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INTEGRAL MANIFOLDS AND NONLINEAR OSCILLATIONS

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In the last few years, the theory of nonlinear oscillations for ordinary differential equations has been developed at an outstanding rate. Many problems remain to be solved but some aspects of the theory are very well understood. Concurrently with the development of this theory, there has been renewed interest in oscillatory phenomena for differential-difference equations. The purpose of the present paper is to state some of the known results for ordinary differential equations together with some extensions to differential-difference equations and the difficulties involved in such extensions. Due to lack of space, periodic solutions are not treated in detail and the applications have been suppressed.

1 Almost periodic solutions

Consider the system of equations

$$
\begin{aligned}
\dot{x} &= \varepsilon[Ax + X(t, x, y, z, \varepsilon) + X_1(t, x, y, z, \varepsilon)]\,, \\
\dot{y} &= By + Y(t, x, y, z, \varepsilon)\,, \\
\varepsilon\dot{z} &= Cz + Z(t, x, y, z, \varepsilon)\,,
\end{aligned}
\tag{1}
$$

where x, y, z are vectors and the following hypotheses are satisfied:

(H_1) For each fixed ε, all functions are almost periodic in t uniformly with respect to x, y, z for $\|x\| \leqq R$, $\|y\| \leqq R$, $\|z\| \leqq R$, where R is a positive constant.

(H_2) X, Y, Z are continuous in t, x, y, z, ε, Lipschitzian in x, y, z for $-\infty < t < \infty$, $\|x\|, \|y\|, \|z\| \leqq R$, $0 < \varepsilon \leqq \varepsilon_1$ and the Lipschitz constant approaches zero as $\|x\|, \|y\|, \|z\|, \varepsilon \to 0$. Furthermore, for $x = 0$, $y = 0$, $z = 0$, the functions X, Y, Z are bounded by a continuous function $M(\varepsilon)$ with $M(0) = 0$.

(H_3) The eigenvalues of each of the matrices A, B, C have nonzero real parts.

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(H_4) $X_1(t, x, y, z, \varepsilon)$ is continuous together with its first partial derivatives with respect to x, y, z for $-\infty < t < \infty$, $\|x\|, \|y\|, \|z\| \leqq R$, $0 < \varepsilon \leqq \varepsilon_1$ and

$$\lim_{T \to \infty} T^{-1} \int_0^T X_1(\tau, x, y, z, \varepsilon)\, d\tau = 0 .$$

Theorem 1. *If system* (1) *satisfies hypotheses* (H_1)–(H_4) *then there exist* $\varepsilon_2 > 0$, $\sigma > 0$ *such that, for each* ε, $0 < \varepsilon \leqq \varepsilon_2$ *system* (1) *has a solution which is almost periodic in* t *and approaches zero uniformly in* t *as* $\varepsilon \to 0$ *and is unique in the region*

$$0 \leqq \|x\| + \|y\| + \|z\| \leqq \sigma .$$

If all the eigenvalues of A, B, C *have negative real parts, this almost periodic solution is asymptotically stable. If one eigenvalue has a positive real part, it is unstable.*

In the determination of almost periodic solutions of nonlinear differential equations which contain a parameter, the differential equations in many cases are not given in the form (1). However, by appropriate transformation of variables, many problems reduce to a study of (1). For applications and a discussion of these transformations, see the book of N. N. Bogolyubov and Yu. A. Mitropol'skiĭ [4] or their survey paper [5]. The term X_1 is included in system (1) in order to obtain results on almost periodic solution by an application of the method of averaging (see [4]). Notice that system (1) also has a singular perturbation term since for $\varepsilon = 0$ some of the high derivatives disappear.

1.1 *Idea of the proof.* By a well known lemma of N. N. Bogolyubov [3] (or [4]), hypothesis (H_4) implies there is a function $u(t, x, y, z, \varepsilon)$, which is almost periodic in t, such that the transformation

$$x \to x + \varepsilon\, u(t, x, y, z, \varepsilon)\,, \quad y \to y\,, \quad z \to z$$

applied to system (1) yields a system of the form (1) satisfying (H_1)–(H_3) and $X_1 \equiv 0$. In the following, we therefore assume $X_1 \equiv 0$. Without loss of generality, we can assume that each of the matrices A, B, C have the form

$$A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}, \quad B = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}, \quad C = \begin{pmatrix} C_+ & 0 \\ 0 & C_- \end{pmatrix},$$

where all the eigenvalues of the matrices designated by $+$ (by $-$) have positive real parts (negative real parts). For any matrix A of this type, define the matrix $J_A(t)$, $-\infty < t < \infty$ by the relations

$$(2) \qquad J_A(t) = \begin{pmatrix} e^{-A_+t} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for} \quad t > 0,$$

$$J_A(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{-A_-t} \end{pmatrix}, \quad \text{for} \quad t < 0,$$

$$J_A(-0) - J_A(+0) = I, \quad \text{the identity}.$$

The matrix $J_A(t) \to 0$ exponentially as $|t| \to \infty$. Let $\mathfrak{C}(D)$ be the class of functions

$$\mathfrak{C}(D) =$$

$$= \{x, y, z;\ x(t), y(t), z(t) \text{ are continuous and bounded by } D \text{ for } -\infty < t < \infty, \text{ where } D \text{ is a given positive constant}\}.$$

For any (x, y, z) in $\mathfrak{C}(D)$, define the transformation $\mathfrak{J}$ taking (x, y, z) into (u, v, w) by the relation

$$(3) \qquad u(t) = \varepsilon \int_{-\infty}^{\infty} J_A(\varepsilon\tau)\, X[t + \tau, x(t + \tau), y(t + \tau), z(t + \tau), \varepsilon]\, d\tau,$$

$$v(t) = \int_{-\infty}^{\infty} J_B(\tau)\, Y[t + \tau, x(t + \tau), y(t + \tau), z(t + \tau), \varepsilon]\, d\tau,$$

$$w(t) = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} J_C\left(\frac{\tau}{\varepsilon}\right) Z[t + \tau, x(t + \tau), y(t + \tau), z(t + \tau), \varepsilon]\, d\tau.$$

Now, it is not very difficult to show that there exist $\varepsilon_2 > 0$ and $D(\varepsilon)$, continuous in ε for $0 \leqq \varepsilon \leqq \varepsilon_2$, such that $\mathfrak{J}$ has a fixed point in $\mathfrak{C}[D(\varepsilon)]$ for each ε, $0 < \varepsilon \leqq \varepsilon_2$. Furthermore, this fixed point satisfies our differential system and is almost periodic in t.

A detailed analysis of the stability of this almost periodic solution yields the uniqueness.

1.2 *A generalization.* The conclusions of Theorem 1 remain valid in some cases when B, C depend upon t, say, $B = B(t)$, $C = C(t)$. In fact, hypothesis (H_3) can be replaced by the assumption that the zero solution of the system

$$\dot{y} = B(t)\, y$$

is uniformly asymptotically stable and that $C(t)$ has a continuous derivative with all of its eigenvalues, $\lambda(t)$ satisfying $\operatorname{Re} \lambda(t) \leqq -2\gamma < 0$, $-\infty < t < \infty$, where γ is a constant. This new hypothesis implies that if $\Phi(t)$, $\Psi(t)$ are fundamental solutions of $\dot{y} = B(t)\, y$, $\varepsilon\dot{z} = C(t)\, z$ respectively, then

$$\|\Phi(t)\| \leqq \beta e^{-\gamma(t-t_0)}, \quad \|\Psi(t)\| \leqq \beta e^{-\gamma(t-t_0)/\varepsilon}, \quad \beta = \text{const.}$$

for ε sufficiently small. The proof of the theorem proceeds as before with $J_B(\tau)$, $J_C(\tau/\varepsilon)$ replaced by $\Phi(t)$, $\Psi(t)$ respectively, and the limits in the last two integrals in (3) replaced by $-\infty$ to t.

As a final remark, it can be shown that the conclusions of Theorem 1 remain valid if the equation involving $\dot{y}$ in (1) contains a linear term $D(t)\,z$ where $D(t)$ is almost periodic. In the theory of relaxation oscillations a term of this type is encountered.

1.3 *Periodic solutions.* In case all functions in system (1) are periodic in t of period T, then the hypothesis (H_3) on the matrices A, B, C are too restrictive. In fact, hypothesis (H_3) can be replaced by the following:

(H'_3) A, C are constant matrices with $\det A \neq 0$ and all eigenvalues of C with nonzero real parts. The matrix $B = B(t)$ is periodic in t of period T and there is no periodic solution of $\dot{y} = B(t)\,y$ of period T except $y = 0$.

If $C = C(t)$ is periodic in t of period T, then the assumption on C in (H'_3) can be replaced by: the eigenvalues $\lambda(t)$ of the matrix $C(t)$ satisfy $\operatorname{Re}\lambda(t) \leqq -2\gamma < 0$, $0 \leqq \leqq t \leqq T$ with γ a constant.

Of course, the proof of the theorem on existence of a perodic solution of (1) under hypothesis (H'_3) must proceed in a different way from the one outlined above concerning almost periodic solutions. As before, we can assume $X_1 \equiv 0$. The class $\mathfrak{C}(D)$ is taken to be the class of periodic functions of period T, which are bounded by D. The transformation $\mathfrak{J}$ taking (x, y, z) into (u, v, w) is given by

$$u = \varepsilon(e^{-\varepsilon A t} - I)^{-1} \int_t^{t+T} e^{\varepsilon A(t-\tau)}\, X[\tau, x(\tau), y(\tau), z(\tau), \varepsilon]\, d\tau\,,$$

$$v = \int_t^{t+T} \{\Phi(\tau)\,[\Phi^{-1}(T) - I]\,\Phi^{-1}(t)\}^{-1}\, Y[\tau, x(\tau), y(\tau), z(\tau), \varepsilon]\, d\tau\,,$$

$$w = \frac{1}{\varepsilon}(e^{-CT/\varepsilon} - I)^{-1} \int_t^{t+T} e^{C(t-\tau)/\varepsilon}\, Z[\tau, x(\tau), y(\tau), z(\tau), \varepsilon]\, d\tau\,,$$

where $\Phi(t)$, $\Phi(0) = I$, is the principal matrix solution of $\dot{y} = B(t)\,y$. The remainder of the proof is similar to the previous one. Of course, a fixed point of $\mathfrak{J}$ is not necessarily a stable periodic solution of (1). On the other hand, it is not difficult to show that $\mathfrak{J}$ has a unique fixed point in a neighborhood of $x = 0$, $y = 0$, $z = 0$ and the fixed points of $\mathfrak{J}$ coincide with the periodic solutions of (1).

For periodic solutions of nonlinear differential equations, hypothesis (H'_3) is even too restrictive. Many methods which do not use (H'_3) have been devised for obtaining periodic solutions of nonlinear differential equations containing a small parameter. Some authors who have contributed to this question are Cesari, Hale and Gambill (see the book of Cesari [6] and the forthcoming monograph of Hale [16] for a description of this method and references), Malkin [26], Sibuya [30], Golomb [10], [11], Bass [1], [2]. Cesari [7] also has given a procedure for obtaining periodic solutions even when the differential equations do not contain a small parameter. These methods will not be discussed here due to lack of space. The recent work of Halanay [12] and

Šimanov [29] seem to indicate that the methods for the existence of periodic solutions of differential equations can be extended to differential-difference equations.

1.4 *Differential-difference equations.* In this section, we give some of the possible extensions of Theorem 1 to differential-difference equations. Consider the equation

$$\dot{x}(t) = \sum_{k=1}^{r} A_k\, x(t - \tau_k)\,, \tag{4}$$

where A_k, τ_k, x are scalars, $\tau_1 > \tau_2 > \ldots > \tau_r$, $0 \leqq \tau_j \leqq \beta$. If we let $\eta(\vartheta)$ be the step function defined by

$$\eta(\vartheta) = \begin{cases} 0, & -\beta \leqq \vartheta \leqq -\tau_1\,, \\ A_1 + A_2 + \ldots + A_j, & -\tau_j \leqq \vartheta < -\tau_{j+1},\ j = 1, 2, \ldots, r-1\,, \\ A_1 + A_2 + \ldots + A_r, & -\tau_r \leqq \vartheta \leqq 0\,, \end{cases}$$

then (4) can be written as

$$\dot{x}(t) = \int_{-\beta}^{0} x(t + \vartheta)\, d\eta(\vartheta)\,, \tag{5}$$

where the integral is in the sense of Stieltjes. We will discuss equations of the form (5) where it is not necessarily required that $\eta(\vartheta)$ be a step function as above. For this purpose it is convenient to introduce some notation.

Let R^n be the space of n-vectors and for $x \in R^n$, let $\|x\|$ be any vector norm. Let $\mathfrak{C}_n$ denote the space of continuous vector functions mapping the interval $\langle -\beta, 0\rangle$ into R^n and for φ in $\mathfrak{C}_n$, let $\|\varphi\| = \sup_{-\beta \leqq \vartheta < 0} \|\varphi(\vartheta)\|$. For any continuous function $x(u)$ defined on $-\beta \leqq u \leqq A$, $A > 0$, and any fixed t, $0 \leqq t \leqq A$, the symbol x_t will denote the function $x(t + \vartheta)$, $-\beta \leqq \vartheta \leqq 0$, that is, the function x_t is in $\mathfrak{C}_n$ and is that "segment" of the function $x(u)$ defined by letting u range in the interval $t - \beta \leqq \leqq u \leqq t$.

Let $X(\varphi, t)$ be a function defined for φ in $\mathfrak{C}_n$, $\|\varphi\| \leqq H$, $0 \leqq t < \infty$; let $\dot{x}_t(0)$ denote the right hand derivative of the function $x(u)$ at $u = t$, and consider the equation

$$\dot{x}_t(0) = X(x_t, t)\,. \tag{6}$$

For any $t_0 \geqq 0$ and any function φ in $\mathfrak{C}_n$, $\|\varphi\| < H$, a function $x_t(t_0, \varphi)$ is said to be a *solution of* (6) *with initial function* φ *at* $t = t_0$ if there exists a number $A > 0$ such that

i) for each fixed t, $t_0 \leqq t \leqq t_0 + A$, $x_t(t_0, \varphi)$ is in $\mathfrak{C}_n$ and $\|x_t(t_0, \varphi)\| \leqq H$;
ii) $x_{t_0}(t_0, \varphi) = \varphi$;
iii) x_t satisfies (6).

If $X(\varphi, t)$ is continuous in t and Lipschitzian in φ, then for any given φ in $\mathfrak{C}_n$, $\|\varphi\| \leqq H_1 < H$, there always exists a solution of (6) with initial function φ at $t = t_0$ and it is unique.

Up to the present time, no one has succeeded in proving a theorem for differential-difference equations which is as general as Theorem 1 for ordinary differential equations. After a little thought, some of the difficulties become apparent. First of all, for differential-difference equations, the transformation used in the proof of Theorem 1 to eliminate the term X_1 seems to be very difficult. If the lags, that is, β, is of order ε, then such a transformation can be applied (see Halanay [13]). However, if β is not "small" then the application of the transformation of Bogolyubov has not yielded satisfactory results. A second difficulty arises from the fact that solutions of differential-difference equations cannot be continued, in general, to the left of the initial time t_0. On the other hand, for the existence of almost periodic solutions, it is not necessary to discuss all solutions for $t < t_0$. Because of this fact, it may be possible to extend the proof used for Theorem 1 to differential-difference equations, but this has not been accomplished.

We now proceed to state a result that can be proved for differential-difference equations. Consider the system

$$\begin{aligned} \dot{x}_t(0) &= \varepsilon[a(x_t) + X(t, x_t, y_t, z_t, \varepsilon) + h(t)]\,, \\ \dot{y}_t(0) &= b(t, y_t) + d(t, z_t) + Y(t, x_t, y_t, z_t, \varepsilon)\,, \\ \varepsilon\, \dot{z}_t(0) &= c(t, z_t) + Z(t, x_t, y_t, z_t, \varepsilon)\,, \end{aligned} \tag{7}$$

where x, y, z are k, m, n-vectors, respectively, and the following hypotheses are satisfied:

(P_1) For each fixed ε, all functions are almost periodic in t uniformly with respect to φ, ψ, η for $\varphi \in \mathfrak{C}_k$, $\psi \in \mathfrak{C}_m$, $\eta \in \mathfrak{C}_n$, $\|\varphi\|, \|\psi\|, \|\eta\| \leqq R$, where R is a positive constant.

(P_2) Each of the functions $X(t, \varphi, \psi, \eta, \varepsilon)$, $Y(t, \varphi, \psi, \eta, \varepsilon)$, $Z(t, \varphi, \psi, \eta, \varepsilon)$ is continuous in $t, \varphi, \psi, \eta, \varepsilon$, Lipschitzian in φ, ψ, η for $-\infty < t < \infty$, $\|\varphi\|, \|\psi\|, \|\eta\| \leqq R$, $0 \leqq \varepsilon \leqq \varepsilon_1$, and the Lipschitz constant approaches zero as $\|\varphi\|, \|\psi\|, \|\eta\|, \varepsilon \to 0$. Furthermore, for $\varphi = 0$, $\psi = 0$, $\eta = 0$, the functions X, Y, Z are bounded by a continuous function $M(\varepsilon)$ with $M(0) = 0$.

(P_3) The functions $a(\varphi)$, $b(t, \psi)$, $c(t, \eta)$, $d(t, \eta)$, $h(t)$ are continuous in t, φ, ψ, η for all $-\infty < t < \infty$, $\varphi \in \mathfrak{C}_k$, $\psi \in \mathfrak{C}_m$, $\eta \in \mathfrak{C}_n$ and linear in φ, ψ, η.

(P_4) There exist constants $K \geqq 1$, $\alpha > 0$ such that every solution $x_t(t_0, \varphi)$ of the system $\dot{x}_t = a(x_t)$ satisfies

$$\|x_t(t_0, \varphi)\| \leqq Ke^{-\alpha(t-t_0)}\|\varphi\| \quad \text{for all} \quad t \geqq t_0, \ \varphi \in \mathfrak{C}_k\,.$$

(P_5) There exist constants $K \geqq 1$, $\alpha > 0$ such that every solution $y_t(t_0, \varphi)$ of the system $\dot{y}_t = b(t, y_t)$ satisfies

$$\|y_t(t_0, \psi)\| \leqq Ke^{-\alpha(t-t_0)}\|\psi\| \quad \text{for all} \quad t \geqq t_0, \ \psi \in \mathfrak{C}_m\,.$$

(P_6) There exist constants $K \geqq 1$, $\alpha > 0$, $L > 0$ such that

$$|c(t_1, \eta) - c(t_2, \eta)| \leqq L\|\eta\| \cdot |t_1 - t_2|$$

for all t_1, t_2 and $\eta \in \mathfrak{C}_n$ and for each s, $-\infty < s < \infty$, every solution $z_t(t_0, \eta, s)$ of the system $\dot{z}_t(0) = c(s, z_t)$ satisfies $\|z_t(t_0, \eta, s\| \leqq Ke^{-2\alpha(t-t_0)}\|\eta\|$ for all $t \geqq t_0$, $\eta \in \mathfrak{C}_n$.

Theorem 2. *If system* (7) *satisfies* $(P_1)-(P_6)$ *and*

$$\lim_{T\to\infty} T^{-1} \int_0^T h(\tau)\,d\tau = 0\,, \tag{8}$$

then there exist $\varepsilon_2 > 0$, $\sigma > 0$, *such that for each* ε, $0 < \varepsilon \leqq \varepsilon_2$, *system* (7) *has a solution which is almost periodic in* t *and approaches zero uniformly in* t *as* $\varepsilon \to 0$ *and is unique in the region* $0 \leqq \|\varphi\| + \|\psi\| + \|\eta\| \leqq \sigma$. *Furthermore, this solution is uniformly asymptotically stable.*

Theorem 2 generalizes many known results in differential-difference equations. See, for example, Cooke [8] for the case where the vectors y, z are absent in (7), Halanay [14], Krasovskiĭ [24] and Šimanov [28] for the case where the vectors x, z are absent in (7), Klimuščev and Krasovskiĭ [23] and Klimuščev [22] for the case where x is absent in (7). The most interesting part of the generalization of previous work is that hypothesis (P_6) is sufficient.

The method of proof used in [17] for a special case of this result may easily be extended to prove Theorem 2. The basic idea is to construct appropriate Lyapunov functions and then solve a system of differential inequalities. The use of Lyapunov functions allows one to exploit the implications of stability without knowing the behavior of the solutions for large negative values of t (as in the proof of Theorem 1).

2 Integral manifolds — Averaging

The present section is devoted to a theoretical discussion of integral manifolds and a method of averaging. The theorem stated below is applicable to many problems (see, for example, [4], [5], [16]).

First of all, we give an analytic definition of an integral manifold of a system of differential equations

$$\dot{x} = X(t, x) \tag{9}$$

where x, X are n-vectors, $X(t, x)$ is continuous in t, x for $-\infty < t < \infty$, x in U, an open set in E^n, Euclidean n-space.

In the (x, t) space, suppose there exists a surface S which may be described parametrically by means of the equations

$$S = \{(x, t); x = f(t, C_1, \ldots, C_s), -\infty < t < \infty\}\,, \tag{10}$$

where $s \leqq n$, $f(t, C_1, \ldots, C_s)$ is a continuous function of t, $C_1, \ldots, C_s$ in the whole range of their variation.

The surface S will be called an *integral manifold* of system (9) if any solution of (10), $x(t, t_0, x_0)$, $x(t_0, t_0, x_0) = x_0$, with (x_0, t_0) in S has the property that $(x(t, t_0, x_0), t)$ is in S for all t, $-\infty < t < \infty$.

The simplest type of integral manifold of system (9) would be the set S consisting of those points (x, t) for which $x(t, t_0, x_0)$, $x(t_0, t_0, x_0) = x_0$ is a solution of (9) which is defined for $-\infty < t < \infty$. Another more interesting one is the following. Suppose $X(t, x)$ in (9) is independent of t; that is, consider the system

$$\dot{x} = X(x) \tag{11}$$

and suppose that this equation has a nonconstant periodic solution $x = x^0(t)$ of period 2π. Since (11) is autonomous, the function $x = x^0(t + \varphi)$ is also a periodic solution of (11) for every arbitrary constant φ. Furthermore, for any φ, the pair $(x^0(t + \varphi), t)$ lies on the cylinder S in $(n + 1)$-dimensional (x, t)-space defined parametrically by the equation

$$S = \{(x, t);\ x = x^0(\vartheta),\ 0 \leqq \vartheta \leqq 2\pi,\ -\infty < t < \infty\}\,. \tag{12}$$

Furthermore, any solution of (11), $x(t, t_0, x_0)$, $x(t_0, t_0, x_0) = x_0$, with (x_0, t_0) in S must coincide with one of the periodic motions above and the cylinder S is an integral manifold of (11).

Now, consider the perturbed system

$$\dot{x} = X(x) + \varepsilon\, X^*(t, x)\,, \tag{13}$$

where for $\varepsilon = 0$ the system has a periodic solution $x^0(t)$ of period 2π and the perturbation term $X^*(t, x)$ is a bounded function for $-\infty < t < \infty$, x in U. Under what conditions on the function $X(x)$ will the solution of (11) and (13) be "essentially" the same for ε sufficiently small? Of course, one cannot begin to answer such a question without first clarifying the word "essentially". Suppose, for example, that the periodic solution, $x^0(t)$ of (11) is exponentially asymptotically orbitally stable (more precisely, $n - 1$ of the characteristic exponents of the linear variational equations associated with this periodic solution have negative real parts). Then, the cylinder S in (12) is exponentially asymptotically stable. Since the cylinder S is filled with a one parameter family of periodic solutions differing only by a shift in phase, one could not hope that under small general perturbations $\varepsilon\, X^*(t, x)$, each particular periodic solution on S enjoys a property of stability. However, it seems reasonable to suppose that there is another integral manifold S_ε of (13) which is stable and $S_\varepsilon \to S$ as $\varepsilon \to 0$.

Under appropriate hypotheses, one can introduce "polar" coordinates (ϑ, ϱ) in a neighborhood of the cylinder S, to reduce the above problem to a discussion of the system

$$\begin{aligned} \dot{\vartheta} &= 1 \;\; + \Theta(t, \vartheta, \varrho, \varepsilon)\,, \\ \dot{\varrho} &= A\varrho + R(t, \vartheta, \varrho, \varepsilon)\,, \end{aligned} \tag{14}$$

where the eigenvalues of the matrix A have negative real parts, Θ, R are periodic in ϑ and are $O(\|\varrho\|^2 + |\varepsilon|)$ as $\|\varrho\|$, $|\varepsilon| \to 0$. For these equations, the problem is to determine a function $f(t, \vartheta, \varepsilon)$, $f(t, \vartheta, 0) = 0$, $f(t, \vartheta, \varepsilon)$ bounded in t, such that $\varrho = f(t, \vartheta, \varepsilon)$ is an integral manifold. The existence of such a function is a consequence of Theorem 3 below.

By a consideration of other types of problems in nonlinear oscillations (see [4], [5], [9], [16]), one is led to an investigation of systems of equations of the form

$$
\begin{aligned}
\dot{\vartheta} &= d(\varepsilon) + \Theta(t, \vartheta, x, y, \varepsilon), \\
\dot{x} &= \varepsilon C x + \varepsilon X(t, \vartheta, x, y, \varepsilon), \\
\dot{y} &= A y + Y(t, \vartheta, x, y, \varepsilon),
\end{aligned}
\tag{15}
$$

where ε is a real parameter, $d(\varepsilon)$ is a constant vector continuous in ε for $0 < \varepsilon \leqq \varepsilon_0$, ϑ, x, y are k, m, n-vectors, respectively. For any σ, μ define the set $\Sigma_{\sigma,\mu}$ as follows:

$$
\Sigma_{\sigma,\mu} = \{(t, \vartheta, x, y); t \in E, \vartheta \in E^k, 0 \leqq \|x\| \leqq \sigma, 0 \leqq \|y\| \leqq \mu\}. \tag{16}
$$

The following hypotheses will also be needed:

(K_1) There exist positive constants ϱ_1, ϱ_2 such that the functions Θ, X, Y are continuous on $\Sigma_{\varrho_1,\varrho_2} \times (0, \varepsilon_0\rangle$ and are multiply periodic in Θ with vector period ω; that is, if $\Theta = \operatorname{col}(\vartheta_1, \ldots, \vartheta_k)$, $\omega = (\omega_1, \ldots, \omega_k)$, $\omega_j > 0$, then Θ, X, Y are periodic in ϑ_j with period ω_j, $j = 1, 2, \ldots, k$.

(K_2) There exists a function $M(\varepsilon)$ continuous in ε for $0 < \varepsilon \leqq \varepsilon_0$ such that $M(\varepsilon) \to 0$ as $\varepsilon \to 0$ and each of the functions, Θ, X, Y are bounded by $M(\varepsilon)$ on $\Sigma_{0,0}$.

(K_3) There exists a function $\eta(\varepsilon, \sigma, \mu)$, continuous in ε, σ, μ for $0 < \varepsilon \leqq \varepsilon_0$, $0 \leqq \sigma \leqq \varrho_1$, $0 \leqq \mu \leqq \varrho_2$, $\eta(\varepsilon, \sigma, \mu) \to 0$ as $\varepsilon, \sigma, \mu \to 0$, $\eta(\varepsilon, 0, 0) = o(\varepsilon)$ as $\varepsilon \to 0$ such that the function Θ is Lipschitzian in ϑ, x, y on $\Sigma_{\sigma,\mu} \times (0, \varepsilon_0\rangle$ with Lipschitz constant $\eta(\varepsilon, \sigma, \mu)$. If the vector x is absent in (15) then the condition $\eta(\varepsilon, 0, 0) = o(\varepsilon)$ as $\varepsilon \to 0$ is unnecessary.

(K_4) There exists a function $\lambda(\varepsilon, \sigma, \mu)$ continuous in ε, σ, μ for $0 < \varepsilon \leqq \varepsilon_0$, $0 \leqq \sigma \leqq \varrho_1$, $0 \leqq \mu \leqq \varrho_2$, $\lambda(\varepsilon, \sigma, \mu) \to 0$ as $\varepsilon, \sigma, \mu \to 0$ such that the functions X, Y are Lipschitzian in ϑ, x, y on $\Sigma_{\sigma,\mu} \times (0, \varepsilon_0\rangle$ with Lipschitz constant $\lambda(\varepsilon, \sigma, \mu)$.

(K_5) The eigenvalues of the constant matrices A, C have nonzero real parts.

Theorem 3. *If system* (15) *satisfies condition* (K_1)$-$(K_5), *then there exist an* $\varepsilon_1 > 0$, *scalar functions* $D(\varepsilon)$, $\Delta(\varepsilon)$, *and vector functions* $f(t, \vartheta, \varepsilon)$, $g(t, \vartheta, \varepsilon)$ *of dimensions* m, n *respectively, which are continuous in* t, ϑ, ε *for* $t \in E$, $\vartheta \in E^k$, $0 < \varepsilon \leqq \varepsilon_1$; $D(\varepsilon) \to 0$, $\Delta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$; f, g *multiply periodic in* ϑ *of vector period* ω;

$$
\begin{gathered}
\|f(t, \vartheta, \varepsilon)\| \leqq D(\varepsilon), \quad \|g(t, \vartheta, \varepsilon)\| \leqq D(\varepsilon), \\
\|f(t, \vartheta^1, \varepsilon) - f(t, \vartheta^2, \varepsilon)\| \leqq \Delta(\varepsilon) \|\vartheta^1 - \vartheta^2\|, \\
\|g(t, \vartheta^1, \varepsilon) - g(t, \vartheta^2, \varepsilon)\| \leqq \Delta(\varepsilon) \|\vartheta^1 - \vartheta^2\|,
\end{gathered}
$$

for all $t \in E$, $\vartheta, \vartheta^1, \vartheta^2 \in E^k$, $0 < \varepsilon \leqq \varepsilon_1$; *such that*

$$
x = f(t, \vartheta, \varepsilon), \quad y = g(t, \vartheta, \varepsilon) \tag{17}
$$

is an integral manifold of system (15). *The behavior of the solutions on this integral manifold are obtained by solving the system*

$$\dot{\vartheta} = d(\varepsilon) + \Theta(t, \vartheta, f(t, \vartheta, \varepsilon), g(t, \vartheta, \varepsilon), \varepsilon),$$

$\vartheta(t_0) = \vartheta_0$ *arbitrary.*

If, for each fixed ε, $0 < \varepsilon \leqq \varepsilon_0$, all functions in (15) are almost periodic in t uniformly with respect to (ϑ, x, y) in $\Sigma_{\varrho_1,\varrho_2}$, then f, g are almost periodic in t uniformly with respect to ϑ for each fixed ε, $0 < \varepsilon \leqq \varepsilon_1$.

The complete proof of this theorem may be found in [18] and the basic idea of the proof is modeled after the one given by N. Bogolyubov and Yu. Mitropol'skiĭ [4]. This result has also been obtained by Diliberto [9] when the functions are periodic in t. One can also discuss the stability properties of the manifold given in Theorem 3 and the result is that they are the same as the stability properties of the solution $x = 0$, $y = 0$ of the system $\dot{x} = Cx$, $\dot{y} = Ay$.

In the applications, there often occur terms Θ^*, X^* in the first two equations of (15) which are not small in the sense of hypotheses (K_3), (K_4) but have average zero with respect to t and some of the components of the vector ϑ. The specific type of average may be found in Diliberto [9] and Hale [18]. A transformation similar to the one mentioned in the proof of Theorem 1 can be applied to reduce such a system to the one discussed above (see [4], [18]).

To the author's knowledge, no extension of Theorem 3 has been given for differential-difference equations. However, if equations (15) have the form

$$\begin{aligned}
\dot{\vartheta}(t) &= d(\varepsilon) + \Theta(t, \vartheta(t), x_t, y_t, \varepsilon), \\
\dot{x}_t(0) &= \varepsilon\, a(x_t) + \varepsilon\, X(t, \vartheta(t), x_t, y_t, \varepsilon), \\
\dot{y}_t(0) &= b(y_t) + Y(t, \vartheta(t), x_t, y_t, \varepsilon),
\end{aligned}$$

where $a(\varphi)$, $b(\psi)$ are linear functionals with the solutions of $\dot{x}_t(0) = a(x_t)$, $\dot{y}_t(0) = b(y_t)$ uniformly asymptotically stable, and the functions Θ, X, Y satisfy conditions similar to those imposed on system (15) with the additional hypothesis that they are periodic in t, then the conclusions of Theorem 3 are valid. Of course, the functions f, g in this theorem are elements of a function space. The proof of this fact can be modeled after the one given by Diliberto [9] for ordinary differential equations.

The main reason that one can extend Theorem 3 to this case is the fact that ϑ is a vector in Euclidean space rather than in function space. To show that these equations are realistic, let us consider a very particular problem. Suppose system (11) has a nonconstant periodic solution $x^0(t)$ with $n - 1$ of the characteristic exponents of the linear variational equation having negative real parts. By defining $x^0_\vartheta \in \mathfrak{C}$ by $x^0_\vartheta(\gamma) = x^0(\vartheta + \gamma)$, $-\beta \leqq \gamma \leqq 0$ the function x^0_ϑ defines a closed curve in $\mathfrak{C}$. In a neighborhood of this closed curve one can introduce new coordinates in $\mathfrak{C}$ by a relation of the form $\varphi = x^0_\vartheta + P(\vartheta)\,\varrho$, ϑ a scalar, ϱ in $\mathfrak{C}$ such that

$$\begin{aligned}
\dot{\vartheta}(t) &= 1 + \Theta[t, \vartheta(t), \varrho_t, \varepsilon], \\
\dot{\varrho}_t(0) &= A\, \varrho_t(0) + R[t, \vartheta(t), \varrho_t, \varepsilon],
\end{aligned}$$

which is of the desired form. It would be very interesting to determine appropriate changes of coordinates for an exponentially asymptotically orbitally stable periodic solution of a general autonomous differential-difference equation

$$\dot{x}_t(0) = X(x_t).$$

3 Behavior near integral manifolds

In this section, we consider some of implications of assuming that a certain type of integral manifold in asymptotically stable. More specifically, we consider the system

$$\dot{z} = Z(z), \tag{18}$$

where z, Z are $(n + k + 1)$-dimensional vectors, and assume that there is a $(k+1)$-parameter family of periodic solutions of (18) given by

$$z = z^0[g(x)(t + \varphi), x], \quad z^0(\omega + \pi, x) = z^0(\omega, x), \tag{19}$$

with $x = (x_1, \dots, x_k)$, φ constant, $g(x) > 0$, $x \in U$ an open set or a single point and $-\infty < \varphi < \infty$. We also assume that $Z(z)$, $z^0(\omega, x)$, $g(x)$ have continuous second derivatives with respect to their arguments for all $x \in U$, $-\infty < \omega < \infty$ and $z \in V$ an open set containing the solution (19),

$$\operatorname{rank}\left[\frac{\partial z^0(\omega, x)}{\partial \omega}, \frac{\partial z^0(\omega, x)}{\partial x}\right] = k + 1 \tag{20}$$

and n of the characteristic exponents, $\lambda_1(x), \dots, \lambda_n(x)$, of the linear variational equations

$$\dot{z} = \frac{\partial Z[z^0(g(x)(t + \varphi), x]}{\partial z}\, z \tag{21}$$

have negative real parts (not necessarily bounded away from zero). Hypothesis (20) implies there are $k + 1$ characteristic exponents of (21) equal to zero.

In the (z, t)-space, such a family of periodic solutions defines a k-parameter family of cylinders C_x or a $(k + 2)$-dimensional integral manifold $\mathfrak{M}$ of (18). Under the above hypotheses, it is known [21] that the manifold $\mathfrak{M}$ is asymptotically stable. Also, every solution of (18) with initial value sufficiently close to $\mathfrak{M}$ approaches one of the cylinders C_x ($\mathfrak{M}$ is asymptotically stable with asymptotic amplitude). Finally, for any solution $z(t)$ of (18) with $z(0)$ sufficiently close to $\mathfrak{M}$, there exist a vector x_0 and a constant φ_0 such that $z(t) - z^0[g(x_0)(t + \varphi_0), x_0] \to 0$ as $t \to \infty$, where z^0 is defined in (19). ($\mathfrak{M}$ is asymptotically stable with asymptotic amplitude and phase.) These terms are defined more precisely in [21].

A natural question to ask for such an integral manifold $\mathfrak{M}$ is the following: Under what conditions on the function $Z^*(t, z)$, will the manifold $\mathfrak{M}$ satisfy the same stability properties as those above relative to the solutions of the perturbed equation

$$\dot{z} = Z(z) + Z^*(t, z)? \tag{22}$$

In [21] the following result is proved.

Theorem 4. *Suppose system* (18) *satisfies the conditions enumerated above and let* $\mathfrak{M}$ *be the set in* (z, t)*-space defined by the functions* (19). *If there exist continuous functions* $K(y)$, $L(t)$ *and a* $T \geqq 0$ *such that*

$$\|Z^*(t, z)\| \leqq L(t)\, K(\|z\|) \tag{23}$$

for all $x, t \geqq T$, *and*

$$\int^{\infty} L(t)\, \mathrm{d}t < \infty\,, \tag{24}$$

then $\mathfrak{M}$ *is asymptotically stable with asymptotic amplitude. If, in addition*

$$\int_T^{\infty} \int_t^{\infty} L(u)\, \mathrm{d}u\, \mathrm{d}t < \infty\,, \tag{25}$$

then $\mathfrak{M}$ *is asymptotically stable with asymptotic amplitude and phase.*

If $\mathfrak{M}$ consists of only one cylinder C_x, and $Z^*(t, z)$ satisfies (23) with $L(t) \to 0$ then $\mathfrak{M}$ is asymptotically stable. If $\int^{\infty} L(t)\, \mathrm{d}t < \infty$, then $\mathfrak{M}$ is asymptotically stable with asymptotic phase.

In [21], it was always assumed that $L(t)$ in (23) approached zero as $t \to \infty$. However, an investigation of the proof in [21] shows that this is not necessary. Some other results on the asymptotic behavior of trajectories of system of differential equations of the form (22) have been recently given by Opial [27] and Yoshizawa [31]. No assumption about the existence of an integral manifold of the type above is made in these papers, but the asymptotic behavior is discussed only in the z-space under hypotheses similar to (23), (24).

Theorem 4 is proved by introducing local coordinates (ω, x, y) where ω is a scalar, x is a k-vector, y is an $n-(k+1)$ vector, in a neighborhood of the manifold $\mathfrak{M}$ in such a way that the manifold is given by $y = 0$, $x =$ constant, $\omega = g(x)\,(t + \varphi)$. The differential equations in the new variables ω, x, y have the general form given by

$$\begin{aligned} \dot{\omega} &= g(x) + W(\omega, x, y, t)\,, \\ \dot{x} &= X(\omega, x, y, t)\,, \\ \dot{y} &= h(x)\, y + Y(\omega, x, y, t)\,, \end{aligned} \tag{26}$$

where all the eigenvalues of $h(x)$ have negative real parts and each of the functions W, X, Y are bounded by $K_1(\|y\|^2 + L(t))$ in a neighborhood of $y = 0$ and K_1 is a constant, L given in (23). It is then shown that hypothesis (24) implies that the solutions $\omega(t)$, $x(t)$, $y(t)$ satisfy the property that $y(t) \to 0$, $x(t) \to c$, a constant, as $t \to \infty$

and $\omega(t)$ is defined for all t provided the initial value of $y(t)$ at some time T is sufficiently small. Also, it is shown that hypothesis (25) implies $y(t) \to 0$, $x(t) \to c$, a constant, $\omega(t) - g(c)\, t \to d$, a constant, as $t \to \infty$. In [21], more general systems than (26) are also discussed.

For system of the form (26), Lykova [25] has obtained results concerning the existence of local integral manifolds which generalize Theorem 3 of section 2.

3.1 *A class of differential-difference equations.* In this section, we discuss possible generalizations of the results of the previous section to differential-difference equations. The notation for differential-difference equations is given in section 1.4 above.

Suppose that Z^* in (22) depends on the value of z at some past time, say $Z^* = Z^*(t, z(t - \beta))$, $\beta > 0$. Does the conclusion of Theorem 4 remain valid? One can introduce the local coordinates as before to obtain a system of differential-difference equations of the form (26). For this special case the proof used in [21] can be extended to show that Theorem 4 is still true but it is of independent interest to study more general systems of the form (26) for differential-difference equations. Some results along this line have been previously obtained by Halanay [15], but under very restrictive hypothesis. We summarize below the more general results of [19].

Consider the system

$$
\begin{aligned}
\dot{\omega}_t(0) &= g(x_t) + W(t, \omega_t, x_t, y_t)\,, \\
\dot{x}_t(0) &= X(t, \omega_t, x_t, y_t)\,, \\
\dot{y}_t(0) &= f(t, \omega_t, x_t, y_t) + Y(t, \omega_t, x_t, y_t)\,,
\end{aligned}
\tag{27}
$$

where $g(\varphi)$, $W(t, \eta, \varphi, \psi)$, $X(t, \eta, \varphi, \psi)$, $f(t, \eta, \varphi, \psi)$, $Y(t, \eta, \varphi, \psi)$ are continuous in the region $\Sigma = \{(t, \eta, \varphi, \psi);\ t \geqq 0,\ \eta \in \mathfrak{C}_k,\ \varphi \in \mathfrak{C}_m,\ \psi \in \mathfrak{C}_n,\ \|\varphi\| \leqq H,\ \|\psi\| \leqq H,\ H > 0\}$, and satisfy any additional hypotheses which will insure the existence and uniqueness of a solution of (27) with initial value (η, φ, ψ) at $t = t_0$. We also will need some of the following properties of these functions in the statements of the theorems:

(Q_1) $g(\varphi) > 0$; $f(t, \eta, \varphi, \psi)$ are Lipschitzian in φ in Σ and $f(t, \eta, \varphi, \psi)$ is linear in ψ.

(Q_2) There exist positive constants $\mu_1, \mu_2, \mu_3, K_1, K_2, K_3$ and continuous nonnegative functions $g_1(t)$, $g_2(t)$, $g_3(t)$ such that, in Σ,

$$
\begin{aligned}
\|W(t, \eta, \varphi, \psi\| &\leqq K_1\|\psi\|^{\mu_1} + g_1(t)\,, \\
\|X(t, \eta, \varphi, \psi\| &\leqq K_2\|\psi\|^{\mu_2} + g_2(t)\,, \\
\|Y(t, \eta, \varphi, \psi)\| &\leqq K_3\|\psi\|^{1+\mu_3} + g_3(t)\,.
\end{aligned}
$$

(Q_3) There exist constants $K \geqq 1$, $\beta > 0$ such that, for every continuous function $\eta(t)$ defined for $t \geqq 0$ and every $\varphi \in \mathfrak{C}_m$, $\|\varphi\| \leqq H$, $\psi \in \mathfrak{C}_n$ the solution

$$y_t(t_0, \eta, \varphi, \psi)\,, \quad y_{t_0}(t_0, \eta, \varphi, \psi) = \psi$$

of the equation

$$\dot{y}_t(0) = f(t, \eta_t, \varphi, y_t)$$

satisfies the relation

$$\|y_t(t_0, \eta, \varphi, \psi)\| \leqq Ke^{-\beta(t-t_0)}\|\psi\|, \quad t \geqq t_0.$$

(Q_4) There exist constants $K \geqq 1$, $\beta > 0$ such that, for every continuous function $\eta(t)$ defined for $t \geqq 0$ and every constant function $c \in \mathfrak{C}_m$, $\|c\| \leqq H$, $\psi \in \mathfrak{C}_n$, the solution $y_t(t_0, \eta, c, \psi)$, $y_{t_0}(t_0, \eta, c, \psi) = \psi$, of the system

$$\dot{y}_t(0) = f(t, \eta_t, c, y_t)$$

satisfies the relation

$$\|y_t(t_0, \eta, c, \psi)\| \leqq Ke^{-\beta(t-t_0)}\|\psi\|, \quad t \geqq t_0.$$

Properties (Q_1), (Q_2) specify the smoothness and smallness of the functions in (27), whereas (Q_3), (Q_4) specify the "strongness" of the stability of the linear system

$$\dot{y}_t(0) = f(t, \eta, \varphi, y_t). \tag{29}$$

Notice that (Q_3) implies that the solution of this system is exponentially asymptotically stable for all φ and η whereas (Q_4) implies this same type of stability only for all φ which are constant functions and all η. In the simple problem mentioned at the beginning of this section, the function $f(t, \eta, \varphi, \psi)$ has the form $h(\varphi(0)\,\psi(0))$ where h is given in (26). Consequently, in this special case, only property (Q_4) applies. It will be clear after reading the theorems below that both (Q_3) and (Q_4) yield very similar conclusions.

Theorem 5. *If system (27) satisfies (Q_1), (Q_2), (Q_3) and $\int^\infty g_j(t)\,dt < \infty$, $j = 1, 2, 3$, then, for any $\gamma < H$, there exist positive constants T, δ such that if $\eta \in \mathfrak{C}_k$, $\varphi \in \mathfrak{C}_m$, $\|\varphi\| \leqq \gamma$, $\psi \in \mathfrak{C}_n$, $\|\psi\| \leqq \delta$ are given, then there is a constant function $c \in \mathfrak{C}_m$, $\|c\| < H$, such that the solution $\omega_t(T, \eta, \varphi, \psi)$, $x_t(T, \eta, \varphi, \psi)$, $y_t(T, \eta, \varphi, \psi)$ of (27) with initial value η, φ, ψ at $t = T$ exists for $t \geqq T$ and $\|x_t - c\| \to 0$, $\|y_t\| \to 0$ as $t \to \infty$. If, in addition, $\int_0^\infty \int_t^\infty g_j(\tau)\,d\tau\,dt < \infty$, $j = 1, 2, 3$, then $\omega_t(0) - g(c) \to d$, a constant, as $t \to \infty$.*

If the variable x is absent in (27) and $g_j(t) \to 0$ as $t \to \infty$, then the solution $\omega_t(T, \eta, \varphi)$, $y_t(T, \eta, \varphi)$ exists for $t \geqq T$ and $\|y_t\| \to 0$ as $t \to \infty$. If $\int^\infty g_j(t)\,dt < \infty$, then $\omega_t(0) - g(t) \to d$, a constant, $\|y_t\| \to 0$ as $t \to \infty$.

Theorem 6. *If system (27) satisfies (Q_1), (Q_2), (Q_4), $\int^\infty g_j(t)\,dt < \infty$, $j = 1, 2, 3$, and c_1 is a given constant function, $c_1 \in \mathfrak{C}_m$, $\|c_1\| < H$, then there exist positive constants T, δ, ε, such that if $\eta \in \mathfrak{C}_k$, $\varphi \in \mathfrak{C}_m$, $\|\varphi - c_1\| \leqq \varepsilon$, $\psi \in \mathfrak{C}_n$, $\|\psi\| \leqq \delta$, then there exists a constant function $c \in \mathfrak{C}_m$, $\|c\| < H$, such that the solution $\omega_t(T, \eta, \varphi, \psi)$, $x_t(T, \eta, \varphi, \psi)$, $y_t(T, \eta, \varphi, \psi)$ of (27) with initial values η, φ, ψ at $t = T$ exists for*

$t \geqq T$ *and* $\|x_t - c\| \to 0$, $\|y_t\| \to 0$ *as* $t \to \infty$. *If, in addition,* $\int_0^\infty \int_t^\infty g_j(u)\, du\, dt < \infty$, $j = 1, 2, 3$, *then* $\omega_t(0) - g(c)\, t \to d$, *a constant, as* $t \to \infty$.

If the variable x is absent in (27), then similar remarks as in Theorem 5 apply.

The proofs of these results involve Lyapunov functionals and are therefore different from (and also more elementary) than in [21]. For the linear system (28) one constructs an appropriate Lyapunov functional $V(t, \varphi, \psi)$ and then uses a Perron type of argument on the perturbed coupled systems (27). The proof of Theorem 6 is not too difficult since one can calculate the derivative of $V(t, c, \psi)$ along the solutions of (27) with a given constant function. Consequently, the dependence of $V(t, \varphi, \psi)$ on φ is not needed. On the other hand, for Theorem 5, this is no longer possible and one needs information concerning the Lipschitz constant of V with respect to φ. This information is rather difficult to obtain, but is given in [18], where some other applications are also given.

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