}{\sqrt{|X'(t)|}} \cdot Y[X(t)]$$

is an integral of R; c denotes an arbitrary constant ($\neq 0$). The function X is called transformator of Q into R, whereas $c:\sqrt{|X'(t)|}$ is the multiplicator of the transformation in question. The transformators X are exactly the integrals of the following equation which is, for historical reasons, called Kummer's equation

$$(QR) \qquad -\{X,t\} + Q(X)X'^2 = R(t).$$

The set of these transformators is therefore the general integral I_{QR} of (QR) ([1]).

The inverse of X, denoted X^{-1}, is a transformator of R into Q and therefore an integral of (RQ). Denoting by I_{QR}^{-} the set formed by the inverses of the elements of I_{QR}, we have: $I_{RQ} = I_{QR}^{-}$ and $I_{QR} = I_{RQ}^{-}$.

The theory of Kummer's equation forms the kernel of our considerations concerning Kummerian transformations. It is based on a theorem on existence and unicity of integrals of the equation (QR), which is of Cauchyan type ([1]). The integrals of (QR) are called general dispersions of the equations Q,R so as to suggest their connection with the dislocation (dispersion) of zeros of the integrals of Q and R ([1]).

Finally, let us remark that, according to a classical result of P. Stäckel, the transformations of the form $Y(T) \to w(t)Y[X(t)]$ (w is, in our case, with regard to the special form of Jacobian equations, the previously mentioned multiplicator) are the most general transformations conserving the linearity and order ($n \geqslant 2$) of the transformed equations.

3. Groups of dispersions. The gateway leading to the realm of algebra is the notion of groups of dispersions.

By a dispersion (of the first kind) of the equation Q we understand any transformator of Q into Q or, in other words, any integral of the equation (QQ). We can show that the set of all dispersions of Q, i.e., the general integral I_{QQ} of (QQ) is a group (the group operation being given by composing functions). It is called group of dispersions of Q. Notation: $\mathfrak{D}_Q$. The group $\mathfrak{D}_Q$ is evidently stationary as it does not change the equation Q.

Let us return to Kummer's equation (QR). A substantial progress in the theory of the latter is given by the fact that for any integral X of (QR) there holds ([3]):

$$(1) \qquad I_{QR} = \mathfrak{D}_Q X = X\mathfrak{D}_R.$$

We see, first, that the group $\mathfrak{B}_R$ is the image of $\mathfrak{B}_Q$ by the inner automorphism of $\mathfrak{G}$, generated by any integral X of (QR); in other words, $\mathfrak{B}_R$ is conjugate with $\mathfrak{B}_Q$ by X:

$$\mathfrak{B}_R = X^{-1} \mathfrak{B}_Q X.$$

Hence, the system of subgroups of $\mathfrak{G}$, formed by the groups $\mathfrak{B}_R$ ($R \in \underline{M}$) is composed of groups pairwise conjugate. We can also show that the mapping $R \to \mathfrak{B}_R$ is bijective and the intersection of all the groups of dispersions is the group $\{t\}$.

Another important consequence of (1) is: I_{QR} is the right coset of $\mathfrak{B}_Q$ containing X and, at the same time, the left coset of $\mathfrak{B}_R$ containing X. More precisely,

$$I_{QR} = \bar{Q}_d \cap \bar{R}_g; \tag{2}$$

$\bar{Q}_d$ denotes the right decomposition of $\mathfrak{G}$, formed by the right cosets of $\mathfrak{B}_Q$, and, similarly, $\bar{R}_g$ stands for the left decomposition of $\mathfrak{G}$, generated by $\mathfrak{B}_R$ ([3]).

Starting from (2) we are, by further considerations which we cannot deal with here in detail, led to results concerning common integrals of two Kummerian equations. Let us introduce at least some results concerning the notions of inverse equations and blocks of equations.

4. Inverse equations and blocks of equations. We consider the equation Q. The equation R^- is called inverse to R ($\in \underline{M}$) with regard to Q or, simply: inverse to R, if there exist common integrals of the equations (QR), (R^-Q).

This notion is evidently symmetrical with regard to the equations R, R^-. We can show that the set of equations inverse to R is non-empty. The latter is called the block of equations inverse to R or, simply: a block. It can be proved that the set composed of all blocks is a decomposition of $\underline{M}$ (cf. the case $Q = -1$ in [7]). To each block $\bar{u}$ there corresponds precisely a block $\bar{u}^-$ called inverse to $\bar{u}$, formed by all equations inverse to any element of $\bar{u}$; there holds $(\bar{u}^-)^- = \bar{u}$. Two equations are inverse to each other iff they are contained in blocks inverse to each other. Every equation R^* contained in the same block as R is called associated with R (with regard to Q).

Formulae expressing each equation inverse to R, R^-, or each equation associated with R, R^*, are

$$R^-(t) = Q(t) + [Q(X^{-1})-R(X^{-1})]\cdot X^{-1\prime 2}(t) \qquad \text{for } X \in I_{QR},$$

$$R^*(t) = Q(t) - [Q(X)-R(X)]\cdot X'^2(t) \qquad \text{for } X \in I_{QQ}.$$

5. Adjoint groups. With every equation Q we shall intrinsically associate a figure composed of four groups called adjoint to Q, briefly: adjoint groups:

$$\mathfrak{N}_Q \supset \mathfrak{D}_Q \supset \mathfrak{D}_Q^+ \supset \mathfrak{C}_Q ;$$

$\mathfrak{D}_Q$ is the group of dispersions of Q, $\mathfrak{D}_Q^+$ is its subgroup of all increasing dispersions, $\mathfrak{C}_Q$ the center of $\mathfrak{D}_Q^+$ and, finally, $\mathfrak{N}_Q$ is the normalizer of $\mathfrak{C}_Q$ in the group $\mathfrak{G}$.

The structural properties of the adjoint groups and the stationary character of the latters are mainly these:

$\mathfrak{D}_Q$ is homomorphic with the group composed of all unimodular matrices 2x2 with real elements ([1]).

$\mathfrak{D}_Q^+$ is an invariant subgroup of $\mathfrak{D}_Q$ with the index 2. The coset of $\mathfrak{D}_Q^+$ in $\mathfrak{D}_Q$ is formed by all decreasing dispersions of Q. The normalizer of $\mathfrak{D}_Q^+$ in $\mathfrak{G}$ is $\mathfrak{D}_Q$: $N\mathfrak{D}_Q^+ = \mathfrak{D}_Q$ (cf. the proof of $N\mathfrak{D}_{-1} = \mathfrak{D}_{-1}$ in [3]). The elements of $\mathfrak{D}_Q^+$ keep the orientation of the integral curves of Q.

$\mathfrak{C}_Q$ is an infinite cyclic group

$$\dots < \Phi_{-2}(t) < \Phi_{-1}(t) < \Phi_0(t) = t < \Phi_1(t) < \Phi_2(t) < \dots \quad (t \in \underline{R})$$

whose elements, called central dispersions of Q, may be constructively described as follows: the value $\Phi_\nu(t)$ is the ν th conjugate point with t lying to the right or to the left of t respectively for $\nu > 0$ or $\nu < 0$ (ν integer). The functions Φ_{-1} and Φ_1 are the generators of $\mathfrak{C}_Q$; in particular Φ_1 is called the fundamental dispersion of Q and there evidently holds: $\Phi_\nu(t) = \Phi_1^\nu(t)$. The group $\mathfrak{C}_Q$ is also called the center of Q; two equations with the same center are called concentric. The elements of $\mathfrak{C}_Q$ are exactly the dispersions of Q which transform every integral Y of Q into an integral linearly dependent on Y.

The elements of $\mathfrak{A}_Q$ are exactly the transformators of Q into some equation concentric with Q.

Proof. Let Φ be the fundamental dispersion of Q. Let R be the carrier of the transformed equation Q by some $X \in \mathfrak{G}$, and Ψ its fundamental dispersion. Then we have ([1] p.176): $X\Psi = \Phi^{\operatorname{sgn} X'}X$. $X \in \mathfrak{A}_Q$ implies, by definition of $\mathfrak{A}_Q$: $X\mathfrak{C}_Q = \mathfrak{C}_Q X$ and thus ([4]): $X\Phi = \Phi^{\operatorname{sgn} X'}X$. Hence it is $\Psi = \Phi$ and this yields $\mathfrak{C}_R = \mathfrak{C}_Q$. The rest of the assertion may be proved by similar arguments.

We call the elements of $\mathfrak{A}_Q$ <u>co-dispersions</u> of Q; $\mathfrak{A}_Q$ is of course the group of co-dispersions of Q.

We know that if R is, by the transformator X, the Kummerian image of Q, then $\mathfrak{B}_R$ is conjugate with $\mathfrak{B}_Q$ by the same X. In this case, any group adjoint to R is conjugate, by X, with the corresponding group adjoint to Q. We say that the Kummerian transformations of the equations $Q \in \underline{M}$ are accompanied by inner automorphisms acting on the corresponding adjoint groups.

6. <u>The inclusion theorems.</u> Groups adjoint to two equations Q,R are in certain mutual relations of partly dual character. These relations are described in the following three theorems that we call inclusion theorems. The symbols Ψ and Φ denote the fundamental dispersions of R and Q, respectively.

<u>Theorem 1.</u> The relation $\mathfrak{A}_R \supset \mathfrak{C}_Q$ implies $\Psi\Phi = \Phi\Psi$ and vice versa.

Proof. If $\mathfrak{A}_R \supset \mathfrak{C}_Q$ then $\Phi\mathfrak{C}_R = \mathfrak{C}_R\Phi$ and this yields $\Phi\Psi = \Psi\Phi$. The second part of the proof is obvious.

<u>Corollary.</u> If $\mathfrak{A}_R \supset \mathfrak{C}_Q$ then $\mathfrak{A}_Q \supset \mathfrak{C}_R$.

<u>Theorem 2.</u> The relation $\mathfrak{A}_R \supset \mathfrak{B}_Q^+$ implies $\mathfrak{C}_R \subset \mathfrak{C}_Q$ and vice versa.

Proof. If $\mathfrak{A}_R \supset \mathfrak{B}_Q^+$ then $X\mathfrak{C}_R = \mathfrak{C}_R X$ for every $X \in \mathfrak{B}_Q^+$. Hence we conclude $X\Psi = \Psi X$ and this implies $\mathfrak{B}_Q^+\Psi = \Psi\mathfrak{B}_Q^+$. This yields $\Psi \in N\mathfrak{B}_Q^+$ ($= \mathfrak{B}_Q$) and, since Ψ increases, $\Psi \in \mathfrak{B}_Q^+$; because Ψ commutes with every $X \in \mathfrak{B}_Q^+$ we find $\Psi \in \mathfrak{C}_Q$. The result: $\mathfrak{C}_R \subset \mathfrak{C}_Q$. The rest of the proof follows by similar arguments.

<u>Theorem 3.</u> The relation $\mathfrak{B}_R \supset \mathfrak{C}_Q$ implies $[R(\Phi) - Q(\Phi)]\Phi'^2(t) = R(t) - Q(t)$ $(t \in \underline{R})$ and vice versa.

Proof. Φ being the fundamental dispersion of Q it satisfies the equation (QQ). If $\mathfrak{Z}_R \supset \mathfrak{C}_Q$ then Φ also satisfies the equation (RR) and the above relation follows. The rest of the proof is obvious.

The inclusion theorems are a source of interesting problems concentrated about the properties of equations R, Q satisfying the conditions given by the above theorems. Consider an arbitrary equation $Q \in \underline{M}$. Let $\mathscr{A}_Q$, $\mathscr{B}_Q$, $\mathscr{C}_Q$ be the classes of equations R characterized by the following properties:

Class $\mathscr{A}_Q$: The fundamental dispersion of any equation $R \in \mathscr{A}_Q$ is commuting with Φ .

Class $\mathscr{B}_Q$: The fundamental dispersion of any equation $R \in \mathscr{B}_Q$ is a central dispersion of Q.

Class $\mathscr{C}_Q$: Between the carriers $R \in \mathscr{C}_Q$, Q and the fundamental dispersion Φ we have the relation indicated in Theorem 3.

The above classes $\mathscr{A}_Q$, $\mathscr{B}_Q$ have been studied in case Q = -1 ([5], [6], [4]). As to the class $\mathscr{C}_{-1}$ let us remark that it is composed of all equations with π-periodic carriers.

III. SPECIALIZED THEORY

7. The way of specialization. The general theory we have, so far, spoken about changes its aspect if we choose, arbitrarily, some equation $Q \in \underline{M}$, called canonical, as a representation of the system $\underline{M}$. This equation Q and the right decomposition Q_d of the group of phases, $\mathfrak{G}$, generated by the group $\mathfrak{B}_Q$, enter the center of the theory: Any equation $R \in \underline{M}$ is a Kummerian image of Q. The transformators X of Q into R form an element of $\bar{Q}_d$; $\bar{Q}_d$ is composed of general integrals I_{QR} of Kummer's equations (QR) associated with the single equations $R \in \underline{M}$. The groups adjoint to R arise from the corresponding groups adjoint to Q by inner automorphisms of $\mathfrak{G}$, generated by the transformators $X \in I_{QR}$.

For Q, representing the system $\underline{M}$, it is convenient to choose the equation -1, namely $Y'' = -Y$ $(t \in \underline{R})$, whose simplicity yields an advantage in calculations.

8. The equation $Y'' = -Y$ $(t \in \underline{R})$. In the following formulae ν (integer), a,b,c denote constants; $t \in \underline{R}$.

Integrals: $Y(t) = c \cdot \sin(a+t)$; $0 \leq a < \pi$, $c \neq 0$.

Dispersions: $\epsilon(t) = {}_{\nu}\tan^{-1}(c\cdot\frac{\sin(a+t)}{\sin(b+t)})$;

$\epsilon(-a) = \nu\pi$; $0 \leqslant a,b < \pi$; $c(b-a) \neq 0$.

Increasing dispersions: The last formula with $c(b-a) > 0$.

Central dispersions: $\Phi_{\nu}(t) = t + \nu\pi$.

Fundamental dispersion: $\Phi_1(t) = t + \pi$.

Co-dispersions: $h(t) = \delta t + d(t)$,

$\delta = \pm 1$; $d \in C^3_{\underline{R}}$; $d(t+\pi) = d(t)$; $-\delta\cdot d'(t) < 1$.

The adjoint groups of -1 are also denoted by $\mathfrak{H}$, $\mathfrak{E}$, $\mathfrak{E}^+$, $\mathfrak{Z}$ so that $\mathfrak{A}_{-1} = \mathfrak{H}$, $\mathfrak{B}_{-1} = \mathfrak{E}$, $\mathfrak{B}^+_{-1} = \mathfrak{E}^+$, $\mathfrak{C}_{-1} = \mathfrak{Z}$.

$\mathfrak{E}$ is called the fundamental group; its elements ϵ are called special dispersions. $\mathfrak{H}$ is called the group of elementary phases; its elementa h are, of course, elementary phases.

9. Theory specialized by the choice $Q = -1$. In this case every equation $R \in \underline{M}$ is regarded as a Kummerian image of the equation -1. The transformators X, transforming -1 into R, form the element $I_{-1R} \in \bar{\bar{E}}_d$, $\bar{\bar{E}}_d$ naturally being the right decomposition of $\mathfrak{G}$, generated by the fundamental group $\mathfrak{E}$. Since X are integrals of the equation (-1R): $-\{X,t\} - X'^2 = R(t)$, they coincide exactly with the phases of R. The decomposition $\bar{\bar{E}}_d$ consists, therefore, of phases of single equations of the system $\underline{M}$. Every equation $R \in \underline{M}$ is a Kummerian image of -1 by the phases of R. That is the role of phases in the theory we are dealing with. Note that by a phase of R we understand any phase function A given by the formula $A(t) = {}_{\nu}\tan^{-1}(U(t):V(t))$; U, V denote linearly independent integrals of R. Note, in particular, that the phases of the equation R depend only on R; they originate, so to say, from the interior of R ([11]).

It is evident that the above objects associated with any equation $R \in \underline{M}$, e.g. integrals, adjoint groups, etc., may be expressed by the corresponding objects of the equation -1 and the phases of R.

Let A denote a phase of R. Then, for the integrals and the adjoint groups of R, we have:

$$Y(t) = \frac{c}{\sqrt{|A'(t)|}}\cdot\sin(a+A(t));\ 0 \leqslant a < \pi;\ c \neq 0\ (a,c = \text{const.})$$

$$\mathfrak{A}_R = A^{-1}\mathfrak{H}A\ ,\quad \mathfrak{B}_R = A^{-1}\mathfrak{E}A\ ,\quad \mathfrak{B}^+_R = A^{-1}\mathfrak{E}^+A,\quad \mathfrak{C}_R = A^{-1}\mathfrak{Z}A\ ,$$

etc. Thus we find a powerful analytic instrument well adapted to research in the considered domain and functioning excellently.

Let us be satisfied with this information without a detailed consideration of the above theory.

IV. EQUATIONS WITH π-PERIODIC CARRIERS

10. Introduction. In the above considerations we have met with equations with π-periodic carriers. There naturally arises a question concerning the relations between the classical theory of Floquet and the theory we have just exposed. As a matter of fact, these relations open a way to extend Floquet's theory in case of 2nd order equations. - We speak, simply, about π-periodic equations.

11. A brief outline of the algebraic theory of π-periodic equations. With regard to the above algebraic theory of oscillatory equations we may extend the classical theory of π-periodic equations in two directions: On one hand, by new notions, e.g., dispersions, inverse equations, etc., in case of π-periodic equations. On the other hand, by relations between Floquet's theory and the new notions we have just mentioned. In what follows we present a brief aspect of the region surrounding Floquet's theory in the case of 2nd order equations.

Let $\mathscr{P}$ (= $\mathscr{P}_{-1}$) be the class composed of all π-periodic carriers (equations). For $R \in \underline{M}$ let R^- or R^* be an inverse or an associated carrier (equation) of R with regard to -1, respectively.

Proposition 1. If $R \in \mathscr{P}$ then $\mathfrak{C}_R \subset \mathfrak{H}$.

Proof. If $R \in \mathscr{P}$ then $\mathfrak{B}_R \supset \mathfrak{Z}$, by the 3rd inclusion theorem; this and $\mathfrak{A}_R \supset \mathfrak{B}_R$ imply $\mathfrak{A}_R \supset \mathfrak{Z}$. This implies $\mathfrak{C}_R \subset \mathfrak{H}$, by the 1st inclusion theorem.

Proposition 2. If $R \in \mathscr{P}$ then $\mathfrak{C}_{R^-} \subset \mathfrak{C}$ and vice versa.

For the proof, see [5].

Proposition 3. If $R \in \mathscr{P}$ then $R^* \in \mathscr{P}$.

For the proof, see [5].

Proposition 4 (the conservation law of periodicity factors). The periodicity factors of any two associated π-periodic equations are the same.

For the proof, see [7].

Proposition 5. The periodicity factors of any two equations inverse to concentric equations with center lying in $\mathfrak{C}$, are the same.

For the proof, see [11].

For more detail and results concerning the algebraic theory of π-periodic equations, see [8], [9], [10].

V. FINAL REMARK

Further development of the theory of differential linear equations will render it possible to judge the influence of the theory of global properties of 2nd order equations on the progress of the theory of linear equations. We mean, in particular, the progress of the theory of the 2nd order equations (complex domain, numerical methods, etc.) as well as the problem of global equivalence for $n > 2$. In any case the remarkable results in the field of the latter, presented in the recent papers of F. Neuman, are most encouraging ([2]).

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