Gliceria Godini

Operators in normed almost linear spaces

In: Zdeněk Frolík and Vladimír Souček and Marián J. Fabián (eds.): Proceedings of the 14th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1987. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 14. pp. [309]--328.

Persistent URL: http://dml.cz/dmlcz/702146

OPERATORS IN NORMED ALMOST LINEAR SPACES

G. Godini

1. INTRODUCTION

The notion of normed almost linear space (nals) is a generalization of the notion of normed linear space. Such a space satisfies some of the axioms of a linear space and the norm satisfies all the axioms of a norm on a linear space, as well as an additional one, which is useless in a normed linear space. An example of a nals is the set X of all nonempty, bounded and convex subsets A of a (real) normed linear space E for the addition $A_1+A_2=\{a_1+a_2:a_1\varepsilon A_1, a_2\varepsilon A_2\}$, the element zero of X the set $\{0\}$, the multiplication by reals $\lambda A=\{\lambda a:a\varepsilon A\}$ and the norm $|||A|||=\sup_{a\varepsilon A}||a||$. Besides the axioms of an usual norm on a linear space, the above norm $|||\cdot|||$ satisfies also the following condition: if $A_1=-A_1$ then $|||A|||\leq|||A+A_1|||$ for each $A\varepsilon X$.

The normed almost linear spaces were introduced in [3] as a natural framework for the theory of best simultaneous approximation in normed linear spaces. In [3] and the subsequent papers [4]-[6] we have also begun to develop a theory for the normed almost linear spaces similar with that of the normed linear spaces. It turned out that some results from the latter theory were true in our more general framework. Here we mention that we have introduced the "dual" of a nals X, denoted X*, (where the functionals are no longer linear but "almost linear"), which is also a nals, and when X is a normed linear space then X* is the usual dual space of X (see [3], [4]). In a nals X for each $x\varepsilon X$ there exists $f\varepsilon X^*$, $|||f|||=1$ such that $f(x)=|||x|||$ ([6]), though the result which states that in a normed linear space X, given a linear subspace $M\subset X$ and $f\varepsilon M^*$ there exists a norm-preserving extension to X is not true if we replace "linear" by "almost linear" (see examples in [4]). The main tool for the theory of normed almost linear spaces was given in ([6], Theorem 3.2) where we proved that any nals X can be "embedded" in a normed linear space E_X. Though the embedding mapping is not one-to-one, it has enough properties to permit us the use of normed linear spaces tech-

This paper is in final form and no version of it will be submitted for publication elsewhere.

niques to prove certain problems in a nals.

The present paper is a continuation of the above cited papers, providing results from the theory of bounded linear operators in normed linear spaces which can be formulated and proved in normed almost linear spaces.

When X and Y are two normed almost linear spaces, the definition of a bounded linear operator $T:X \to Y$ may be given as in the case when X and Y are normed linear spaces, but the set of all such operators may be the only operator $T=0$. Moreover, for $Y=R$ we do not obtain the dual space X^*. To avoid these unpleasant facts we shall work with bounded *almost linear operators* with respect to a convex cone $C \subset Y$ (see Section 4). The set of all such operators, denoted by $L(X,(Y,C))$, is $\neq\{0\}$ when $C\neq\{0\}$, $X^*=L(X,(R,R_+))$ and when X,Y are normed linear spaces, $L(X,(Y,C))$ is the set of all bounded linear operators $T:X \to Y$. Though $L(X,(Y,C))$ has some relevant properties, it is not a nals for arbitrary $C \subset Y$. For convex cones C having a certain property (P) in Y (see Section 3), $L(X,(Y,C))$ is a nals. Though property (P) of C is not necessary for $L(X,(Y,C))$ to be a nals, it is in a certain sense the best possible (see Theorem 4.15).

In order to prove the extensions of some results from the theory of bounded linear operators in normed linear spaces, the main tool is given in Theorem 5.6, where we "embed" $L(X,(Y,C))$ in the space of bounded linear operators $T:E_X \to E_Y$. As applications we prove the Banach-Steinhaus Theorem and the inverse mapping Theorem in our more general framework (Section 6).

2. PRELIMINARIES

For an easy understanding of this paper, in this section we recall definitions and results from [3], [4], [6] which will be used in the next sections. Some notations and general assumptions can be also found here. The main assumption is that *all spaces are over the real field* R. Let us denote by R_+ the set $\{\lambda\in R:\lambda\geq 0\}$ and by N the set $\{1,2,\dots\}$.

An *almost linear space* (als) is a set X together with two mappings $s:X\times X \to X$ and $m:R\times X \to X$ satisfying (L_1)-(L_8) below. We denote $s(x,y)$ by $x+y$ (or $x\dot{+}y$) and $m(\lambda,x)$ by $\lambda\circ x$ (or λx). Let $x,y,z\in X$ and $\lambda,\mu\in R$. (L_1) $x+(y+z)=(x+y)+z$; (L_2) $x+y=y+x$; (L_3) There exists an element $0\in X$ such that $x+0=x$ for each $x\in X$; (L_4) $1\circ x=x$; (L_5) $0\circ x=0$; (L_6) $\lambda\circ(x+y)=\lambda\circ x+\lambda\circ y$; (L_7) $\lambda\circ(\mu\circ x)=(\lambda\mu)\circ x$; (L_8) $(\lambda+\mu)\circ x=\lambda\circ x+\mu\circ x$ for

$\lambda, \mu \in R_+$.

In an als X the following two sets play an important role:

$$V_X = \{x \in X : x + (-1 \circ x) = 0\}$$

$$W_X = \{x \in X : x = -1 \circ x\} \ (= \{x + (-1 \circ x) : x \in X\})$$

They are *almost linear subspaces* of X (i.e., closed under addition and multiplication by scalars), and by $(L_1)-(L_8)$, V_X is a linear space. Plainly, an als X is a linear space iff $X = V_X$, iff $W_X = \{0\}$.

In an als X we shall always use the notation $\lambda \circ x$ for $m(\lambda, x)$, the notation λx being used only in a linear space.

An als X satisfies the *law of cancellation* if the relations $x, y, z \in X$, $x+y=x+z$ imply $y=z$.

In what follows a *cone* in an als X is a set $C \subset X$ such that $\lambda \circ x \in C$ for each $x \in X$ and $\lambda \in R_+$. The definition of a *convex* set in an als X is similar with that in a linear space.

A *norm* on the als X is a functional $|||\cdot||| : X \to R$ satisfying $(N_1)-(N_4)$ below. Let $x, y \in X$, $w \in W_X$ and $\lambda \in R$. (N_1) $|||x+y||| \le |||x||| + |||y|||$; (N_2) $|||x|||=0$ iff $x=0$; (N_3) $|||\lambda \circ x||| = |\lambda| \, |||x|||$; (N_4) $|||x||| \le |||x+w|||$. By $(N_1)-(N_4)$ it follows that $|||x||| \ge 0$, $x \in X$. A *normed almost linear space* (nals) is an als X together with $|||\cdot||| : X \to R$ satisfying $(N_1)-(N_4)$. Here we note that in [3]-[5] we gave another equivalent definition for the norm, the above one being used in [6].

In a nals X the following inequality holds:

$$(2.1) \qquad \big| |||x||| - |||y||| \big| \le |||x+y||| \qquad (x, y \in X)$$

2.1. REMARK. Let X be a nals and $x, y \in X$. The function $\varphi(\lambda) = |||x + \lambda \circ y|||$ is convex on $[0, \infty)$ and $(-\infty, 0]$.

The next result is from ([3]).

2.2. LEMMA. *Let* X *be a nals and* $x, y, z \in X$.

(i) *If* $x+y=x+z$ *then* $|||y|||=|||z|||$.

(ii) *If* $x+y \in V_X$ *then* $x, y \in V_X$.

Let X,Y be two almost linear spaces. A mapping $T : X \to Y$ is called a *linear operator* if $T(\lambda_1 \circ x_1 + \lambda_2 \circ x_2) = \lambda_1 \circ T(x_1) + \lambda_2 \circ T(x_2)$, $x_i \in X$, $\lambda_i \in R$, $i=1,2$.

The main tool for the theory of normed almost linear spaces is the following theorem ([6], Theorem 3.2).

2.3. THEOREM. *For any nals* $(X,|||\cdot|||)$ *there exist a normed linear space* $(E_X,||\cdot||_{E_X})$ *and a mapping* $\omega_X:X\to E_X$ *with the following properties:*

(i) *The set* $X_1=\omega_X(X)$ *is a convex cone of* E_X *such that* $E_X=X_1-X_1$, *and* X_1 *can be organized as an als where the addition and the multiplication by non-negative reals are the same as in* E_X.

(ii) *For each* $z\varepsilon E_X$ *we have:*

$$(2.2)\qquad ||z||_{E_X}=\inf\{|||x_1|||+|||x_2|||:x_1,\ x_2\varepsilon X,\ z=\omega_X(x_1)-\omega_X(x_2)\}$$

and the als X_1 *together with this norm is a nals.*

(iii) *The mapping* ω_X *from* X *onto the nals* X_1 *is a linear operator and* $||\omega_X(x)||_{E_X}=|||x|||$ *for each* $x\varepsilon X$.

In the sequel we shall not use the subscript X (resp. E_X) for E_X and ω_X (resp. $||\cdot||_{E_X}$) when these will not lead to misunderstandings.

2.4. REMARK. We have $\omega(W_X)=W_{X_1}$ and $\omega(V_X)=V_{X_1}$.

2.5. REMARK. If $\omega:X\to X_1$ is one-to-one then $\omega^{-1}:X_1\to X$ is a linear operator.

The proof of the following lemma is contained in the proof of ([6], Theorem 3.2, (iv), fact (I)).

2.6. LEMMA. *Let* $(X,|||\cdot|||)$ *be a nals and* $x,y\varepsilon X$ *such that* $\omega(x)=\omega(y)$. *Then for each* $\varepsilon>0$ *there exist* $x_\varepsilon,y_\varepsilon,u_\varepsilon\varepsilon X$ *such that* $|||x_\varepsilon|||+|||y_\varepsilon|||<\varepsilon$ *and* $x+y_\varepsilon+u_\varepsilon=y+x_\varepsilon+u_\varepsilon$.

A consequence of Theorem 2.3 is the following result ([6], Corollary 3.4).

2.7. COROLLARY. *For any nals* $(X,\ |||\cdot|||)$ *the function*

$$\rho(x,y)=\rho_X(x,y)=||\omega(x)-\omega(y)||\qquad (x,y\varepsilon X)$$

is a semi-metric on X *and we have:*

$$(2.3)\qquad \rho(-1ox,\ -1oy)=\rho(x,y)\qquad (x,y\varepsilon X)$$

In a nals X the semi-metric ρ generates a topology on X (which is not Hausdorff in general) and in the sequel any topological concept such as closeness, completion, continuity, will be understood for this topology. Clearly ρ is a metric on X iff ω is one-to-one. Notice that even when ρ is not a metric on X we can use sequences instead of nets. Moreover the topology on the normed linear space $(V_X, |||\cdot|||)$ generated by ρ is the same as the topology generated by the norm.

2.8. REMARK. If A is a closed subset of the nals $(X, |||\cdot|||)$ then $\omega(A)$ is a closed subset of the nals $(X_1, ||\cdot||)$.

We recall now the definition of the dual space of a nals X and some of its properties used in the next sections.

Let X be an als. A functional $f:X \to R$ is called an *almost linear functional* if f is additive, positively homogeneous and $f(w) \geq 0$ for each $w \in W_X$. Let $X^{\#}$ be the set of all almost linear functionals on X. Define the addition in $X^{\#}$ by $(f_1+f_2)(x)=f_1(x)+f_2(x)$, $x \in X$ and the multiplication by reals $(\lambda \circ f)(x)=f(\lambda \circ x)$, $x \in X$. The element $0 \in X^{\#}$ is the functional which is 0 at each $x \in X$. Then $X^{\#}$ is an als. When X is a nals, for $f \in X^{\#}$ define $|||f|||=\sup\{|f(x)|:|||x|||\leq 1\}$, and let $X^* = \{f \in X^{\#}:|||f|||<\infty\}$. Then X^* is a nals ([3]) called the *dual* space of the nals X. The dual space X^* is $\neq\{0\}$ if $X\neq\{0\}$ since the corollary of Hahn-Banach Theorem extends to a nals (see the introduction and the reference cited there). The next corollary is an immediate consequence of the above mentioned result and ([4], Proposition 3.15). We give another direct proof using only the extension of the corollary of Hahn-Banach Theorem.

2.9. COROLLARY. *If* X *is a nals such that* $X \neq V_X$ *then* $W_{X^*} \neq \{0\}$.

Proof. Let $w \in W_X$, $|||w|||=1$ and let $f \in X^*$, $|||f|||=1$ such that $f(w)=|||w|||$. Define for $x \in X$, $f_1(x)=f(x+(-1\circ x))/2$. Then $f_1 \in W_{X^*}$ and $|||f_1|||=1$.

We conclude this section with some examples from [3], [4] which will be used in the next sections.

2.10. EXAMPLE. Let $X=\{(\alpha,\beta)\in R^2:\beta \in R_+\}$. Define the addition and the multiplication by non-negative reals as in R^2 and define $-1\circ(\alpha,\beta)=(-\alpha,\beta)$. The element zero of X is $(0,0)\in R^2$. Then X is an als and we have $V_X=\{(\alpha,0):\alpha\in R\}$ and $W_X=\{(0,\beta):\beta\in R_+\}$. Define for $(\alpha,\beta)\in X$, $|||(\alpha,\beta)|||=|\alpha|+\beta$. The als X together with this norm is a nals.

2.11. EXAMPLE. Let $X=R_+$. Define $x \dot{+} y=\max\{x,y\}$ and for $\lambda\neq 0$, $\lambda\circ x=x$ and $0\circ x=0$. The element $0\in X$ is $0\in R_+$. Then X is an als such that $W_X=X$. There exists no norm on X.

2.12. EXAMPLE. Let X=R. Define the addition and the element $0\in X$ as in R and define $\lambda\circ x=|\lambda|x$. Then X is an als such that $W_X=X$.

There exists no norm on X.

If otherwise not stated, an als (nals) X will be supposed $\neq\{0\}$.

3. CONES WITH PROPERTY (P) IN A NORMED ALMOST LINEAR SPACE

Let $(X,|||\cdot|||)$ be a nals and C a convex cone of X.

3.1. DEFINITION. The convex cone C has *property* (P) *in* X if the relations $x,y\in X$, $x+y\in C$ and $c\in C$ imply that

$$\max\{|||x|||,|||y|||\}\leq\max\{|||x+c|||,|||y+c|||\} \tag{3.1}$$

Note that if C',C are convex cones of X, $C'\subset C$ and C has property (P) in X then C' has also property (P) in X.

Clearly the cone $C=W_X$ has property (P) in X. The next result gives more information about the existence of cones with property (P) in a nals X.

3.2. PROPOSITION. *In any nals* X *there exists a maximal convex cone* $C\neq\{0\}$ *having property* (P) *in* X *and such that* $W_X\subset C$.

<u>Proof</u>. Suppose $W_X\neq\{0\}$. As we observed above W_X has property (P) in X. Let F be the set of all convex cones $C\subset X$, having property (P) in X and such that $W_X\subset C$. It is a partially ordered set, ordered by set-inclusion, and by Zorn's Lemma the conclusion follows.

Suppose $W_X=\{0\}$. Then X is a normed linear space. Let $x_o\in X$, $|||x_o|||=1$ and let $C_o=\{\lambda x_o:\lambda\in R_+\}$. Then C_o has property (P) in X. Indeed, let $x,y\in X$ such that $x+y\in C_o$ and let $c\in C_o$. If $x+y=0$ then (3.1) is obvious. If $x+y=\lambda_o x_o$, $\lambda_o>0$, suppose $|||y|||\leq|||x|||$. Let $c=\lambda_1 x_o$, $\lambda_1\in R_+$ and let $\lambda=\lambda_1/\lambda_o$. We have $|||x|||=(1+\lambda)|||x|||-\lambda|||x|||\leq$ $\leq(1+\lambda)|||x|||-\lambda|||y|||\leq|||(1+\lambda)x+\lambda y|||=|||x+c|||$, whence (3.1) follows. As in the case $W_X\neq\{0\}$ (replacing W_X by C_o), the assertion from the proposition follows by Zorn's Lemma.

The next proposition yields a necessary condition for a convex cone to have property (P) in X.

3.3. PROPOSITION. *If* C *is a convex cone having property* (P) *in the nals* X *then*:

$$|||c_1|||\leq|||c_1+c_2||| \qquad (c_1,c_2\in C) \tag{3.2}$$

<u>Proof</u>. Let $c_1,c_2\in C$. We can suppose $0\neq|||c_2|||\leq|||c_1|||$.
<u>Case 1</u>. $|||c_2|||<|||c_1|||$. Choose $0<\lambda<1$ such that $(1+\lambda)\cdot$

$\cdot |||c_2|||<|||c_1|||$. Since $c_1+c_2 \varepsilon C$, by property (P) of C in X we have:

$$|||c_1|||\leq \max\{|||c_1+\lambda\circ c_2|||,\ |||c_2+\lambda\circ c_2|||\}$$

By the choice of λ we must have $|||c_1|||\leq|||c_1+\lambda\circ c_2|||$, and (3.2) follows now by Remark 2.1.

Case 2. $|||c_2|||=|||c_1|||$. Let $0<\mu<1$. Then $|||\mu\circ c_2|||<|||c_1|||$ and by the above case we get $|||c_1|||\leq|||c_1+\mu\circ c_2|||$. Again by Remark 2.1 we obtain (3.2).

The necessary condition for property (P) given above is not sufficient as the following example shows.

3.4. EXAMPLE. Let X be the nals described in Example 2.10. Let $C=\{(\alpha,\beta)\varepsilon X:\alpha,\beta\varepsilon R_+\}$. Then (3.2) is satisfied for $c_1,c_2\varepsilon C$ but C has not property (P) in X. Indeed, let $0<\varepsilon<1/2$ and let $x=(-\varepsilon,1)$, $y=c=$ $=(\varepsilon,0)\varepsilon C$. We have $x+y\varepsilon C$, $|||y|||<|||x|||=1+\varepsilon$, $|||x+c|||=1$ and $|||y+c|||=2\varepsilon<1$ and so (3.1) fails.

Let $(X,|||\cdot|||)$ be a nals and $(E,||\cdot||)$, ω, X_1 and ρ be given by Theorem 2.3 and Corollary 2.7.

3.5. LEMMA. *Let* $(X,|||\cdot|||)$ *be a nals satisfying the law of cancellation and let* $C\subset X$ *be a convex cone having property* (P) *in* X *and such that* $W_X\subset C$.

(i) $C_1=\omega(C)$ *is a convex cone having property* (P) *in* X_1.

(ii) *The closure* $\overline{C}$ *of* C *in* X *is a convex cone having property* (P) *in* X.

Proof. (i). By the properties of ω given in Theorem 2.3, C_1 is a convex cone. Let now $\overline{x},\overline{y}\varepsilon X_1$ such that $\overline{x}+\overline{y}=\overline{c}_1\varepsilon C_1$ and let $\overline{c}\varepsilon C_1$. Let $x,y\varepsilon X$, $c,c_1\varepsilon C$ such that $\omega(x)=\overline{x}$, $\omega(y)=\overline{y}$, $\omega(c)=\overline{c}$ and $\omega(c_1)=\overline{c}_1$. Then $\omega(x+y)=\omega(c_1)$. By Lemma 2.6 and since X satisfies the law of cancellation, for each $\varepsilon>0$ there exist $x_\varepsilon,y_\varepsilon\varepsilon X$ such that $|||x_\varepsilon|||+$ $+|||y_\varepsilon|||\leq\varepsilon$ and $x+y+y_\varepsilon=c_1+x_\varepsilon$. Hence, using the hypothesis $W_X\subset C$, we get $x+y+y_\varepsilon+(-1\circ x_\varepsilon)\varepsilon C$, and by (2.1) and the property (P) of C in X we obtain

$$\begin{aligned}&\max\{|||x|||-|||y_\varepsilon|||,\ |||y|||-|||x_\varepsilon|||\}\leq\\ &\leq\max\{|||x+y_\varepsilon|||,\ |||y+(-1\circ x_\varepsilon)|||\}\leq\\ &\leq\max\{|||x+y_\varepsilon+c|||,\ |||y+(-1\circ x_\varepsilon)+c|||\}\leq\\ &\leq\max\{|||x+c|||+|||y_\varepsilon|||,\ |||y+c|||+|||x_\varepsilon|||\}\end{aligned}$$

Letting $\varepsilon\to0$ we get (3.1), and the conclusion that C_1 has property

(P) in X_1 follows by the properties of ω.

(ii) Clearly $\overline{C}$ is a convex cone of X. Let now $x,y\in X$ such that $x+y\in\overline{C}$ and let $c\in\overline{C}$. For $\varepsilon>0$ there exist $c',c''\in C$ such that $\rho(x+y,c'')<\varepsilon$ and $\rho(c,c')<\varepsilon$. Since $||\omega(x)+\omega(y)-\omega(c'')||<\varepsilon$, by (2.2) there exist $x_1,y_1\in X$ such that $\omega(x)+\omega(y)-\omega(c'')=\omega(x_1)-\omega(y_1)$ and $|||x_1|||+|||y_1|||<$ $<\varepsilon$. Then $\omega(x+y+y_1)=\omega(x_1+c'')$ and as in (i) above we find $x_\varepsilon,y_\varepsilon\in X$ with $|||x_\varepsilon|||+|||y_\varepsilon|||\le\varepsilon$ and such that $x+y+y_1+y_\varepsilon=x_1+c''+x_\varepsilon$. Hence $x+y+y_1+$ $+y_\varepsilon+(-1ox_1)+(-1ox_\varepsilon)\in C$. Using property (P) of C in X and (2.1) we get:

$$\begin{aligned}&\max\{|||x|||,\ |||y|||\}-2\varepsilon\le\max\{|||x+y_1+y_\varepsilon|||,|||y+(-1ox_1)+(-1ox_\varepsilon)|||\}\le\\ (3.3)\quad &\le\max\{|||x+y_1+y_\varepsilon+c'|||,\ |||y+(-1ox_1)+(-1ox_\varepsilon)+c'|||\}\le\\ &\le\max\{|||x+c'|||,\ |||y+c'|||\}+2\varepsilon\end{aligned}$$

Now $|||x+c'|||-|||x+c|||=||\omega(x)+\omega(c')||-||\omega(x)+\omega(c)||\le||\omega(c')-\omega(c)||=$ $=\rho(c',c)<\varepsilon$ and similarly $|||y+c'|||-|||y+c|||<\varepsilon$. By (3.3) we obtain:

$$\max\{|||x|||,\ |||y|||\}-2\varepsilon\le\max\{|||x+c|||,\ |||y+c|||\}+3\varepsilon$$

Letting $\varepsilon\to 0$ we obtain (3.1), i.e., $\overline{C}$ has property (P) in X.

We have not an example to show that the assumption on X to satisfy the law of cancellation is not superfluous in the above lemma.

We conclude this section with the following remark.

3.6. REMARK. Let $C_1\subset X_1$ be a convex cone having property (P) in X_1. Then $\omega^{-1}(C)=\{x\in X,\ \omega(x)\in C_1\}$ is a convex cone having property (P) in X.

4. ALMOST LINEAR OPERATORS

Let X,Y be two almost linear spaces and C a convex cone of Y.

4.1. DEFINITION. A mapping $T:X\to Y$ is called an *almost linear operator with respect to* C if the following three conditions hold:

(4.1) $\quad T(x_1+x_2)=T(x_1)+T(x_2) \quad (x_1,x_2\in X)$

(4.2) $\quad T(\lambda ox)=\lambda oT(x) \quad (x\in X,\ \lambda\in R_+)$

(4.3) $\quad T(W_X)\subset C$

We denote by $L(X,(Y,C))$ the set of all $T:X\to Y$ satisfying

(4.1)-(4.3). We organize $\mathcal{L}(X,(Y,C))$ as an als in the following way: for $T_1,T_2,T\in L(X,(Y,C))$ and $\lambda\in R$ we define $T_1+T_2\in L(X(Y,C))$ and $\lambda\circ T\in L(X,(Y,C))$ by

$$(T_1+T_2)(x)=T_1(x)+T_2(x) \qquad (x\in X)$$

$$(\lambda\circ T)(x)=T(\lambda\circ x) \qquad (x\in X)$$

The element $0\in L(X,(Y,C))$ is the operator which is zero at any $x\in X$. It is straightforward to show that $L(X,(Y,C))$ is an als.

4.2. REMARK. If C', C are convex cones of Y such that $C'\subset C$ then $L(X,(Y,C'))$ is an almost linear subspace of $L(X,(Y,C))$.

Let us also denote by $L(X,Y)$ the set $L(X,(Y,\{0\}))$ and by $\Lambda(X,Y)$ the set of all linear operators $T:X\to Y$. By Remark 4.2 $L(X,Y)$ is an almost linear subspace of $L(X,(Y,C))$ for every $C\subset Y$. It is easy to construct examples of $T\in L(X,(Y,C))$ which are not linear operators (see Example 4.7 below). Clearly if $T\in L(X,(Y,C))$ then we have $T\in\Lambda(X,Y)$ iff $S=-1\circ T$ where $S:X\to Y$ is defined by $S(x)=-1\circ(T(x))$, $x\in X$. Here we also note that the inclusion $\Lambda(X,Y)\subset L(X,(Y,C))$ can fail, but we always find cones $C\subset Y$ when it holds, as the following remark shows.

4.3. REMARK. The set $\Lambda(X,Y)$ is an almost linear subspace of $L(X,(Y,W_Y))$.

4.4. REMARK. We have:

$$\Lambda(X,V_Y)\subset V_{L(X,(Y,C))}=L(X,Y) \tag{4.4}$$

$$\Lambda(V_X,V_Y)\subset L(V_X,(Y,C)) \tag{4.5}$$

$$\Lambda(V_X,V_Y)=L(V_X,(V_Y,C)) \tag{4.6}$$

$$L(X,(R,R_+))=X^{\#} \tag{4.7}$$

Formula (4.5) shows that Definition 4.1 generalizes the notion of a linear operator between two linear spaces and (4.6) shows that when X and Y are linear spaces then the cone C is superfluous and Definition 4.1 is equivalent with the definition of a linear operator $T:X\to Y$. Formula (4.7) shows that Definition 4.1 generalizes the notion of an almost linear functional on an als X.

4.5. REMARK. Let $T\in L(X,(Y,C))$. We have $T\in W_{L(X,(Y,C))}$ iff $T(x)=T(-1\circ x)$ for each $x\in X$. Consequently if $T\in W_{L(X,(Y,C))}$ then $T(X)\subset C$.

4.6. REMARK. If $T\in\Lambda(X,Y)$ then $T(X)$ is an almost linear subspace of Y. If $T\in L(X,(Y,C))$ then $T(X)$ is a convex cone of Y which can be not an almost linear subspace of Y as the following example shows.

4.7. EXAMPLE. Let $Y=\{(\alpha,\beta)\in R^2:\beta\in R_+\}$ be the als described in Example 2.10 and let X be the almost linear subspace of Y defined by $X=\{(\alpha,\beta)\in Y:\beta\geq|\alpha|\}$. We have $W_X=W_Y=\{(0,\beta):\beta\in R_+\}$. Let $T\in L(X,(Y,W_Y))$ be defined by $T((\alpha,\beta))=(\alpha,\alpha+\beta)$, $(\alpha,\beta)\in X$. Then $T(X)=\{(\alpha,\beta)\in Y:\beta\geq 2\alpha\}$ which is not an almost linear subspace of Y since $(-1,0)\in T(X)$ and $-1\circ(-1,0)=(1,0)\notin T(X)$. Clearly $T\notin\Lambda(X,Y)$.

When Y is a nals then we can improve some of the above statements.

4.8. REMARK. When Y is a nals, condition (4.2) in Definition 4.1 can be given only for $\lambda\in R_+\setminus\{0\}$. The fact that it holds for $\lambda=0$ is an immediate consequence of (4.1) and Lemma 2.2 (i). This is no more true when Y is not a nals.

4.9. EXAMPLE. Let $X=R_+$ be the als described in Example 2.11. Let $Y=C=X$ and define $T:X\to X$ by $T(x)=\max\{1,x\}$, $x\in X$. Then T satisfies (4.1), (4.3) and (4.2) for $\lambda\neq 0$ but $T\notin L(X,(X,X))$ since $T(0)=1$.

4.10. REMARK. Let Y be a nals. We have:

$$L(X,Y)=\Lambda(X,V_Y) \tag{4.8}$$

$$\{T|V_X:T\in L(X,(Y,C))\}\subset\Lambda(V_X,V_Y) \tag{4.9}$$

$$\Lambda(V_X,V_Y)=L(V_X,(Y,C)) \tag{4.10}$$

The formulas (4.8)-(4.10) are not true when Y is not a nals.

4.11. EXAMPLE. Let X be the linear space R and let $Y=R$ be the als described in Example 2.12. Since $V_Y=\{0\}$ we have $\Lambda(X,V_Y)=\Lambda(V_X,V_Y)=\{0\}$. Define $T:X\to Y$ by $T(x)=x$. Then (4.8)-(4.10) do not hold for this T.

Suppose now that X and Y are two normed almost linear spaces. For $T\in L(X,(Y,C))$ define

$$|||T|||=\sup\{|||T(x)|||:|||x|||\leq 1\} \tag{4.11}$$

and let $L(X,(Y,C))=\{T\in \mathcal{L}(X,(Y,C)):|||T|||<\infty\}$. It is easy to show that $|||\cdot|||$ defined by (4.11) satisfies $(N_1)-(N_3)$, whence $L(X,(Y,C))$ is an als. It is not always a nals for arbitrary convex cones $C\subset Y$ (see Proposition 4.14 or the example given in the proof of Theorem 4.15 below). Though we shall avoid the word "norm" when (N_4) does not hold, in the sequel we shall always consider the als $L(X,(Y,C))$ eqquiped with the $|||\cdot|||$ defined by (4.11).

4.12. REMARK. If $C\neq\{0\}$ then $L(X,(Y,C))\neq\{0\}$. Indeed, let $c\in C\setminus\{0\}$ and let $f\in X^*\setminus\{0\}$. Define $T(x)=f(x)c$, $x\in X$. Then $T\in \mathcal{L}(X,(Y,C))$ and $|||T|||=|||f|||\,|||c|||<\infty$ and $|||T|||\neq 0$, i.e., $T\in L(X,(Y,C))\setminus\{0\}$. If $C=\{0\}$ then $L(X,(Y,C))$ may be $\{0\}$ (e.g., when $X=W_X$). We also note here that if $C=\{0\}$ then $L(X,(Y,C))$ may be $\neq\{0\}$ (e.g., when X and Y are normed linear spaces).

4.13. REMARK. It is easy to show that if $T\in \mathcal{L}(X,(Y,C))$ and T is continuous then $T\in L(X,(Y,C))$. The converse will be proved in Remark 5.5 in the next section.

We conclude this section with some necessary and (or) sufficient conditions on the convex cone $C\subset Y$ in order that $L(X,(Y,C))$ be a nals. As we observed above, if X is a linear space then the cone $C\subset Y$ is superfluous and *up to the end of this section we suppose* $X\neq V_X$.

4.14. PROPOSITION. *Let* C *be a convex cone of the nals* Y. *In order that* $L(X,(Y,C))$ *be a nals it is necessary that the elements of* C *satisfy* (3.2). *If* $X=W_X$ *then this condition is also sufficient.*

Proof. Suppose $L(X,(Y,C))$ a nals and suppose there are $c_1,c_2\in C$ such that $|||c_1+c_2|||<|||c_1|||$. By Corollary 2.9 there exists $f\in W_{X^*}$, $|||f|||=1$. Define $T_i(x)=f(x)c_i$, $x\in X$, $i=1,2$. By Remark 4.5, $T_1,T_2\in W_{L(X,(Y,C))}$ and we have $|||T_1|||=|||c_1|||$, $|||T_1+T_2|||=|||c_1+c_2|||$ and so (N_4) is not satisfied, contradicting the hypothesis that $L(X,(Y,C))$ is a nals.

The other statement is obvious, since if $X=W_X$ then for each $T\in L(X,(Y,C))$ we have $T(X)\subset C$ and (N_4) follows by (3.2).

Now we show that property (P) of C in Y introduced in Section 3 is a sufficient condition in order that $L(X,(Y,C))$ be a nals. Though this condition is not always necessary (see example below), it is in a certain sense the best possible, as one can see in the next result.

4.15. THEOREM. *Let* C *be a convex cone of the nals* Y. $L(X,(Y,C))$ *is a nals for each nals* X *iff* C *has property* (P) *in* Y.

Proof. Suppose C has property (P) in Y. Let $T\in L(X,(Y,C))$, $T_1\in W_{L(X,(Y,C))}$ and $x\in X$, $|||x|||\leq 1$. By Remark 4.5 we have

$T_1(x)=T_1(-1\circ x)\in C$. Since $T(x)+T(-1\circ x)\in C$ and by hypothesis we get

$$\max\{|||T(x)|||,\ |||T(-1\circ x)|||\}\le$$
$$\le\max\{|||T(x)+T_1(x)|||,\ |||T(-1\circ x)+T_1(x)|||\}\le|||T+T_1|||$$

whence (N_4) follows, i.e., $L(X,(Y,C))$ is a nals.

If C has not property (P) in Y, there exist $y_1,y_2\in Y$, $|||y_2|||\le|||y_1|||$ and $c\in C$ such that $y_1+y_2\in C$ and $\max\{|||y_1+c|||,\ |||y_2+c|||\}<|||y_1|||$. Let X be the almost linear subspace of the als described in Example 2.10, defined by $X=\{(\alpha,\beta)\in R^2:\beta\ge|\alpha|\}$. Define $|||(\alpha,\beta)|||=\beta$ for $(\alpha,\beta)\in X$. Then $(X,|||\cdot|||)$ is a nals. Let $T\in L(X,(Y,C))$, $T_1\in W_{L(X,(Y,C))}$ be defined by

$$T((\alpha,\beta))=\frac{\alpha+\beta}{2}y_1+\frac{\beta-\alpha}{2}y_2 \qquad ((\alpha,\beta)\in X)$$

$$T_1((\alpha,\beta))=\beta c \qquad ((\alpha,\beta)\in X)$$

Since $\beta\ge|\alpha|$ for $(\alpha,\beta)\in X$, we have:

$$|||T((\alpha,\beta))|||\le\frac{\alpha+\beta}{2}|||y_1|||+\frac{\beta-\alpha}{2}|||y_2|||\le\beta|||y_1|||$$

and since $|||T((1,1))|||=|||y_1|||$ it follows that $|||T|||=|||y_1|||$. Furthermore

$$|||(T+T_1)((\alpha,\beta))|||=|||\frac{\alpha+\beta}{2}(y_1+c)+\frac{\beta-\alpha}{2}(y_2+c)|||\le$$
$$\le\beta\max\{|||y_1+c|||,\ |||y_2+c|||\}$$

whence $|||T+T_1|||\le\max\{|||y_1+c|||,\ |||y_2+c|||\}<|||y_1|||=|||T|||$ which shows that $L(X,(Y,C))$ is not a nals.

We give now the example promised before Theorem 4.15.

4.16. EXAMPLE. Let X be the nals described in Example 2.10 and let $C=\{(\alpha,\beta)\in R^2:\alpha,\beta\in R_+\}$. In Example 3.4 we showed that C has not property (P) in X. Let $v=(1,0)\in V_X$, $w=(0,1)\in W_X$. For $(\alpha,\beta)\in X$ we have $(\alpha,\beta)=\alpha\circ v+\beta\circ w$. Let $T_1\in L(X,(X,C))$ and $T_2\in W_{L(X,(X,C))}$. By (4.9) and Remark 4.5 we get $T_i((\alpha,\beta))=\alpha\circ T_i(v)+\beta\circ T_i(w)$, $T_i(v)\in V_X$, $i=1,2$ and $T_2(v)=0$. Let $T_1(v)=(\gamma_o,0)$ and $T_i(w)=(\gamma_i,\delta_i)$, $\gamma_i,\delta_i\in R_+$, $i=1,2$. Then $T_1((\alpha,\beta))=(\alpha\gamma_o+\beta\gamma_1,\beta\delta_1)$ and $T_2((\alpha,\beta))=(\beta\gamma_2,\beta\delta_2)$. Let $(\alpha,\beta)\in X$, $|||(\alpha,\beta)|||\le 1$. If $\alpha\gamma_o\ge 0$ then $|||T_1((\alpha,\beta))|||=\alpha\gamma_o+\beta\gamma_1+\beta\delta_1\le$ $\le|||(T_1+T_2)((\alpha,\beta))|||\le|||T_1+T_2|||$. If $\alpha\gamma_o<0$ then $|\alpha\gamma_o+\beta\gamma_1|<-\alpha\gamma_o+\beta\gamma_1$ and by the above case we get $|||T_1((\alpha,\beta))|||\le|||T_1((-\alpha,\beta))|||\le$

$\leq |||T_1+T_2|||$, i.e., we have (N_4) and so $L(X,(X,C)$ is a nals.

4.17. REMARK. By Proposition 4.14 and Theorem 4.15 we immediately obtain another proof for Proposition 3.3.

5. MAIN RESULT

Let X and Y be two normed almost linear spaces and C a convex cone of Y. *Up to the end of this paper we shall use the following notation*:

$$X_1=\omega_X(X)$$
$$Y_1=\omega_Y(Y)$$
$$C_1=\omega_Y(C)$$

Even when $L(X,(Y,C))$ is not a nals, it has certain properties which we give below.

5.1. LEMMA. (i) *For each* $T\varepsilon L(X,(Y,C))$ *there exists (a unique)* $\tilde{T}\varepsilon L(X_1,(Y_1,C_1))$ *such that* $\omega_Y T=\tilde{T}\omega_X$ *and* $||\tilde{T}||=|||T|||$,

(ii) *The mapping* $I:L(X,(Y,C)) \to L(X_1,(Y_1,C_1))$ *defined by* $I(T)=\tilde{T}$, *is a linear operator such that* $||I(T)||=|||T|||$, $T\varepsilon L(X,(Y,C))$.

(iii) *If* $L(X_1,(Y_1,C_1))$ *is a nals, then* $L(X,(Y,C))$ *is a nals.*

(iv) *If* ω_Y *is one-to-one then* I *is one-to-one and onto* $L(X_1,(Y_1,C_1))$, *and* $L(X,(Y,C))$ *is a nals iff* $L(X_1,(Y_1,C_1))$ *is a nals.*

(v) *We have* $I(L(X,(Y,C))\cap\Lambda(X,Y))\subset L(X_1,(Y_1,C_1))\cap\Lambda(X_1,Y_1)$ *and the equality sign holds if* ω_Y *is one-to-one.*

Proof. (i) Let $T\varepsilon L(X,(Y,C))$. For $\bar{x}\varepsilon X_1$ let $\tilde{T}(\bar{x})=\omega_Y(T(x))$, $x\varepsilon\omega_X^{-1}(\bar{x})$. To show that $\tilde{T}$ is well defined, let $x_1,x_2\varepsilon X$ such that $\omega_X(x_1)=\omega_X(x_2)=\bar{x}$ and let $\varepsilon>0$. By Lemma 2.6, there exist $x'_\varepsilon,x''_\varepsilon,u_\varepsilon\varepsilon X$ such that $|||x'_\varepsilon|||+|||x''_\varepsilon|||<\varepsilon$ and $x_1+x''_\varepsilon+u_\varepsilon=x_2+x'_\varepsilon+u_\varepsilon$. Hence $T(x_1)+T(x''_\varepsilon)+T(u_\varepsilon)=T(x_2)+T(x'_\varepsilon)+T(u_\varepsilon)$ and so $\omega_Y(T(x_1))+\omega_Y(T(x''_\varepsilon))=\omega_Y(T(x_2))+\omega_Y(T(x'_\varepsilon))$. Then $||\omega_Y(T(x_1))-\omega_Y(T(x_2))||=||\omega_Y(T(x'_\varepsilon))-\omega_Y(T(x''_\varepsilon))||\leq\leq|||T|||(|||x'_\varepsilon|||+|||x''_\varepsilon|||)<|||T|||\varepsilon$, whence since $\varepsilon>0$ was arbitrary, we obtain $\omega_Y(T(x_1))=\omega_Y(T(x_2))$, i.e., $\tilde{T}$ is well defined. Using the fact that $\omega_X(W_X)=W_{X_1}$, it is easy to show that $\tilde{T}\varepsilon L(X_1,(Y_1,C_1))$. Since for $x\varepsilon\omega_X^{-1}(\bar{x})$ we have $||\tilde{T}(\bar{x})||=||\omega_Y(T(x))||=|||T(x)|||$ and $||\bar{x}||=|||x|||$, it follows that $||\tilde{T}||=|||T|||<\infty$.

(ii) By (i) above we have $||I(T)||=|||T|||$ for each $T\varepsilon L(X,(Y,C))$. It is straightforward to show that I is a linear operator.

(iii) If $T\varepsilon W_{L(X,(Y,C))}$ then by Remark 4.5 we get that

$I(T)\in W_{L(X_1,(Y_1,C_1))}$. Now (N_4) for $|||\cdot|||$ on $L(X,(Y,C))$ follows by (N_4) for the norm of $L(X_1,(Y_1,C_1))$ using (ii).

(iv) Suppose ω_Y one-to-one. Plainly, I is also one-to-one and to show that I is onto $L(X_1,(Y_1,C_1))$, let $\tilde{T}\in L(X_1,(Y_1,C_1))$. Define

$$(5.1) \qquad T(x)=\omega_Y^{-1}(\tilde{T}(\omega_X(x))) \ , \qquad (x\in X)$$

By Remark 2.5, $T\in \mathcal{L}(X,(Y,C))$ and since $|||T(x)|||=||\tilde{T}(\omega_X(x))||\leq$ $\leq||\tilde{T}||\,|||x|||$ for each $x\in X$, it follows $|||T|||\leq||\tilde{T}||<\infty$, i.e., $T\in L(X,(Y,C))$. By the definition of T we have that $I(T)=\tilde{T}$, i.e., I is onto $L(X_1,(Y_1,C_1))$. For the last assertion in (iv), by (iii) above it remains to show that $L(X_1,(Y_1,C_1))$ is a nals if $L(X,(Y,C))$ is a nals. The proof is similar with the proof of (iii), observing that if $\tilde{T}\in W_{L(X_1,(Y_1,C_1))}$ and $T\in L(X,(Y,C))$ is such that $I(T)=\tilde{T}$ then $T\in W_{L(X,(Y,C))}$.

(v) Let $T\in L(X,(Y,C))\cap\Lambda(X,Y)$ and let $I(T)=\tilde{T}\in L(X_1,(Y_1,C_1))$. Let $\bar{x}\in X_1$ and $x\in X$ such that $\omega_X(x)=\bar{x}$. We have $\tilde{T}(-1\circ\bar{x})=\tilde{T}(\omega_X(-1\circ x))=$ $=\omega_Y(T(-1\circ x))=-1\circ\omega_Y(T(x))=-1\circ\tilde{T}(x)$, i.e., $\tilde{T}\in\Lambda(X_1,Y_1)$. If ω_Y is one-to-one and $\tilde{T}\in L(X_1,(Y_1,C_1))\cap\Lambda(X_1,Y_1)$ then T defined by (5.1) belongs to $L(X,(Y,C))\cap\Lambda(X,Y)$ and we have $I(T)=\tilde{T}$.

5.2. REMARK. Let $\Lambda_b(X,Y)=\{T\in\Lambda(X,Y):|||T|||<\infty\}$ where $|||T|||$ is given by (4.11). Using Remark 4.3 and the fact that $L(X,(Y,W_Y))$ is a nals (by Theorem 4.15), it follows that $\Lambda_b(X,Y)=\Lambda(X,Y)\cap$ $\cap L(X,(Y,W_Y))$ is a nals. By Lemma 5.1 (v) for $C=W_Y$ we have that $I:\Lambda_b(X,Y)\to\Lambda_b(X_1,Y_1)$ is a linear operator such that $||I(T)||=$ $=|||T|||$, $T\in\Lambda_b(X,Y)$, and when ω_Y is one-to-one, then I is one-to-one and onto $\Lambda_b(X_1,Y_1)$.

Let $\mathcal{K}$ be the convex cone of the linear space $\mathcal{L}(E_X,E_Y)$ defined by

$$\mathcal{K}=\{T\in\mathcal{L}(E_X,E_Y):T(X_1)\subset Y_1 \ , \ T(W_{X_1})\subset C_1\}$$

and let

$$K=\mathcal{K}\cap L(E_X,E_Y)$$

5.3. LEMMA. *For* $T\in\mathcal{K}$ *let* $\tilde{T}=T|X_1$. *Then* $\tilde{T}\in\mathcal{L}(X_1,(Y_1,C_1))$ *and* $||\tilde{T}||=||T||$.

<u>Proof</u>. Clearly $\tilde{T}\in\mathcal{L}(X_1,(Y_1,C_1))$ and $||\tilde{T}||\leq||T||$. Let now $z\in E_X$ $||z||<1$. There exist $\bar{x}_1,\bar{x}_2\in X_1$ such that $z=\bar{x}_1-\bar{x}_2$ and $||\bar{x}_1||+||\bar{x}_2||\leq 1$.

We have $||T(z)||\le||T(\bar{x}_1)||+||T(\bar{x}_2)||=||\tilde{T}(\bar{x}_1)||+||\tilde{T}(\bar{x}_2)||\le$ $\le||\tilde{T}||(||\bar{x}_1||+||\bar{x}_2||)\le||\tilde{T}||$, whence $||T||\le||\tilde{T}||$.

5.4. LEMMA (i) *The cone* K *can be organized as an als where the addition and the multiplication by non-negative reals are as in* $L(E_X,E_Y)$.

(ii) K *is an almost linear subspace of* K *and the als* K *together with the norm* $||\cdot||$ *of* $L(E_X,E_Y)$ *satisfy* $(N_1)-(N_3)$.

(iii) *The mapping* $J:K\to L(X_1,(Y_1,C_1))$ *defined by* $J(T)=T|X_1$, $T\in K$, *is a linear operator such that* $||J(T)||=||T||$, $T\in K$, *and* J *is one-to-one and onto* $L(X_1,(Y_1,C_1))$.

(iv) $(K,||\cdot||)$ *is a nals iff* $L(X_1,(Y_1,C_1))$ *is a nals.*

Proof. (i) Observing that if $T_1,T_2,T\in K$ and $\lambda\in R_+$ then $T_1+T_2\in K$ and $\lambda\circ T=\lambda T\in K$, it remains to define $-1\circ T\in K$. For $z\in E_X$, $z=\bar{x}_1-\bar{x}_2$, $\bar{x}_i\in X_1$, i=1,2, let $(-1\circ T)(z)=T(-1\circ\bar{x}_1)-T(-1\circ\bar{x}_2)\in E_Y$. It is easy to show that $-1\circ T$ is well defined and that $-1\circ T\in K$. Now a simple verification shows that K is an als.

(ii) Let $T\in K$. Since $(-1\circ T)|X_1=-1\circ(T|X_1)$, by Lemma 5.3 it follows that $||-1\circ T||=||(-1\circ T)|X_1||=||T|X_1||=||T||<\infty$. The proof of the assertions in (ii) is now obvious.

(iii) By Lemma 5.3, for $T\in K$ we have $J(T)\in L(X_1,(Y_1,C_1))$ and $||J(T)||=||T||$. It is straightforward to show that J is a linear operator which is one-to-one. Let now $\tilde{T}\in L(X_1,(Y_1,C_1))$ and for $z\in E_X$, $z=\bar{x}_1-\bar{x}_2$, $\bar{x}_i\in X_1$, i=1,2, define $T(z)=\tilde{T}(\bar{x}_1)-\tilde{T}(\bar{x}_2)\in E_Y$. This mapping is well defined and $T\in L(E_X,E_Y)$. Clearly $T\in K$ and $T|X_1=\tilde{T}$. By Lemma 5.3 we get $||T||=||\tilde{T}||<\infty$, i.e., $T\in K$ and since $J(T)=\tilde{T}$ it follows that J is onto $L(X_1,(Y_1,C_1))$.

(iv) Using Remark 4.5 and the definition of $-1\circ T$ for $T\in K$ it is easy to show that $T\in W_K$ iff $J(T)\in W_{L(X_1,(Y_1,C_1))}$. The assertions of (iv) follow now immediately.

We can now prove the converse statement in Remark 4.13.

5.5. REMARK. If $T\in L(X,(Y,C))$ then T is continuous. Indeed, let $T_1=J^{-1}I(T)\in K$, where I and J are given by Lemmas 5.1 and 5.4. Then $I(T)=J(T_1)=T_1|X_1$. Now let $x_n,x\in X$ such that $\lim_{n\to\infty}\rho_X(x_n,x)=0$. We have $\rho_Y(T(x_n),T(x))=||\omega_Y(T(x_n))-\omega_Y(T(x))||=||I(T)(\omega_X(x_n))-I(T)(\omega_X(x))||=$ $=||T_1(\omega_X(x_n))-T_1(\omega_X(x))||\to 0$, since $T_1\in L(E_X,E_Y)$ and $||\omega_X(x_n)-$ $-\omega_X(x)||=\rho_X(x_n,x)\to 0$.

The main result of this paper is the next theorem which gives $(E,||\cdot||)$ and ω from Theorem 2.3 for $L(X,(Y,C))$ when it is a nals. Unfortunately we are able to prove it under the stronger assumption (in view of Lemma 5.1 (iii)) that $L(X_1,(Y_1,C_1))$ is a nals. Let I and J be given by Lemmas 5.1 and 5.4, and denote by K_1 the following subset of $L(E_X,E_Y)$:

$$K_1=J^{-1}I(L(X,(Y,C)))$$

5.6. THEOREM. *If* $L(X_1,(Y_1,C_1))$ *is a nals, then for the nals* $L(X,(Y,C))$ *the following assertions are true:*

(i) $E_{L(X,(Y,C))}$ *is a linear subspace of* $L(E_X,E_Y)$ *and we have* $E_{L(X,(Y,C))}=K_1-K_1$. *The norm on* $E_{L(X,(Y,C))}$ *is defined for* $T\varepsilon E_{L(X,(Y,C))}$ *by*

$$||T||_{E_{L(X,(Y,C))}}=\inf\{||T_1||_{L(E_X,E_Y)}+||T_2||_{L(E_X,E_Y)}\}$$

where the inf is taken over all $T_1,T_2\varepsilon K_1$ *such that* $T=T_1-T_2$. *Moreover*

$$||T||_{E_{L(X,(Y,C))}}=||T||_{L(E_X,E_Y)} \qquad (T\varepsilon K_1)$$

(ii) *We have* $\omega_{L(X,(Y,C))}=J^{-1}I$ *and* $\omega_{L(X,(Y,C))}(L(X,(Y,C)))=K_1$ *is an almost linear subspace of the als* K *such that* $(K_1,||\cdot||_{L(E_X,E_Y)})$ *is a nals.*

(iii) *If* ω_Y *is one-to-one then the conclusions of* (i) *and* (ii) *hold for* $K_1=K$ *and the mapping* $\omega_{L(X,(Y,C))}$ *is now one-to-one.*

<u>Proof</u>. As we have noted above, since $L(X_1,(Y_1,C_1))$ is a nals, by Lemma 5.1 (iii), $L(X,(Y,C))$ is also a nals. Using Lemmas 5.1 and 5.4 together with the observation that since $J^{-1}I$ is a linear operator then K_1 is an almost linear subspace of K, it is easy to show that the linear space K_1-K_1 endowed with the norm defined at (i) above, and the linear operator $J^{-1}I$ satisfy all the requirements of Theorem 2.3 for the nals $L(X,(Y,C))$, as well as (i)-(iii) above.

Even when ω_Y is one-to-one, we have not the equality sign in the inclusion $K-K\subset L(E_X,E_Y)$, as the following example shows.

5.7. EXAMPLE. Let X be the nals described in Example 2.10, $Y=R^2$ endowed with the Euclidean norm and $C\subset Y$ be the convex cone $\{(\alpha,0):\alpha\varepsilon R_+\}$. Since C has property (P) in Y, by Theorem 4.15, $L(X,(Y,C))$ is a nals. We have $X=X_1$, $Y=Y_1=E_Y$ and $E_X=R^2$ endowed with the norm $||(\alpha,\beta)||=|\alpha|+|\beta|$, $(\alpha,\beta)\varepsilon R^2$. Let $T\varepsilon L(E_X,E_Y)$ be defined by $T((\alpha,\beta))=(\alpha,\beta)$, $(\alpha,\beta)\varepsilon E_X$. Suppose $T=T_1-T_2$, $T_i\varepsilon K$, i=1,2. Then for the element $(0,1)\varepsilon W_X$, we must have $T_i((0,1))=(\alpha_i,0)\varepsilon C$, i=1,2. Hence $T((0,1))=(0,1)=T_1((0,1))-T_2((0,1))=(\alpha_1-\alpha_2,0)$, which is not possible.

6. APPLICATIONS

The aim of this section is to obtain certain classical theorems from the the theory of operators in normed linear spaces, within the framework of normed almost linear spaces. For the proofs we shall use Theorem 5.6, the corresponding theorem known in normed linear spaces, as well as the following result.

6.1. LEMMA. *A nals* $(X,|||\cdot|||)$ *is complete iff* $(E_X, ||\cdot||)$ *is a Banach space and* X_1 *is norm-closed in* E_X.

Proof. Suppose X complete. Then X_1 is complete in the $||\cdot||$ of E_X and so closed in E_X. We show now that E_X is a Banach space. Let $\{z_n\}_{n=1}^{\infty} \subset E_X$ be a Cauchy sequence. We can suppose (passing to a subsequence if necessary) that for each $n\in N$ we have

$$||z_n - z_{n+p}|| < \frac{1}{2^{n+1}} \qquad \text{for each } p\geq 1$$

Let $z_1 = x_1 - y_1$, $x_1, y_1 \in X_1$. Since $||z_2 - z_1|| < 1/2^2$, there exist $x_2, y_2 \in X_1$ such that $z_2 - z_1 = x_2 - y_2$ and $||x_2|| + ||y_2|| < 1/2^2$. Then $z_2 = (x_1 + x_2) - (y_1 + y_2)$ where $||x_2|| < 1/2^2$, $||y_2|| < 1/2^2$. By induction on n we find two sequences $\{x_i\}_{i=1}^{\infty}$, $\{y_i\}_{i=1}^{\infty} \subset X_1$ such that for each $n\in N$ we have $z_n = (\sum_{i=1}^n x_i) - (\sum_{i=1}^n y_i)$ and for $n\geq 2$ we have $||x_n|| < 1/2^n$, $||y_n|| < 1/2^n$. For each $n\in N$, let $\bar{x}_n = \sum_{i=1}^n x_i \in X_1$ and $\bar{y}_n = \sum_{i=1}^n y_i \in X_1$. Clearly, $\{\bar{x}_n\}_{n=1}^{\infty}$ and $\{\bar{y}_n\}_{n=1}^{\infty}$ are Cauchy sequences and since X_1 is complete, there exist $\bar{x}, \bar{y} \in X_1$ such that $\lim_{n\to\infty} ||\bar{x}_n - \bar{x}|| = 0$ and $\lim_{n\to\infty} ||\bar{y}_n - \bar{y}|| = 0$. Then for $z = \bar{x} - \bar{y} \in E_X$ we have $\lim_{n\to\infty} ||z_n - z|| = 0$, i.e., E_X is a Banach space. The "if" part is obvious.

Simple examples show that the assumption $(E_X, ||\cdot||)$ be a Banach space does not imply that X_1 is norm-closed in E_X.

We can now prove e.g. the extensions of Banach-Steinhaus Theorem and the inverse mapping theorem from the theory of normed linear spaces.

6.2. THEOREM. *Let* X *be a complete nals,* Y *a nals such that* ω_Y *is one-to-one and* $C\subset Y$ *a closed convex cone such that* $L(X,(Y,C))$ *is a nals. Let* $\{T_n\}_{n=1}^{\infty}$ *be a sequence in* $L(X,(Y,C))$ *such that* $\lim_{n\to\infty} \rho_Y(T_n(x), T(x)) = 0$ *for each* $x\in X$. *Then the sequence* $\{|||T_n|||\}_{n=1}^{\infty}$ *is bounded and* $T\in L(X,(Y,C))$.

Proof. Since ω_Y is one-to-one and C closed, it is easy to show that $T\in L(X,(Y,C))$. Now for each $x\in X$, $|||x|||\leq 1$ we have $|||T(x)||| = ||\omega_Y(T(x))|| \leq ||\omega_Y(T(x)) - \omega_Y(T_n(x))|| + ||\omega_Y(T_n(x))|| = \rho_Y(T_n(x), T(x)) + |||T_n(x)||| \leq \rho_Y(T_n(x), T(x)) + |||T_n|||$ for each $n\in N$, and so if we show that $\{|||T_n|||\}_{n=1}^{\infty}$ is bounded, then $T\in L(X,(Y,C))$. Since ω_Y is one-to-one, by hypothesis and Lemma 5.1 (iv), $L(X_1,(Y_1,C_1))$ is a nals. By Theorem 5.6, $\omega_{L(X,(Y,C))}(T_n)\in K$, $n\in N$. Then $\omega_{L(X,(Y,C))}(T_n)|X_1=\tilde{T}_n\in L(X_1,(Y_1,C_1))$ and $\omega_Y T_n=\tilde{T}_n\omega_X$, $n\in N$. Hence and by hypothesis we have for each $x\in X$ that $0=\lim_{n\to\infty}\rho_Y(T_n(x),T(x))=$ $=\lim_{n\to\infty}||\omega_Y(T_n(x))-\omega_Y(T(x))||=\lim_{n\to\infty}||\tilde{T}_n(\omega_X(x))-\omega_Y(T(x))||$ and so for each $\bar{x}\in X_1$ the sequence $\{T_n(\bar{x})\}_{n=1}^{\infty}$ converges to an element of Y_1. Let $z\in E_X$, $z=\bar{x}_1-\bar{x}_2$, $\bar{x}_i\in X_1$, $i=1, 2$. Then $\omega_{L(X,(Y,C))}(T_n)(z)=$ $=\tilde{T}_n(\bar{x}_1)-\tilde{T}_n(\bar{x}_2)$ and so the sequence $\{\omega_{L(X,(Y,C))}(T_n)(z)\}_{n=1}^{\infty}$ converges to an element of E_Y. By Lemma 6.1, E_X is a Banach space, whence by Banach-Steinhaus Theorem the sequence $\{||\omega_{L(X,(Y,C))}(T_n)||\}_{n=1}^{\infty}$ is bounded. Since $||\omega_{L(X,(Y,C))}(T_n)||=|||T_n|||$ for each $n\in N$, the sequence $\{|||T_n|||\}_{n=1}^{\infty}$ is bounded.

6.3. THEOREM. *Let* X,Y *be two complete normed almost linear spaces such that both* ω_X *and* ω_Y *are one-to-one. If* $T\in L(X,(Y,W_Y))$ *is one-to-one and onto* Y *and* $T(W_X)=W_Y$, *then the inverse operator* $T^{-1}\in L(Y,(X,W_X))$.

<u>Proof</u>. By Remark 2.4 we have $\omega_X(W_X)=W_{X_1}$ and $\omega_Y(W_Y)=W_{Y_1}$. By Theorem 4.15, $L(X,(Y,W_Y))$, $L(X_1,(Y_1,W_{Y_1}))$, $L(Y,(X,W_X))$ and $L(Y_1,(X_1,W_{X_1}))$ are normed almost linear spaces. Let $T\in L(X,(Y,W_Y))$ be one-to-one and onto Y and $T(W_X)=W_Y$, and let $T_1=\omega_{L(X,(Y,W_Y))}(T)\in K$. Then $T_1|X_1=\tilde{T}\in L(X_1,(Y_1,W_{Y_1}))$ and $\tilde{T}\omega_X=\omega_Y T$. We show that the bounded linear operator $T_1:E_X\to E_Y$ is one-to-one and onto E_Y. Let $z_1,z_2\in E_X$ such that $T_1(z_1)=T_1(z_2)$. Let $x_i\in X$, $1\le i\le 4$, such that $z_1=\omega_X(x_1)-\omega_X(x_2)$ and $z_2=\omega_X(x_3)-\omega_X(x_4)$. Then $T_1(z_1)=\tilde{T}(\omega_X(x_1))-\tilde{T}(\omega_X(x_2))=\omega_Y(T(x_1))-$ $-\omega_Y(T(x_2))$, and similarly, $T_1(z_2)=\omega_Y(T(x_3))-\omega_Y(T(x_4))$, and so $\omega_Y(T(x_1+x_4))=\omega_Y(T(x_2+x_3))$. Since ω_Y and T are one-to-one, it follows that $x_1+x_4=x_2+x_3$, whence $z_1=\omega_X(x_1)-\omega_X(x_2)=\omega_X(x_3)-\omega_X(x_4)=z_2$, i.e., T_1 is one-to-one. Let now $u\in E_Y$ and $y_1,y_2\in Y$ such that $u=\omega_Y(y_1)-\omega_Y(y_2)$. Since T is onto Y there exist $x_1,x_2\in X$ such that $y_i=T(x_i)$, $i=1,2$. Let $z=\omega_X(x_1)-\omega_X(x_2)\in E_X$. We have $T_1(z)=\tilde{T}(\omega_X(x_1))-\tilde{T}(\omega_X(x_2))=$ $=\omega_Y(T(x_1))-\omega_Y(T(x_2))=\omega_Y(y_1)-\omega_Y(y_2)=u$, i.e., T_1 is onto E_Y. By the inverse mapping theorem, there exists $T_1^{-1}\in L(E_Y,E_X)$ such that $T_1^{-1}(T_1(z))=z$ for each $z\in E_X$. We show now that the following inclusions hold:

$$(6.1)\qquad T_1^{-1}(Y_1)\subset X_1$$

$$(6.2)\qquad T_1^{-1}(W_{Y_1})\subset W_{X_1}$$

For the proof of (6.1), let $\bar{y}\in Y_1$ and $z\in E_X$ such that $T_1^{-1}(\bar{y})=z$. Let $y\in Y$ such that $\bar{y}=\omega_Y(y)$ and let $x\in X$ such that $T(x)=y$. Then $T_1(z)=\bar{y}=$ $=\omega_Y(T(x))=\tilde{T}(\omega_X(x))=T_1(\omega_X(x))$ and since T_1 is one-to-one, it follows that $z=\omega_X(x)\in X_1$. For the proof of (6.2), let $\bar{w}_1\in W_{Y_1}$. By (6.1) we get $T_1^{-1}(\bar{w}_1)=\bar{x}\in X_1$. By Remark 2.4, there exists $w_1\in W_Y$ with $\bar{w}_1=\omega_Y(w_1)$. By hypothesis there exists $w\in W_X$ such that $w_1=T(w)$. We have $T_1(\bar{x})=\bar{w}_1=$ $=\omega_Y(T(w))=\tilde{T}(\omega_X(w))=T_1(\omega_X(w))$, and since T_1 is one-to-one, we get $\bar{x}=\omega_X(w)$. Again by Remark 2.4, $\bar{x}\in W_{X_1}$.

Using (6.1), (6.2) and the hypothesis that ω_X is one-to-one, by Theorem 5.6, there exists $T'\in L(Y,(X,W_X))$ such that $\omega_{L(Y,(X,W_X))}(T')=$ $=T_1^{-1}$. It remains to show that for each $x\in X$ we have $T'(T(x))=x$, i.e., $T'=T^{-1}$. Let us denote by $I':L(Y,(X,W_X)) \to L(Y_1,(X_1,W_{X_1}))$ the mapping given by Lemma 5.1 (ii). Let $x\in X$ and $y=T(x)$. We have $\omega_X(T'(T(x))=$ $=\omega_X(T'(y))=(I'(T'))(\omega_Y(y))=\omega_{L(Y,(X,W_X))}(T')(\omega_Y(y))=T_1^{-1}(\omega_Y(y))=$

$=T_1^{-1}(\omega_Y(T(x)))=T_1^{-1}(\tilde{T}(\omega_X(x)))=T_1^{-1}(T_1(\omega_X(x)))=\omega_X(x)$. Since ω_X is one--to-one, we get $T'(y)=x$, which completes the proof.

As one can see in the above Theorems 5.2 and 5.3, the formulations in our more general setting of some results known in the theory of operators in normed linear spaces is not difficult. The above method may be used to prove other results. We can not prove or disprove in the framework of normed almost linear spaces the closed graph theorem and the open mapping theorem. We also do not know whether a nals $L(X,(Y,C))$ is complete if Y is complete. It is easy to show that if V_Y is a Banach space then $V_{L(X,(Y,C))}$ is a Banach space.

REFERENCES

1. DAY, M.M. "Normed Linear Spaces", 3rd ed., Springer-Verlag, New York-Heidelberg-Berlin, 1973.

2. DUNFORD, N. and SCHWARTZ, J.T. "Linear Operators". Part I. General Theory, Pure and Applied Mathematics, 7, New York, London Interscience, 1958.

3. GODINI, G. "A framework for best simultaneous approximation: normed almost linear spaces", J. Approximation Theory, 43 no. 4 (1985), 338-358.

4. GODINI, G. "An approach to generalizing Banach spaces: normed almost linear spaces", Proceedings of the 12th Winter School on Abstract Analysis (Srni 1984). Suppl. Rend. Circ. Mat. Palermo Serie

II, no. 5 (1984), 33-50.

5. GODINI, G. "Best approximation in normed almost linear spaces", in "Constructive Theory of Functions". Proceedings of the International Conference on Constructive Theory of Functions (Varna 1984). Publishing House of the Bulgarian Academy of Sciences, Sofia, 1984, 356-363.

6. GODINI, G. "On normed almost linear spaces", Preprint Series Math., INCREST, Bucureşti, 38 (1985).

7. SCHAEFER, H. "Topological Vector Spaces", New York, London, 1966.

G. GODINI
DEPARTMENT OF MATHEMATICS
INCREST, Bd. PACII 220
79622 BUCHAREST, ROMANIA