Compact Boolean Algebras

WIESŁAW GŁOWCZYŃSKI

Lublin

Received 11. March 2005

We give the following characterization of compact Boolean algebras:
A complete Boolean algebra B is T_2, compact in the order sequential topology τ_s if and only if it is homeomorphic with the power algebra $\mathcal{P}(\kappa)$ where $\kappa \leq \omega$.

1. Introduction.

An aspect of metrizability of the order sequential topology τ_s on complete Boolean algebra B was investigated in [M], [B-G-J] and [B-J-P].

In [M], D. Maharam pointed out that the order sequential topology τ_s on a complete Boolean algebra B is metrizable, precisely in the case when there exists a strictly positive Maharam submeasure on the Boolean algebra B.

A T_2, compact complete Boolean algebra (B, τ_s) is a metrizable space. An example of complete not purely atomic Boolean algebra B such that (B,τ_s) is a T_2, compact space gives a negative answer to the famous control measure problem. In this paper we shown that there is no T_2, compact, complete not purely atomic Boolean algebra (B, τ_s).

The characterization announced in the abstract is also a consequence of theorem 4.1 and corollary 6.3 of [B-G-P] but we give here a direct proof without any elements of forcing methods.

Mathematical Institute of the Catholic University of Lublin, Konstantynów 1H, 20 708 Lublin, Poland

2. Basic facts.

In this paper we use the same notions as in [B-G-J] and [B-J-P], so we repeat only some basic facts and notions. For the definitions of notions of the Boolean algebras theory see [Ko] or [V].

We say that a sequence $\{b_n\}_{n\in\omega}$ of elements of a σ-complete Boolean algebra B algebraically converges to an element $b \in B$ if and only if

$$b = \bigwedge_{k\in\omega} \bigvee_{n\geq k} b_n = \bigvee_{k\in\omega} \bigwedge_{n\geq k} b_n .$$

We write then $b_n \Rightarrow b$.

The order sequential topology τ_s is the largest topology τ on B with the property: if $b_n \Rightarrow b$ then $b_n \xrightarrow[\tau]{} b$, i.e. converges in the topology τ.

Usually it is not true that a convergence $\xrightarrow[\tau_s]{}$ in the topology τ_s implies the algebraic convergence $\Rightarrow$.

For every σ-complete Boolean algebra B topology τ_s is a T_1 and sequential topology (see [E] for the definition). Hence for every topological space (X, τ), a function $f : B \to X$ is continuous if and only if for every element $b \in B$ and a sequence $b_n \Rightarrow b$ the sequence $f(b_n) \to f(b)$ in (X, τ).

The space (B, τ_s) is homogeneous, so the family of neighborhoods of the minimal element $\mathbf{0}$ of B completely defines the topology τ_s.

We review some basic facts about the power algebra $(\mathscr{P}(\kappa), \tau_s)$. We remark:

(i) $(\mathscr{P}(\kappa), \tau_s)$, for every κ, is a T_2 space in which the algebraic convergence is the same as the convergence in the topology τ_s.

(ii) $(\mathscr{P}(\kappa), \tau_s)$ is a T_3 only for $\kappa \leq \omega$; (see [G]).

(iii) $(\mathscr{P}(\kappa), \tau_s)$ is a compact space only for $\kappa \leq \omega$; for $\kappa > \omega$ the order sequential topology is strictly stronger than te product topology on $\mathscr{P}(\kappa)$, so $(\mathscr{P}(\kappa), \tau_s)$ is not compact; for $\kappa \leq \omega$ the order sequential topology is equal to the product topology on $\mathscr{P}(\kappa)$, so $(\mathscr{P}(\omega), \tau_s)$ is the Cantor set (see [B-G-J]).

It is proved in [B-G-J] that for a complete, ccc Boolean algebra B, if the space (B, τ_s) is T_2, then it is a metrizable space. Hence for every complete ccc Boolean algebra B, if the space (B, τ_s) is T_2, then the space (B, τ_s) is a compact space iff it is sequentially compact.

Let B be σ-complete Boolean algebra. A strictly positive Maharam submeasure on B is a function $v : B \to \mathbf{R}_+$ with the following properties:

(i) $v(b) = 0$ if and only if $b = \mathbf{0}$,

(ii) $v(a) \leq v(b)$ whenever $a \leq b$,

(iii) $v(a \vee b) \leq v(a) + v(b)$,

(iv) $\lim v(b_n) = 0$ for every decreasing sequence $\{b_n\}_{n\in\omega}$ such that $\bigwedge_{n\in\omega} b_n = \mathbf{0}$.

A σ-complete Boolean algebra B with a strictly positive Maharam submeasure v is a complete, ccc algebra. We call it a Maharam algebra and denote by $[B, v]$.

We say that a strictly positive Maharam submeasure $\mu : B \to \mathbf{R}_+$ is a measure on B if for any disjoint a and b, $\mu(a \vee b) = \mu(a) + \mu(b)$. Then, of course, $\mu(\bigvee_{n\in\omega} b_n) = \sum_{n\in\omega} \mu(b_n)$ for every disjoint sequence $\{b_n\}_{n\in\omega}$. A Boolean algebra B is called a measure algebra, if there exists a measure $\mu : B \to \mathbf{R}_+$. We write then (B, μ).

For every strictly positive Maharam submeasure $\nu : B \to \mathbf{R}_+$ the following function $d_\nu : B \times B \to \mathbf{R}_+$ given by formula: $d_\nu(a, b) = \nu(a \triangle b)$, for any $a, b \in B$, is a metric on B and the topology given by d_ν coincides with the order sequential topology; (see [V]; sec. 4.2.5 and 7.1.1). Hence if there exists any strictly positive Maharam submeasure on B, then (B, τ_s) is metrizable. Moreover, any strictly positive Maharam submeasures ν_1, ν_2 on B give the same topology τ_s on B.

3. Control measure.

Let X be a metrizable linear topological space and B a σ-complete Boolean algebra. We call a function $\vec{\mu} : B \to X$ a vector measure on a Boolean algebra B, if $\vec{\mu}(\bigvee_{n\in\omega} b_n) = \sum_{n\in q} \vec{\mu}(b_n)$ for every disjoint sequence $\{b_n\}_{n\in\omega}$.

A measure $\mu : B \to \mathbf{R}_+$ is called a control measure for a vector measure $\vec{\mu} : B \to X$, if $\vec{\mu}(b) = \Theta$ if and only if $\mu(b) = 0$.

Let $\preccurlyeq$ be a partially order relation on a linear space X, which is compatible with the linear operations. We say that $(X, \preccurlyeq)$ is a complete Riesz space, if for every bounded in $(X, \preccurlyeq)$ subset Y of X there exist infY and sup$Y$.

The following lemma is a combination of ideas from [F] and [K]. For completeness we give a version of proof with more details than in a very short outline presented in [K] and with less details than in a very long version presented in the few sections of different chapters (namely, sec. 364, 392, 393 and the appendix 2A5) of [F].

Lemma 3.1. *For every Maharam algebra $[B, \nu]$ there exist a complete Riesz space $L^0(B)$ which is also a metric linear topological space and a continuous in the order sequential topology τ_s vector measure $\vec{\mu} : B \to L^0(B)$.*

Proof. Let S be the Stone space of Boolean algebra B, $\mathcal{M}$ be its σ-ideal of meager sets and Σ let be the σ-algebra of subsets of the Stone space S generated by the family of all clopen subsets of S and the σ-ideal $\mathcal{M}$. By the Loomis-Sikorski theorem there exists an isomorphism $\pi : B \to \Sigma/\mathcal{M}$ of Boolean algebras which is also a homeomorphism of topological spaces (B, τ_s) and $(\Sigma/\mathcal{M}, \tau_s)$ and preserves infinite suprema and infima.

The function $\tilde{\nu} : \Sigma/\mathcal{M} \to \mathbf{R}_+$ defined by the formula $\tilde{\nu}([E]) = \nu(\pi^{-1}[E])$ for every $E \in \Sigma$ is a strictly positive Maharam submeasure on $\Sigma/\mathcal{M}$.

Let $\mathscr{L}^0(S) \subset \mathbf{R}^S$ be the set of all functions $f: S \to \mathbf{R}$ with the linear structure inherited from the linear structure of $\mathbf{R}^S$. For the solid $\mathscr{W} = \{f \in \mathscr{L}^0(S) : : \{x \in S : f(x) \neq 0\} \in \mathscr{M}\}$ the space $L^0(B) = \mathscr{L}^0(S)/\mathscr{W}$ be the quotient linear space with the natural linear structure of the quotient space. The space $L^0(B)$ with the partial order $[f] \preccurlyeq [g]$ defined as $\{x \in S : g(x) \leq f(x)\} \in \mathscr{M}$ is a complete Riesz space.

The functional $\tau : L^0(B) \to \mathbf{R}_+$ determined by the formula

$$\tau([f]) = inf\{\varepsilon > 0 : \tilde{\nu}([x \in S : |f|(x) > \varepsilon\}]) < \varepsilon\}$$

has all properties of F- norm and hence the formula $d([f], [g]) = \tau([f] - [g])$ defines a metric $d : L^0(B) \times L^0(B) \to \mathbf{R}_+$.

The space $(L^0(B), d)$ is a metric topological space in which for every neighborhood G of zero Θ in $L^0(B)$ there exists a neighbourhood H of Θ such that $[g] \in H$ whenever $[g] \preccurlyeq [h]$ and $[h] \in H$ (see [F], 364 M).

Putting $\vec{\mu}_0([E]) = [\chi_E]$, where χ_E is the characteristic function of E, we give a vector measure $\vec{\mu}_0 : \Sigma/\mathscr{M} \to L^0(B)$. Then $\vec{\mu} : B \to L^0(B)$ given by the formula $\vec{\mu}(b) = \vec{\mu}_0(\pi(b))$ is a vector measure on B which is a continuous function in the topology τ_s. Namely:

Let $\{b_n\}_{n\in\omega}$ be any disjoint sequence of elements of B with $b = \bigvee_{n\in\omega} b_n$. By properties of characteristic function we have $\vec{\mu}(\bigvee_{0\leq i\leq n} b_i) = \vec{\mu}(\sum_{0\leq i\leq n} b_i)$. Because $\bigvee_{0\leq i\leq n} b_i \Rightarrow b$, then $\nu(b - \bigvee_{0\leq i\leq n} b_i) \to 0$ and consequently $\tau(\vec{\mu}(b) - \vec{\mu}(\bigvee_{0\leq i<n} b_i)) \to 0$, so $\vec{\mu}(\bigvee_{n\in\omega} b_n)$.

If $\{b_n\}_{n\in\omega}$ is a decreasing sequence in B such that $\bigwedge_{n\in\omega} b_n = \mathbf{0}$ then for $b_n = \bigvee_{i\geq n}(b_i - b_{i+1})$, $\lim \vec{\mu}(b_n) = \lim \vec{\mu}(\bigvee_{i\geq n}(b_i - b_{i+1})) = \lim \sum_{i\geq n} \mu(b_i - b_{i+1}) = = 0.$ □

The Kalton-Roberts theorem 5.1 of [K-R] can be stated in the following form:

Lemma 3.1. *For every complete Boolean algebra B and a metrizable linear topological space X, if $\vec{\mu} : B \to X$ is a vector measure with the compact range $\vec{\mu}(B)$, then there is a control measure $\mu : B \to [0, 1]$.*

Let (B, μ) be a measure algebra such that $\mu(B) \subset [0, 1])$ and (B, τ_s) is a separable space. Let $At(B)$ be the set of all atoms of Boolean algebra B with the cardinality of the set $At(B)$, $|At(B)| = \kappa$. By the Bessaga-Pełczyński theorem (see theorem 7.2 of [B-P], p. 200) we have:

Lemma 3.2.

(i) *For every purely atomic Boolean algebra B, the space (B, τ_s) is homeomorphic with $\mathscr{P}(\kappa)$, in the product topology.*

(ii) *For every not purely atomic Boolean algebra B, the space* (B, τ_s) *is homeomorphic with* $l_2 \times \mathscr{P}(\kappa)$*, where* l_2 *is the Hilbert space.*

4. Compact Boolean algebras

We give the characterization of a complete, T_2 compact Boolean algebras.

Lemma 4.1. *Let B be a complete Boolean algebra. If* (B, τ_s) *is a compact space then B is a ccc Boolean algebra.*

Proof. Assume that B is not a ccc algebra. Then there is an antichain A of cardinality ω_1 in B. Hence B contains (as a complete generated) subalgebra $B[A]$, homeomorphic with the power algebra $(\mathscr{P}(\omega_1), \tau_s)$. Because the subalgebra $B[A]$ is a closed set, it must be a compact subspace, but it is not true. □

Lemma 4.2. *If a complete Boolean algebra* (B, τ_s) *is a* T_2*, compact space, then* (B, τ_s) *is a metrizable space.*

Proof. By Lemma 4.1, a Boolean algebra B is a ccc algebra. If (B, τ_s) is a T_2, then by [B-G-J] (B, τ_s) is a metrizable space. □

By [M], we have:

Lemma 4.3. *If a complete Boolean algebra* (B, τ_s) *is a* T_2*, compact space, then there is a strictly positive Maharam submeasure* $\mu : B \to \mathbf{R}_+$.

For a ccc Boolean algebra B the cardinality of the set $At(B)$ is not greater then ω. For every Maharam algebra $[B, \nu]$, the strictly positive Maharam submeasure $\mu : B \to \mathbf{R}_+$ is a continuous function in the topology τ_s.

Theorem 4.4. *A complete Boolean algebra* (B, τ_s) *is a* T_2*, compact space if and only if, the space* (B, τ_s) *is homeomorphic with power algebra* $(\mathscr{P}(\kappa), \tau_s)$, where $\kappa \leq \omega$.

Proof. If a complete Boolean algebra (B, τ_s) is a T_2, compact space then the space (B, τ_s) is metrizable and there is strictly positive Maharam submeasure $\nu : B \to [0, 1]$ on B.

By Lemma 3.1 there exists a continuous in τ_s, vector measure $\vec{\mu} : B \to L^0(B)$ with the compact range $\vec{\mu}(B) \subset L^0(B)$. By Lemma 3.2 there exists a control measure $\mu : B \to [0, 1]$ for $\vec{\mu}$ (by the construction of $\vec{\mu}$, $\mu(b) = 0$ iff $b = \mathbf{0}$). So (B, μ) is a measure algebra and μ, ν give the same topology τ_s. A compact metrizable space is a separable space. So by Lemma 3.4, if a Boolean algeba B is not purely atomic, the space (B, τ_s) is homeomorphic with the space $l_2 \times \mathscr{P}(\kappa)$, where $\kappa \leq \omega$. Because l_2 is not compact, B is a purely atomic Boolean algebra and the space (B, τ_s) is homeomorphic with $(\mathscr{P}(\kappa), \tau_s)$, where $\kappa \leq \omega$. □

Corollary 4.5. *There is no complete atomless Boolean algebra B, such that the space* (B, τ_s) *is a* T_2*, compact space.*

Does there exist a complete atomless Boolean algebra B such that the space (B, τ_s), is a compact space, which is not T_2? is an open problem (see in [B-J-P] remarks after theorem 4.1).

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