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ON THE PROPERTIES OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER IN THE COMPLEX DOMAIN

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One of the important properties of linear differential equations with analytical coefficients consists in the possibility of their solutions to be continued along every curve, along which their coefficients can be continued. Therefore, all solutions of linear differential equations are arbitrarily continuable in a domain, in which their coefficients are holomorphic. If this domain is simply connected, then those solutions are single-valued, i.e., they are holomorphic functions. It is therefore useful to restrict the study of linear differential equations to equations which are defined in a simply connected domain, and further to second-order equations. This kind of equation can always be put into the form

$$y'' = Q(x)\, y\,, \tag{1}$$

where $Q(x)$ is a holomorphic function in the simply connected domain $T(\infty \notin T)$. We shall consider equation (1) in the important types of the domain T, namely on the open plane E, or in the unit circle K. By a solution of differential equation (1) we shall always understand a non-trivial solution, i.e., $y \not\equiv 0$. We shall further decompose the set of solutions of equation (1) into the classes in such a way that two solutions y_1, y_2 of equation (1) belong to the same class, if and only if they are linearly dependent.

Closely connected with differential equation (1) is Riccati's differential equation

$$w' = Q(x) - w^2$$

and Schwarzian differential equation

$$-\{z, x\} = Q(x)\,, \tag{2}$$

where Schwarzian derivative $\{z, x\} = \frac{1}{2}z'''/z' - \frac{3}{4}(z''/z')^2$. The connection between equations (1) and (2) is characterized by the fact that the set of fractions y_1/y_2 of two linearly independent solutions y_1, y_2 of equation (1) forms a set of all solutions of differential equation (2).

The theory of single-valued analytical functions finds its significant place in the theory of differential equation (1), as described in H. Wittich's monograph "Neuere Untersuchungen über eindeutige analytische Funktionen" and Wiman-Valiron's theory of the maximum term, the application of which can be found in G. Valiron's book "Fonctions analytiques". The application of the methods and results of these theories yield data on the growth of the solutions of differential equation (1).

1 Growth of the solution of differential equation (1) in K and on E

The character of the solution y of differential equation (1) is determined by the function $M(r, y) = \max_{|x| \leqq r} |y(x)|$ and the characteristic function $T(r, y)$. First of all, $M(r, y)$ is given by the estimate

$$\log M(r, y) \leqq \log (|y(0)| + |y'(0)|\, r) + \frac{r^2}{2} M(r, Q),$$

$0 \leqq r < R$, $R = 1$ or $R = +\infty$.

On E this estimate can be improved in the following way. If $Q(x)$ is a polynomial of the degree n ($\geqq 0$), then all the solutions of differential equation (1) are of order $\frac{1}{2}n + 1$ and of intermediate type. If $Q(x)$ is not a polynomial, every solution is of infinite order. In such a case, except for the set of finite logarithmical measure for the solution y the following estimate from above holds:

$$\log M(r, y) < [1 + o(1)] \int_{r_0}^{r} M^{1/2+\varepsilon}(r, Q)\, dr, \quad \varepsilon > 0$$

and the estimate from below

$$T(r, Q) \leqq A \log r + B \log T(r, y), \quad A > 0, \quad B > 0.$$

Though it is true that in K all solutions of differential equation (1) are of the same order, as soon as $T(r, Q) = O[\log (1 - r)^{-1}]$, we can obtain solutions of arbitrary finite order. The differential equation

$$y'' = \frac{(\lambda + 1)^2 + (\lambda + 1)(\lambda + 2)(1 - x)^{\lambda+1}}{(1 - x)^{2\lambda+4}}\, y$$

has a solution $y(x) = \exp [(1 - x)^{-\lambda-1}]$ whose $\log M(r, y) = (1 - r)^{-\lambda-1}$,

$$T(r, y) = \begin{cases} O\left[\left(\dfrac{1}{1 - r}\right)^{\lambda}\right], & \lambda > 0, \\ O\left(\log \dfrac{1}{1 - r}\right), & \lambda = 0. \end{cases}$$

Furthermore for all solutions y of differential equation

$$y'' = P\left(\frac{1}{x - 1}\right) y$$

(where P is a polynomial of the n-th degree) the following statement holds:

If $n = 1$, $\log M(r, y) = O(1)$. If $n = 2$, $\log M(r, y) = O[\log (1 - r)^{-1}]$.
If $n \geqq 3$ and $P(u) = a_3 u^3 + \ldots + a_n u^n$, then $\log M(r, y)$ is of the order $\frac{1}{2}n - 1$.

If $\overline{\lim}_{r \to 1-} [T(r, Q)/\log(1 - r)^{-1}] = \infty$ then all solutions of differential equation (1) are of infinite order. For all these solutions, with the exception of the set M, where $\int_M (1 - r)^{-1} \, dr < \infty$, the estimate from below holds:

$$T(r, Q) \leqq A \log \frac{1}{1 - r} + B \log T(r, y)$$

and there exists a sequence $r_n < r_{n+1} \to 1$, which allows the estimate from above

$$\log M(r_n, y) < A \int_{r_0}^{r_n} M^{1/2}(r, Q) \, dr + B \, M^{1/2}(r_n, Q) \, .$$

As regards the growth of the solution of Schwarzian differential equation, the order (and type on E) of its solution is equal to the order (and type) of the solution of differential equation (1). The solutions of Riccati's differential equation are, with the possible exception of two of them, of the same order as the solutions of differential equation (1). The excepted solutions are of lower order (type).

2 Transformation of differential equation (1)

We shall deal now with the transmission of Prof. O. Borůvka's theory of transformation into the complex domain. Equivalence and local equivalence of two differential equations of the form (1) are the basic concepts of this theory. They are introduced in the following manner:

The differential equation

$$v'' = Q_2(\xi) \, v \tag{1_2}$$

defined in the domain U, is equivalent in the subdomain $U_2 \subset U$ to the differential equation

$$y'' = Q_1(x) \, y \tag{1_1}$$

in the subdomain T_1 of the domain of definition T of this equation, if there are functions $t(x)$, $\xi(x)$ having the properties:

1) $t(x)$ and $\xi(x)$ are holomorphic in T_1, $\xi(T_1) = U_2$;
2) $t(x) \neq 0$, $\xi(x)$ is one-one in T_1

for which the following statement is valid:

If $v(\xi)$ is an arbitrary solution of differential equation (1_2) in the domain U_2, the composed function

$$y(x) = t(x) \, v[\xi(x)] \tag{3}$$

is a solution of differential equation (1_1) in the domain T_1.

More general is the conception of local equivalence, where $t(x)$ and $\xi(x)$ show the following properties:

1) $t(x)$ and $\xi(x)$ are generalized analytical functions in T_1, i.e. unempty systems of analytical elements with their centres in T_1, whereby two arbitrary elements of such a system are connected by a chain of elements of the same system. $t(x)$ and $\xi(x)$ are further defined in T_1 and $\xi(T_1) = U_2$;

2) $t(x) \neq 0$, $\xi(x)$ is locally one-one (every element $\xi(x)$ is invertible).

The difference between the theory of transformation in the real and complex domain is expressed by the fact that in the real domain the concept of local equivalence coincides with that of equivalence of differential equations (1_1) and (1_2).

The relation between equivalence and local equivalence of differential equations (1_1), (1_2) is reflexive and symmetric. The inverse relation to (3) is

$$v(\xi) = \frac{1}{t[x(\xi)]}\, y[x(\xi)]\,,$$

$x(\xi)$ is the system of inverse elements of $\xi(x)$.

The possibility of transforming differential equation (1_2) into the differential equation (1_1) depends only on the distribution of zero-points of the solutions of these equations, owing to the validity of the following theorem: The differential equation (1_2) is equivalent in U_2 with the differential equation (1_1) in T_1, if and only if there exists an isomorphic transformation Z of the set of solutions of the equation (1_2) into a set of solutions of the differential equation (1_1) and a conformal transformation $\xi(x)$ of the domain T_1 onto the domain U_2 in such a way, that if $v(\xi)$ is an arbitrary solution of the differential equation (1_2) and $y(x)$ its image in the transformation Z, then $\xi(x)$ transforms the set of zero-points of the solution $y(x)$ in T_1 into a set of zero-points $v(\xi)$ in the domain U_2. An analogous result holds for local equivalence. Two differential equations (1_1) and (1_2) are always locally equivalent, and the bearers of this equivalence are the generalized analytical functions: $\xi(x)$, which is a solution of the differential equation

$$-\{\xi, x\} + Q_2(\xi)\,\xi'^2 = Q_1(x)$$

and

$$t(x) = \frac{1}{\sqrt{\xi'(x)}}$$

so that formula (3) then takes the form

$$y(x) = \frac{v[\xi(x)]}{\sqrt{\xi'(x)}}\,.$$

The local equivalence of every differential equation (1) with the differential equation $v'' = 0$ implies that the solution $\xi(x)$ of the differential equation $\{\xi, x\} = 0$ transforms the zero-points of every solution of differential equation (1) into the zero-point of the solution of $v'' = 0$. Hence, the c-points of the function $\xi(x)$ are identical with the zero-points of the solution of differential equation (1). Thus an inquiry into the zero-points of the solutions of differential equation (1) can also be carried out by means of the inquiry into the distribution of the values of the fractions y_1/y_2 of two linearly independent solutions of this equation. This result is of great importance for the study of the zero-points of the solutions of differential equation (1). It can be proved directly and with its help we can again derive the local equivalence of two arbitrary differential equations (1).

3 Inquiry into the zero-points of the solutions of differential equation (1) and the zero-points of their derivatives

As already mentioned, the inquiry into the zero-points of the solutions of differential equation (1) can be reduced to the inquiry of the distribution of the values of the fraction y_1/y_2 of two linearly independent solutions of differential equation (1). The zero-points of the derivatives of the solutions are again identical with the c-points of the function y'_1/y'_2. The functions y_1/y_2 and y'_1/y'_2 are of the same order (on E of the same type) as the solutions of differential equation (1). From Picard's and Picard-Borel's theorem we can deduce the following statement: There are at most two classes of solutions y of the differential equation (1), so that the function $N(r, 1/y)$ is of lower order (type on E) than the solution y. Especially, there are at most two classes of solutions of equation (1), which have a finite number of zero-points. An analogous result holds for the zero-points of the derivatives of the solutions. This exceptional solution y is called Picard-Borel's exceptional solution (briefly P.B.e.) of the first or second order, respectively, according as to whether $N(r, 1/y)$ or $N(r, 1/y')$ are of lower order (type) than y. A solution with a finite number of zero-points, or the derivative of which has only a finite number of zero-points, is called Picard's exceptional solution (briefly P.e.) of the first or second kind, respectively. In K there exist solutions of zero order, which are P.e. solutions but not P.B.e. solutions. Apart from this, every P.e. solution is also a P.B.e. solution.

If $Q(x)$ is a polynomial, every solution which is a P.B.e. or P.e. solution of the first kind, resp., is also a P.B.e. or P.e. solution of the second kind, resp., and conversely. Furthermore, every P.B.e. solution is also a P.e. solution. An equation with a coefficient, which is constant and different from zero, has two classes of P. e. solutions. If $Q(x)$ is a polynomial of at least the first degree, there exists at most one class of P.e. solutions. An equation with an entire transcendental coefficient cannot have two linearly independent P.e. solutions of the first kind, of which at least one is also a P.e. solution of the second kind.

In K, every P.B.e. solution of the first kind of finite order is also a P.B.e. solution of the second kind, and conversely. As to the P.e. solutions, by using the principal theorem of conformal transformation we can prove the existence of a differential equation (1) in K, which has an arbitrary number of classes of solutions without a zero-point. Every solution of this equation, which has at least one zero-point, has an infinite number of them.

4 Construction problems of the theory of differential equation (1)

These problems are all derived from a problem set by Prof. O. Borůvka. Assume a sequence $\{a_n\}$ of mutually different points on E with at most one limit point in ∞. Find a differential equation (1) such that one of its solutions has zero-points just in the points a_n. Using Weierstrass' product and Pringsheim's construction of the entire function with prescribed values at the given sequence of points, the following problem can be solved: Find all differential equations (1) with an entire coefficient, one of the solutions of which, together with its derivative, have prescribed zero-points, or respectively, a couple of linearly independent solutions of which have prescribed zero-points. The problem has a positive solution and there exists an uncountable set of differential equations (1) with the given properties. The same holds also for the unit circle and, by the transformation of differential equation (1), we can transmit the result into an arbitrary simply connected domain.

5 The question of the determination of differential equation (1) by the zero-points of its solutions

We have seen that differential equation (1) is not uniquely determined by the zero-points of two of its solutions. The question arises whether it is not uniquely determined by the zero-points of all its solutions, which means that there exists no differential equation (1_1), different from differential equation (1), such that to every solution u of equation (1) there is the solution s of equation (1_1) with the same zero-points as u; and conversely, to every solution t of differential equation (1_1) there exists a solution v of equation (1), which has the same zero-points as t. The properties of conformal transformation imply that equation (1), whose set of classes of solutions without zero-points is at most denumerable, is uniquely determined by the zero-points of its solutions. Thus, a differential equation with an entire coefficient is uniquely determined. This, however, need not be true in a unit circle and we shall deal with it in the next section.

6 Properties of differential equation (1) similar, in a sense, to a differential equation with a constant coefficient

The differential equation

$$y'' = ky\,, \quad 0 \neq k = \text{const}\,, \tag{4}$$

enjoys an interesting property, which we shall call property $\mathscr{A}$. To each of its solutions u there exists a solution v of the same differential equation such that the functions u, v' as well as the functions u', v have the same zero-points. The properties of conformal mapping imply that on E only equation (4) has the property $\mathscr{A}$. However, in the quadrant $0 < \operatorname{Re} x < \infty$, $0 < \operatorname{Im} x < \infty$ the property $\mathscr{A}$ is shown by the differential equation

$$y'' = \tfrac{3}{4}x^{-2}y$$

and in the strip $a < \operatorname{Im} x < b$, $0 < a < b$ for sufficiently large a by the differential equation

$$y'' = (\tfrac{1}{2} + \tfrac{3}{4}\operatorname{tg}^2 x - \tfrac{3}{16}\operatorname{cotg}^2 x)\, y\,.$$

The last equation shows moreover the properties:

1) Every solution is characterized by the fact that either y or y' have no zero-point at all.

2) Every solution, which has at least one zero-point, has infinitely many of them. The same is true of the zero-points of the derivative of the solution.

3) There exist uncountably many solutions of this equation which, as well as their derivatives, have no zero-point at all.

With the exception of equation (4), differential equations (1) with the property $\mathscr{A}$ are not uniquely determined by the zero-points of their solutions. Then the solutions of the differential equation

$$y'' = \left[Q(x) - \left\{\int Q\,\mathrm{d}x, x\right\}\right] y$$

have the same zero-points as the solutions of differential equation (1).

7 Estimates of the modulus and argument of the difference of two zero-points of the solutions of differential equation (1)

The Hille's equality for the arbitrary solution y of differential equation (1)

$$[\overline{y(x)}\, y'(x)]_{x_1}^{x_2} - \int_{x_1}^{x_2} |y'(x)|^2\, \overline{\mathrm{d}x} - \int_{x_1}^{x_2} |y(x)|^2\, Q(x)\,\mathrm{d}x = 0\,,$$

in which $x_1, x_2 \in T$ are two points and the integration follows the regular curve C which lies in T, implies the following estimates: If $|Q(x)| \leq M$ in the convex domain T and x_1, x_2 are two zero-points of the solution y, then $|x_1 - x_2| \geqq \pi M^{-\frac{1}{2}}$. If x_3 is the zero-point of the derivative y', then $|x_1 - x_3| \geqq \frac{1}{2}\pi M^{-\frac{1}{2}}$. If, moreover, the smallest closed convex set L, which contains the set $Q(T)$, has a positive distance m from the point O, and x_3, x_4 are two zero-points of the derivative y', then $|x_3 - x_4| \geqq \pi M^{-1} m^{\frac{1}{2}}$.

From these estimates we can deduce that every solution of differential equation (1) has in T a finite number of zero-points, if $|Q(x)|$ is bounded in a bounded convex domain T. Finally, if L lies in the angle $\alpha - \beta \leqq \operatorname{Arg} x \leqq \alpha + \beta$, where $0 \leqq \beta \leqq \frac{1}{2}\pi$ and x_1 is the zero-point of the function yy', then the other zero-points of this function lie in the angles

$$\tfrac{1}{2}(\pi - \alpha - \beta) \leqq \operatorname{Arg}(x - x_1) \leqq \tfrac{1}{2}(\pi - \alpha + \beta),$$
$$\tfrac{1}{2}(3\pi - \alpha - \beta) \leqq \operatorname{Arg}(x - x_1) \leqq \tfrac{1}{2}(3\pi - \alpha + \beta).$$