ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS AND ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACE

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The theory of elliptic boundary value problems in bounded regions is rather well developed, but in infinite domains these problems have not been studied so intensively. This is a report of some work with S. Agmon on elliptic (and parabolic) differential equations in an infinite cylinder.

The noncompactness of the region introduces some new features; for instance the well known statement, that the set of square integrable solutions of a homogeneous uniformly elliptic equation satisfying reasonable boundary conditions is finite dimensional, is not true in a cylinder. A counterexample is easily constructed with the aid of an example of A. Plis of a homogeneous elliptic equation with nontrivial solutions having compact support. In case the elliptic problem in the cylinder is invariant under translation (parallel to the generator) P. D. Lax has proved a general Phragmén-Lindelöf theorem: that the square integrable solutions decay exponentially; it follows easily that the set of solutions is finite dimensional. We have extended this result to elliptic equations whose coefficients tend to limiting values with sufficient rapidity at infinity.

The paper is concerned with a number of other questions concerning the behavior at infinity for solutions of elliptic problems. In studying these questions it is most convenient to formulate the partial differential equations in the usual way as an ordinary differential equation, in some suitable Banach space X (with norm $| \; |$), of the form

$$Lu = \frac{1}{i}\frac{du}{dt} - Au = f, \tag{1}$$

or as an inequality

$$|Lu| \leqq \Phi(t)\,|u|. \tag{2}$$

Here t represents the variable in the direction of the generator of the cylinder, A, a closed (unbounded) operator in the space, represents a partial differential operator in the other variables with coefficients independent of t, and $u(t)$, for every t, takes its values in the domain of A, $D(A)$, in the Banach space ($D(A)$ depends on the boundary conditions on the cylinder sides). Since the initial value (Cauchy) problem for elliptic equations is not well posed the class of equations (1) which we consider does not fall into the usual class leading to continuous semigroups; i.e. A is not the generator of such a semigroup.

Under various assumptions on the resolvent $R(\lambda) = (\lambda I - A)^{-1}$ of A we treat the following topics in addition to the one on stability mentioned above.

1) Regularity of solutions of $Lu = f$.

2) Asymptotic expansions of solutions of $Lu = f$ in terms of exponential solutions. By an exponential solution is meant one of the form $p(t)\, e^{i\lambda t}$ where $p(t)$ is a polynomial in t with coefficients in X (λ is then necessarily an eigenvalue of A).

3) Completeness of exponential solutions.

4) Unique continuation at ∞ for solutions of $Lu = 0$ and of (2); i.e., showing that solutions which decay very rapidly are zero, and also obtaining lower bounds for solutions.

The conditions we impose on the resolvent $R(\lambda)$ are usually related to those which arise naturally in connection with elliptic problems. For such problems $R(\lambda)$ is meromorphic in the complex plane, and is regular, and has norm $= 0(1/\lambda)$ as $|\lambda| \to \infty$ in an angle

$$\operatorname{Re}\lambda > c\,, \quad |\operatorname{Im}\lambda| \leqq K(\operatorname{Re}\lambda - c)$$

as well as its reflection in the imaginary axis.

The various results are proved using certain standard techniques: Fourier transform, the Paley-Wiener theorem and some complex function theory, such as contour deformation, Phragmén–Lindelöf theorems and the "three lines theorem". For all details as well as illustrations of the techniques see [1].

REFERENCES

[1] S. Agmon and L. Nirenberg: Properties of solutions of ordinary differential equations in Banach space. Comm. Pure Appl. Math. *16* (1963), 121–239.