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ON VARIOUS PROPERTIES OF THE SOLUTIONS OF THIRD- AND FOURTH-ORDER LINEAR DIFFERENTIAL EQUATIONS

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The subject of this paper will be second-order differential equations with a right-hand side and the third- and fourth-order differential equations. It will be assumed that the coefficients of the differential equations and the corresponding derivatives used in the considerations are real functions in the interval (a, ∞) (where $-\infty \leqq a$). By solutions we shall mean only real solutions. We shall call a function $F(x)$ *oscillatory*, when the set of its zero-points is infinite and unbounded from above. Otherwise, we shall call it *non-oscillatory*. The differential equations will be divided into three basic types: A differential equation will be called *non-oscillatory*, when all its solutions are non-oscillatory; *strictly oscillatory*, when all its solutions are oscillatory; and *oscillatory*, when some of its solutions are oscillatory and some non-oscillatory. Non-oscillatory and strictly oscillatory equations will denote *equations having the same character of solution*. We shall say that two differential equations have *the same character*, if both are either non-oscillatory or both strictly oscillatory.

It is well known that all solutions of a second-order linear differential equation have the same character, i.e. either all solutions are oscillatory or all are non-oscillatory. We shall now examine this property while dealing with second-order linear differential equations with a right-hand side, and with third- and fourth-order linear differential equations.

We will first consider the equations

$$z'' + p(x)\, z = f(x)\,, \tag{1}$$

$$y'' + p(x)\, y = 0\,. \tag{2}$$

We can easily find examples of p and f such that the solutions of equation (1) are not all of the same character, as well as examples such that the solutions are all of the same character. There is, for instance, the equation $y'' - y = \sin x$ and the equation $y'' + y = \exp \sin x$. The last equation has only oscillatory solutions. We will first deal with the question of the influence of the right-hand side $f(x)$ on the change of the character of the differential equation (1). If z is the solution of equation (1) and y that of equation (2), then for its wronskian the following equation holds:

$$W(z, y) = c - \int_{x_0}^{x} f(t)\, y(t)\, \mathrm{d}t\,. \tag{3}$$

From this the validity of the following statements follows immediately:

Theorem 1. *If the function $W(z, y)$ is a non-oscillatory function for every c, then the character of equations* (1) *and* (2) *is the same.*

Theorem 2. *If $\widetilde{W}(z, y) = \int_{x_0}^{x} f(t)\, y(t)\, \mathrm{d}t$ is a strictly monotonic function, then the character of equations* (1) *and* (2) *is the same.*

Theorem 3. *If equation* (2) *is non-oscillatory and $f(x)$ shows a constant sign for large values of x, then equation* (1) *is non-oscillatory.*

Theorem 4. *If $f(x) = g(x)\, y(x)$, where $g(x) \neq 0$ and $y(x)$ is the solution of equation* (2), *then the character of equations* (1) *and* (2) *is the same.*

Remark. For equation

$$z'' + az' + bz = f(x) \tag{1'}$$

and

$$y'' + ay' + by = 0 \tag{2'}$$

we obtain

$$W(z, y) = \exp\left\{-\int_{x_0}^{x} a\, \mathrm{d}t\right\}\left[c - \int_{x_0}^{x} f(t)\, y(t) \exp\left\{\int_{x_0}^{t} a\, \mathrm{d}\tau\right\} \mathrm{d}t\right].$$

Theorems 1–4 remain valid.

We shall now confine our attention to the case where $f(x)$ is of constant sign in the interval (a, ∞) and where equation (2) is (strictly)-oscillatory. We now want to find out whether or not there is always a non-oscillatory solution of equation (1) or whether or not there is always an oscillatory solution. The answer in both cases is negative, as can be shown by the following examples. The equation $y'' + y = \exp \sin x$ has only oscillatory solutions, while the equation $y'' + y = e^x$ has only non-oscillatory ones. The kind of solution which can be found for equation (1) depends on both functions $p(x)$ and $f(x)$. We shall point to a method of finding the conditions on p and f, which guarantee the existence of at least one non-oscillatory solution, or the existence of only non-oscillatory solutions, or of only oscillatory solutions.

If f has a constant sign in the interval (a, ∞), then function (3) is a non-constant solution of the differential equation

$$W''' - 2\frac{f'}{f} W'' + \left[p + f \cdot \left(\frac{1}{f}\right)''\right] W' = 0 \tag{4}$$

and the solutions of equation (1) constitute a certain subset of the solutions of the equation

$$u''' - \frac{f'}{f} u'' + pu' + \left(p' - \frac{f'}{f} p\right) u = 0\,. \tag{5}$$

If y_1, y_2 constitute a fundamental system of solutions of equation (2) and z is a solution of equation (1), then y_1, y_2, z constitute a fundamental system of solutions of equation (5). Hence every solution u of equation (5) can be written in the form

$$u = c_1 y_1 + c_2 y_2 + c_3 z \, .$$

This expression implies that every solution of equation (5) is either a solution of equation (2) or a certain multiple of a solution of equation (1). From this fact we can derive the following statement.

Theorem 5. *If equation* (5) *is strictly oscillatory, then so is equation* (1). *If equation* (5) *has a non-oscillatory solution which is not a solution of equation* (2), *then* (1) *has a non-oscillatory solution, too.*

The relation between equations (4) and (5) is such that, if u_1, u_2 vary over the set of all solutions of equation (5), then $W(u_1, u_2)$ varies over the set of all solution of equation (4). The function $w = (1/f)\ W$ is the solution of an equation which is the adjoint to equation (5). Equation (4) evidently has the solution $W =$ const, which corresponds to the wronskian of the two solutions of equation (2). Every other solution of equation (4) (modulo of a multiplication factor) can be obtained as a wronskian of two suitable solutions of equation (5), of which at least one is not a solution of equation (2). Hence we can express Theorem 2 in this form:

Theorem 6. *If equation* (4) *has a strictly monotonic solution, then the solutions of equation* (1) *and equation* (2) *have the same character.*

Equations (4) and (5) are third-order linear differential equations. In certain cases the character of these equations enables us to infer the character of equation (1). This is why the following part will be devoted to third-order linear differential equations and notably to their character.

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As for the character of third-order linear differential equations $L_3 y = 0$, every possible case is represented, i.e. it can be I) *non-oscillatory,* II) *strictly oscillatory,* III) *oscillatory.*

We can easily find an equation of type I among the equations of constant coefficients. As an instance of an equation of type II we can point to equation (5), if $p = 1$, $f = \exp \sin x$. Even an equation of the form $y''' + Q(x)\, y = 0$ can be of type II; this case was mentioned by Kondraťev in [1], which contains sufficient conditions.

With respect to equations of type III, we can distinguish three special cases:

IIIa) There is only one non-oscillatory solution (except for a constant factor of multiplication). There are a number of instances even among equations of constant coefficients, and more will be said about this later.

IIIb) There is a two-parameter set of oscillatory solutions (e.g. equations with the fundamental systems e^x, $\sin x$, $\cos x$).

IIIc) There is only one oscillatory solution (except for a constant factor of multiplication). Take as an example an equation having 1, $\int_0^x \sin t \exp\{\sin t\}\, dt$, $e^{-x}\cos x$ as a fundamental system.

We will now put forward several statements concerning these various types of equations. According to Mammana [2], the necessary and sufficient condition for the equation $L_3 y = 0$ being decomposed into $L_1 L_2 y = 0$ is the fact that there is a pair of solutions y_1, y_2 such that their wronskian $W(y_1, y_2) \neq 0$ for $x > x_0$. This condition and the Theorem 3 imply the validity of

Theorem 7. *If the equation $L_3 y = 0$ has two non-oscillatory solutions, whose wronskian for large values of x differs from zero, then the equation is of type I, i.e. it is non-oscillatory.*

Let us denote by $\bar{L}_3 y = 0$ the adjoint equation to $L_3 y = 0$. The following relation between them is well known: If y_1, y_2 vary over the set of solutions of the equation $L_3 y = 0$ and W is the wronskian of their fundamental set of solutions, then $W(y_1, y_2)/W$ varies over the set of all solutions of the equation $\bar{L}_3 y = 0$. The converse also holds.

On this basis we can assert that the necessary and sufficient condition for the validity of $L_3 y = L_1 L_2 y$ is the existence of a non-oscillatory solution of the adjoint equation $\bar{L}_3 y = 0$. This can be used in inquiries into the properties of the equation $L_3 y = 0$, i.e. in studying the properties of subsets of the solutions of this equation such that they constitute a set of solutions of the equation $L_2 y = 0$. In studying these subsets we can then apply the same resources used for the inquiry into the properties of the second-order linear equations. This has been done for instance by M. Greguš in his papers [3], [4] and [13], where his inquiry dealt with sets of integrals having a zero in the point x_0 (the so-called beam in the point x_0). He derived the conditions under which this set could be described by a differential equation $L_2 y = 0$, found its explicit expression and studied the relations between the beams in various points. He then used the properties which he was able to derive and solved a number of two- or three-point boundary problems.

From Theorem 7 and the properties of the adjoint equation, mentioned above, the validity of the following statements can be maintained:

Theorem 8. *If equation $L_3 y = 0$ is of type I, then $\bar{L}_3 y = 0$ is of type I or II.*

Theorem 9. *If equation $L_3 y = 0$ is of type IIIc), then it cannot be decomposed into the form $L_1 L_2 y = 0$, and $\bar{L}_3 y = 0$ is of type II (i.e., if the equation $L_3 y = 0$ has only one oscillatory solution, then the wronskian of any two of its solutions is an oscillatory function).*

Let equation $L_3 y = 0$ be of type IIIc), let z be its only oscillatory solution, and let u be any other arbitrary solution which is linearly independent of z (it is evidently non-oscillatory). Then decomposition is possible $L_3 y = L_2 L_1 y = 0$, where $L_1 u = 0$. Let $L_1 y = v$, then $L_2 v = 0$. Let $V = c_1 v_1 + c_2 v_2$ be a general solution of the equation $L_2 v = 0$. Then we obtain the solution of the equation $L_3 y = 0$ as a solution of the equation

$$L_1 y = V, \quad \text{i.e.} \quad y' - \frac{u'}{u} y = \frac{1}{u} V.$$

Therefore the solutions of the equation $L_3 y = 0$ have the form

$$y = u \left[c_3 + \int_{x_0}^{x} \frac{1}{u^2} (c_1 v_1 + c_2 v_2) \, dt \right]. \tag{6}$$

There is only one triple of numbers (except for linear dependence) c_1^*, c_2^*, c_3^* such that

$$z = u \left[c_3^* + \int_{x_0}^{x} \frac{1}{u^2} (c_1^* v_1 + c_2^* v_2) \, dt \right]. \tag{7}$$

This, however, means that the expression in the outer brackets is an oscillatory function and hence that $c_1^* v_1 + c_2^* v_2$ is an oscillating integral of the equation $L_2 v = 0$ and therefore that this equation is an oscillatory one. Furthermore, since z is the only oscillatory solution of the equation $L_3 y = 0$ it follows that

$$\lim_{x \to \infty} \left[c_3^* + \int_{x_0}^{x} \frac{1}{u^2} (c_1^* v_1 + c_2^* v_2) \, dt \right] = 0. \tag{8}$$

From these considerations we can derive the statement

Theorem 10. *Let $L_3 y = 0$ be of type IIIc) and z its only oscillatory solution, let y be another of its solutions linearly independent of z. Then $\lim_{x \to \infty} z/y = 0$. If $\lim_{x \to \infty} z \neq 0$, then y is unbounded.*

In case of $L_3 y = 0$ being of type IIIa), i.e. if there is only one non-oscillatory solution (let us call it u), then we obtain for another arbitrary solution y, which is linearly independent of u, the expression (6). From this expression it follows that in such a case necessarily

$$\limsup_{x \to \infty} \left[\int_{x_0}^{x} \frac{1}{u^2} (c_1 v_1 + c_2 v_2) \, dt \right] = \infty,$$

$$\liminf_{x \to \infty} \left[\int_{x_0}^{x} \frac{1}{u^2} (c_1 v_1 + c_2 v_2) \, dt \right] = -\infty$$

for every c_1 and c_2 which are not simultaneously zero.

From this follows

Theorem 11. *Let equation $L_3 y = 0$ be of type IIIa) and u be its only non-oscillatory solution, let y be another arbitrary solution which is linearly independent of u. Then y/u is an unbounded function for large values of x. If y is a bounded solution, then* $\lim_{x \to \infty} u = 0$.

We will now consider the possibility of determining the character or the type of a third-order linear differential equation by using its coefficients. In the literature on the linear equation $L_3 y = 0$ we frequently find an inquiry into the existence of at least one non-oscillatory solution (Kneser [5], Fite [6], Mikusiński [7], Kondraťev [1], Sansone [9], M. Hanan [8]). A whole series of conditions for the existence of a non-oscillatory solution is implied by the following properties which the equation $L_3 y = 0$ can exhibit:

(V_1). *Every solution of the equation $L_3 y = 0$, which has a double zero in the point x_0, has no zero less then x_0.*

(V_2). *Every solution of the equation $L_3 y = 0$, which has a double zero in the point x_0, has no zero larger than x_0.*

In Hanan's paper, mentioned above, the properties of the solutions of the equations showing the property (V_1) or (V_2) have been thoroughly investigated. One important property of the equations $L_3 y = 0$ may be pointed out: *If an equation has the property* (V_1), *then the adjoint equation has the property* (V_2), *and conversely.*

Hence all solutions of an equation with property (V_1) having a zero are of the same character. Zeros of solutions going to the right from a fixed zero, bracket each other their zeros to the right of it.

The definition of property (V_2) points clearly to the fact that an equation with this property has a non-oscillatory solution. In case of property (V_1) the construction of a non-oscillatory solution can proceed in the following manner:

We select a sequence of points $\{x_n\}_{n=1}^{\infty} \to \infty$ and construct a sequence of solutions $\{y_n(x)\}_{n=1}^{\infty}$ such that $y_n(x_n) = y'_n(x_n) = 0$, $y_n^2(x_0) + y'_n(x_0) + y''^2_n(x_0) = 1$. On the basis of the last condition we can prove that there is a subsequence of the sequence $\{y_n(x)\}_{n=1}^{\infty}$ which converges to a certain solution $y(x)$ having either no zero (as for instance in the case of a differential equation which has an oscillatory solution), or having at most one zero its multiplicity being two. If this point is x_1, then every solution has at most two zeros in the interval (x_1, ∞).

To get down to details, we should like to consider first the equation

$$y''' + Ay' + By = 0 . \tag{9}$$

If we multiply this equation by y or y', we can easily derive the following identities which are valid for solutions with a double zero in x_0:

$$y'y'' - \tfrac{1}{2}y'^2 + \tfrac{1}{2}Ay^2 = \int_{x_0}^{x} (\tfrac{1}{2}A' - B)\, y^2 \,\mathrm{d}t\,, \tag{10}$$

$$(11) \qquad y'y'' + \tfrac{1}{2}By^2 = \int_{x_0}^{x} (y''^2 - Ay'^2 + \tfrac{1}{2}B'y^2)\,\mathrm{d}t\,.$$

These identities easily provide the conditions such that equation (10) has the property (V_1) or (V_2) and, hence, a non-oscillatory solution. If we then go into greater detail, we can determine some further properties of the examined non-oscillatory solution or other cases of solutions. (See [4] to [17] and [19].)

The following statements hold:

Theorem 12. *Put $\frac{1}{2}A' - B \leqq 0$ and let the sign $=$ not be valid in any interval. Then equation (9) has the property (V_1) and there is a solution $y(x)$ which has no zero. All solutions having a zero have the same character. The solutions belonging to one beam (i.e. having a common zero) bracket each other their zeros lying behind the common zero. If, moreover, $A \leqq 0$, then it holds for its non-oscillatory solution $y(x)$:*

$$yy'y'' \neq 0\,, \quad \operatorname{sgn} y = \operatorname{sgn} y'' \neq \operatorname{sgn} y'\,, \quad \lim_{x\to\infty} y' = 0\,.$$

Theorem 13. *Put $A' \leqq 0$, $A \leqq 0$, $B \geqq 0$ and let the sign $=$ in the last inequality not be valid in any interval. Then all the statements of the preceding theorem are valid. Furthermore, the following statement is valid: y, y', y'' are monotonic functions, $\lim_{x\to\infty} y = \lim_{x\to\infty} y' = \lim_{x\to\infty} y'' = 0$. All the other solutions, which are not conditioned by $\operatorname{sgn} y = \operatorname{sgn} y'' \neq \operatorname{sgn} y'$, have the same character. If there is an oscillatory solution, then the solution y is the only one that does not oscillate — the equation is therefore of type IIIa).*

Theorems 12 and 13 are generalizations of the theorem proved for the equation $y''' + By = 0$ in [14] and of the theorem by G. Vilari [12] and M. Greguš [13].

Theorem 14. (G. Vilari). *In the equation*

$$(12) \qquad y''' + b_1y'' + b_2y' + b_3y = 0$$

let $b_2 \leqq 0$, $b_3 \geqq 0$. Then all its solutions have the same character, with the contingent exception of such a solution y, for which $yy' < 0$, $y'y'' < 0$.

The statement by M. Greguš is similar to Theorem 13, though its presumptions have the following form: $A \leqq 0$, $\frac{1}{2}A' - B \leqq 0$, $B - \frac{1}{2}A' - |A'| \geqq k > 0$.

Theorem 15. *Let $\frac{1}{2}A' - B \geqq 0$, where the sign $=$ is not valid in any interval. Then equation (9) has a non-oscillatory solution. If $A \leqq 0$ then the non-oscillatory solution y is such that $y'' > 0$, $y' > 0$, $\lim_{x\to\infty} y = \infty$.*

From Theorems 12 and 15 follows

Theorem 16. *A necessary condition for equation* (9) *being of type II, i.e. being strictly oscillatory, is the oscillatory character of the function* $\frac{1}{2}A' - B$.

From the identity (11) we obtain the following statement:

Theorem 17. *Let* $A \leqq 0$, $B \geqq 0$, $B' \geqq 0$ *where in the inequality* $B \geqq 0$ *the sign* $=$ *is not valid in any interval. Then the statements of Theorem 13 are valid.*

From (9) we obtain

Theorem 18. *Let* $A \leqq 0$, $B \leqq 0$. *Then equation* (9) *has a non-oscillatory solution* y *such that* $\lim_{x\to\infty} y = \lim_{x\to\infty} y' = \infty$ *and* $y'' > 0$ *is increasing. Equation* (9) *has the property* (V_2).

Statements of a similar kind and by similar methods can be gained for equation (12).

By using Theorems 12 to 18 and applying them to the equation which is the adjoint to equation (9), i.e. to the equation

$$W''' + AW' + (A' - B)\,W = 0 \tag{13}$$

we immediately obtain the statements (see [9], [15]):

Theorem 19. *If* $A \leqq 0$, $B \geqq 0$, $\frac{1}{2}A' - B \geqq 0$, *then equation* (9) *and equation* (13) *are non-oscillatory.*

Theorem 20. *If* $A \leqq 0$, $A' - B \geqq 0$, $\frac{1}{2}A' - B \leqq 0$, *then equations* (9) *and* (13) *are non-oscillatory.*

If we apply Theorem 18 to the equations (9) and (13), we obtain

Theorem 21. *Let* $A \leqq 0$, $B \leqq 0$, $A' - B \leqq 0$. *Then equations* (9) *and* (13) *are non-oscillatory and every its solution has at most two zeros.*

If we apply the results of Theorem 12 to 21 to equations (4) and (5), which can be transformed into (9), we get a considerable number of conditions for the functions $p(x)$ and $f(x)$, ensuring that equation (1) has a non-oscillatory solution, or that the character of equations (1) and (2) is the same. For instance, we obtain the result: If $f = \text{const}$ and $p' \geqq 0$ or $p' \leqq 0$, then equation (1) always has a non-oscillatory solution.

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We will now turn our attention to the fourth-order linear differential equations. Here again all three possibilities may occur, i.e., all solutions may be oscillatory, all may be non-oscillatory, and also there may be oscillatory and non-oscillatory solutions. We will describe a method of obtaining certain criteria which allow us to determine whether or not the examined equation is one having solutions of the same character.

If we consider Theorem 4, we see immediately that the differential equation $L_4 y = 0$ certainly has all solutions of the same character, if there exists the decomposition

$$L_4 = \lambda L_2 L_2 \tag{14}$$

where $\lambda \neq 0$ for $x > a$. If the computation is carried out and the coefficients compared, we have

Theorem 22. *The necessary condition for the validity of the decomposition* (14) *is the possibility of transforming the equation $L_4 y = 0$ by a transformation of the form $y = F(x)\,u$, $F(x) \neq 0$ into a self-adjoint form, which is equivalent to Halphen's semi-invariant of the operator L_4 being zero.*

Let us therefore consider the equation

$$a_0 y^{(4)} + a_1 y''' + a_2 y'' + a_3 y' + a_4 y = 0\,. \tag{15}$$

If (14) is valid, we must be able to transform this equation by a transformation of the form $y = F(x) u$ into the form

$$u^{(4)} + (P_2 u')' + P_4 u = 0 \tag{16}$$

or the form

$$(\vartheta_0 u'')'' + (\vartheta_2 u')' + \vartheta_4 u = 0\,. \tag{17}$$

It may be pointed out that equation (17) can always be transformed into (16) (see [18]). It can further be shown that

Theorem 23. *The necessary and sufficient condition for the decomposition of equation* (16) *in the form of* (14) *where $L_2 = q_0 D'' + q_1 D + q_2$, is q_0 being the solution of the equation*

$$q_0 q_0^{(4)} - q_0' q_0''' + \tfrac{1}{2} q_0''^2 P_2'' + q_0 q_0' P_2' + (2 q_0 q_0'' - q_0'^2)\, P_2 + \tfrac{1}{2} q_0^2 P_2^2 = 2 q_0^2 P_4 \tag{18}$$

which is in some interval (x_0, ∞) different from zero.

Hence, as a first result for $q_0 = 1$ we obtain

Theorem 24. *If $2P_2'' + P_2^2 - 4P_4 = 0$, then equation* (16) *has all solutions of the same character.*

We obtain further results, if we analyse equation (18). Namely the solutions of this equation constitute a subset of solutions of an equation, whose set of solutions is formed by all functions of the form $W(u_1, u_2)$ if u_1 and u_2 vary over the whole set of solutions of equation (16).

On this basis we can express Theorem 23 in another way:

Theorem 25. *The necessary and sufficient condition that equation* (16) *has all solutions of the same character is that equation* (16) *has solutions u_1 and u_2 such that*

$$u_1(x_0) = u_1'(x_0) = u_2(x_0) = u_2'(x_0) = 0\,, \quad W(u_1, u_2) \neq 0$$

for $x > x_0$.

Solutions u_1 and u_2 of the preceding theorem certainly have the demanded property, if equation (16) has the property

(V_3). *Every solution has at most one double zero.*

Let us consider the following fact: If y_1, y_2 are solutions of equation (15) and $y_1 = F(x)\,u_1$, $y_2 = F(x)\,u_2$, then $W(y_1, y_2) = F^2(x)\,W(u_1, u_2)$. From this we can conclude that the following theorems are true:

Theorem 26. *If the differential equation* (15) *has a Halphen semi-invariant of zero-value and has the property* (V_3), *then all its solutions have the same character.*

Theorem 27. *If equation* (16) *or* (17) *have property* (V_3), *then all the solutions are of the same character.*

Let us now look for conditions on the coefficients ϑ_0, ϑ_2, ϑ_4 so that equation (17) has property (V_3). If we multiply it by u and adjust it, then for the solution $u(x)$ with a double zero in x_0 we get the following identity [18]:

$$u(\vartheta_0 u'')' - \vartheta_0 u' u'' + \vartheta_2 u u' = -\int_{x_0}^{x} [\vartheta_0 u''^2 - \vartheta_2 u'^2 + \vartheta_4 u^2]\,dt\,. \tag{19}$$

Hence

Theorem 28. *Let $\vartheta_0 > 0$, $\vartheta_2 \leqq 0$, $\vartheta_4 \geqq 0$. Then equation* (17) *has property* (V_3) *and, consequently, all its solutions have the same character.*

For $\vartheta_0 = 1$, $\vartheta_2 = 0$, $\vartheta_4 \geqq 0$ this theorem is proved in [14]. For $\vartheta_0 > 0$, $\vartheta_2 = 0$, $\vartheta_4 \geqq 0$ it is proved in [16].

In [17] the following statement is proved:

Lemma 1. *If the equation $y'' + p(x)\,y = 0$, where $p(x) > 0$, is non-oscillatory, then there is a number $c > a$ such that for two arbitrary numbers $c < x_0 < x$ and* *an arbitrary function* $v \in D'(c, \infty)$ *such that* $v(x_0) = 0$ *the following is valid*:

$$\int_{x_0}^{x} p(t)\, v^2(t)\, \mathrm{d}t < \int_{x_0}^{x} v'^2(t)\, \mathrm{d}t\,.$$

On the basis of this Lemma we obtain from (19)

Theorem 29. *Let* $\vartheta_0 \geqq 1$, $\vartheta_4 \geqq 0$ *and* $\vartheta_2 > 0$, *and let the equation*

$$z'' + \vartheta_2 z = 0$$

be non-oscillatory. Then equation (17) *has property* (V_3) *in a certain interval* (c, ∞) *and consequently all its solutions are of the same character.*

Theorem 30. *Let* $\vartheta_0 > 0$, $\vartheta_2 \leqq -1$, $\vartheta_4 < 0$ *and let the equation*

$$z'' + |\vartheta_4| z = 0$$

be non-oscillatory. Then all the solutions of equation (17) *have the same character.*

In their paper [16] Leighton and Nehari assert the following

Lemma 2. *Let* $v(x) \in D''(x_0, \infty)$ *and let* $v(x_0) = v'(x_0) = v(x_1) = v'(x_1)$, $x_0 < x_1$. *Let* $r(x) > 0$, $p(x) > 0$ *and let the equation*

$$(ry'')'' - py = 0 \tag{20}$$

be non-oscillatory in the interval (x_0, ∞). *Then the folloving is true:*

$$\int_{x_0}^{x_1} pv^2\, \mathrm{d}t \leqq \int_{x_0}^{x_1} rv''^2\, \mathrm{d}t\,.$$

Remark. The differential equation is non-oscillatory in (x_0, ∞), if all its solutions in this interval have at most three zeros. If equation (20) is non-oscillatory, then there is a number x_0 such that it is non-oscillatory in (x_0, ∞).

Theorem 31. *Let* $\vartheta_0 > 0$, $\vartheta_2 < 0$, $\vartheta_4 < 0$ *and let equation*

$$(\vartheta_0 z'')'' + \vartheta_4 z = 0$$

be non-oscillatory. Then all the solutions of equation (17) *are of the same, non-oscillatory, character.*

REFERENCES

[1] V. A. KONDRAT'EV: Oscillation of solutions of linear equations of third and fourth order. Trudy Moskov. Mat. Obšč. *8* (1959), 259–281 (Russian).

[2] G. MAMMANA: Decomposizione delle espressioni differenziali lineari omogenee in prodotti di fattori simbolici e applicazione relativa allo studio delle equazioni differenziali lineari. Math. Z. *33* (1931), 186–231.

[3] M. GREGUŠ: O niektorých nových vlastnostiach riešení diferenciálnej rovnice $y''' + Qy' + Q'y = 0$. Publ. Fac. Sci. Univ. Masaryk *365* (1955), 1–13.

[4] M. GREGUŠ: O niektorých vlastnostiach riešení lineárnej diferenciálnej rovnice tretieho rádu. Mat.-Fyz. Časopis Sloven. Akad. Vied 5, *2* (1955), 73–85.

[5] A. KNESER: Untersuchungen über die reellen Nullstellen der Integrale linearer Differentialgleichungen. Math. Ann. *42* (1893), 408–435.

[6] W. B. FITE: Concerning the zeros of the solutions of certain differential equations. Trans. Amer. Math. Soc. *19* (1918), 341–352.

[7] J. MIKUSIŃSKI: On Fite's oscillation theorems. Colloq. Math. II (1949), 34–39.

[8] M. HANAN: Oscillation criteria for third-order linear differential equations. Pacific J. Math. *11* (1961), 919–944.

[9] G. SANSONE: Studii sulle equazioni differenziali lineari omogenee di terzo ordine nel campo reale. Univ. Nac. Tucumán Rev. Ser. A, *6* (1948), 195–253.

[10] M. RÁB: Oscilační vlastnosti integrálu diferenciální lineární rovnice 3. řádu. Práce Brn. Českoslov. Akad. Věd 27, *7* (1955).

[11] M. RÁB: O diferenciální rovnici $y''' + 2A(x)\,y' + [A'(x) + w(x)]\,y = 0$. Mat.-Fyz. Časopis Sloven. Akad. Vied 8, *2* (1958), 115–122.

[12] G. VILARI: Sul carattere oscilatorio delle soluzioni delle equazioni differenziali lineari omogenee del terzo ordine. Boll. Un. Mat. Ital. III. Ser. *13* (1958), 73–78.

[13] M. GREGUŠ: Über einige Eigenschaften der Lösungen der Differentialgleichung $y''' + 2Ay' + (A' + b)\,y = 0$, $A \leqq 0$. Czechoslovak Math. J. 11 (*86*) (1961), 106–114.

[14] M. ŠVEC: Sur une propriété des intégrales de l'équation $y^{(n)} + Q(x)\,y = 0, n = 3, 4$. Czechoslovak Math. J. 7 (*82*) (1957), 450–461.

[15] M. GREGUŠ: Oszillatorische Eigenschaften der Lösungen der linearen Differentialgleichungen dritter Ordnung. Czechoslovak Math. J. 9 (*84*), (1959), 416–428.

[16] W. LEIGHTON and Z. NEHARI: On the oscillation of solutions of selfadjoint linear differential equations of the fourth order. Trans. Amer. Math. Soc. *89* (1958), 325–377.

[17] Z. NEHARI: Oscillation criteria for second-order linear differential equations. Trans. Amer. Math. Soc. *85* (1957), 428–445.

[18] G. SANSONE: Le equazioni differenziali lineari, omogenee, del quarto ordine, nel campo reale. Ann. Scuola Norm. Sup. Pisa Ser. II, XI (1942), 151–195.

[19] M. ZLÁMAL: Asymptotic properties of the solutions of the third-order linear differential equations. Publ. Fac. Sci. Univ. Masaryk *329* (1951), 159–167.