Hurewicz Properties, Non Distinguishing Convergence Properties and Sequence Selection Properties

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We shall compare several properties of a topological space related to the behavior of open coverings or to the behavior of sequences of continuous real-valued functions defined on the space. We shall show that there are closed relationships between them and several of them are mutually equivalent.

1. Introduction

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We shall compare several properties of a topological space related to the existence of some special subcovers of open covers or to the behavior of sequences of continuous real-valued functions defined on the space. All of them were considered in literature in different circumstances, see e.g. [Ar1], [BRR1], [BRR2], [Fr1], [Hu1], [Me], [Ro], [Sc1], [Sc2], and it turned out that there are closed relationships between them. Moreover, several of them are mutually equivalent.

We shall use standard set theoretical terminology and notations, say those of [Je]. Actually, all our reasoning are done in Zermelo-Fraenkel axiomatic set theory with the axiom of choice **ZFC**.

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The set ${}^{\omega}\mathbb{R}$ of all functions from ω into $\mathbb{R}$ is quasi-ordered by the **eventual dominating relation**

$$f \leq^* g \equiv \{n \in \omega; \neg f(n) \leq g(n)\} \text{ is finite.}$$

The set ${}^{\omega}\omega$ considered as a substructure of ${}^{\omega}\mathbb{R}$ has similar combinatorial properties. A set $\mathscr{F} \subseteq {}^{\omega}\mathbb{R}$ is **dominating** if for any $g \in {}^{\omega}\mathbb{R}$ there exists a $f \in \mathscr{F}$ such that $g \leq^* f$. A set $\mathscr{F} \subseteq {}^{\omega}\mathbb{R}$ is **bounded** if there exists an $g \in {}^{\omega}\mathbb{R}$ such that $f \leq^* g$ for any $f \in \mathscr{F}$. For the definition of cardinals $\mathfrak{p}$, $\mathfrak{b}$, $\mathfrak{d}$ and their basic properties see e.g. [BJ], [Je], [Va2]. The cardinal invariants add$(\mathscr{I})$, cof$(\mathscr{I})$, cov$(\mathscr{I})$ and non$(\mathscr{I})$, where $\mathscr{I}$ is a family of sets (usually an ideal), are also defined in [BJ], [Je], [Va2]. We shall mainly deal with the ideal $\mathscr{M}$ of meager subsets of $\mathbb{R}$ and the ideal $\mathscr{N}$ of Lebesgue measure zero subsets of $\mathbb{R}$. For consistency of inequalities between above mentioned cardinals, which we shall use in our examples, see e.g. [BJ], [Va2].

We shall follow the basic topological terminology and notations of [En] with explicity stated exceptions. Generally we do not assume any axiom of separation. By **a real function** f on a topological space X we shall understand a continuous function $f : X \to \mathbb{R}$. The **zero-set of** f is the set

$$\mathrm{Z}(f) = \{x \in X; f(x) = 0\}.$$

A topological space X is said to be **perfectly normal** if X is normal and every closed set is a G_δ-set. Then for every closed set $A \subseteq X$ there is a real function f on X such that $A = \mathrm{Z}(f)$.

An open cover $\mathscr{U}$ of X is a set of open subsets of X such that $\bigcup \mathscr{U} = X$. An open cover $\mathscr{V} \subseteq \mathscr{U}$ is said to be **a subcover** of $\mathscr{U}$. An open cover $\mathscr{U}$ is **a γ-cover** if every point $x \in X$ is in all but finitely many sets from $\mathscr{U}$. Let us remark that a finite cover is a γ-cover. An open cover $\mathscr{U}$ is **an ω-cover** if for every finite $A \subseteq X$ there is a $U \in \mathscr{U}$ such that $A \subseteq U$. A topological space X has **Lindelöf property**[1] if every open cover of X has a countable subcover.

Let $\alpha \leq \mathfrak{c}$ be an uncountable cardinal. A set $X \subseteq \mathbb{R}$ is said to be **an α-Luzin set (an α-Sierpinski set)** if X is not meager (not Lebesgue measure zero set) and for every meager (Lebesgue measure zero) set B we have $|X \cap B| < \alpha$.

2. Hurewicz Properties

We start with recalling two properties of sequences of real functions on a topological space X considered by W. Hurewicz [Hu1]:

H*: *for any sequence $\{f_n\}_{n=0}^{\infty}$ of continuous functions from X into $\mathbb{R}$ the family of sequences of reals $\{\{f_n(x)\}_{n=0}^{\infty}; x \in X\}$ is not dominating.*

[1] We do not ask that the space is regular as R. Engelking [En] does.

H**: *for any sequence* $\{f_n\}_{n=0}^{\infty}$ *of continuous functions from* X *into* $\mathbb{R}$ *the family of sequences of reals* $\{\{f_n(x)\}_{n=0}^{\infty}; x \in X\}$ *is bounded.*
Evidently

$$\mathrm{H}^{**} \rightarrow \mathrm{H}^{*}.$$

W. Hurewicz has proved that for metric separable spaces those properties are equivalent to the following covering properties, respectively:
E*: *for every sequence* $\{\mathcal{U}_n\}_{n=0}^{\infty}$ *of open covers of* X *there exist finite subsets* $\mathcal{V}_n \subseteq \mathcal{U}_n$ *such that* $\{\bigcup \mathcal{V}_n; n \in \omega\}$ *is a cover of* X.
E**: *for every sequence* $\{\mathcal{U}_n\}_{n=0}^{\infty}$ *of open covers of* X *there exist finite subsets* $\mathcal{V}_n \subseteq \mathcal{U}_n$ *such that* $\{\bigcup \mathcal{V}_n; n \in \omega\}$ *is a* γ*-cover of* X.
Again, evidently

$$\mathrm{E}^{**} \rightarrow \mathrm{E}^{*}.$$

Replacing in latter definitions the words 'open covers' by 'countable open covers', we obtain notions of properties $\mathbf{E}^{*}_{\omega}$ and $\mathbf{E}^{**}_{\omega}$, respectively.

W. Hurewicz [Hu1] actually proves that

$$\mathrm{E}^{*}_{\omega} \rightarrow \mathrm{H}^{*}, \qquad \mathrm{E}^{**}_{\omega} \rightarrow \mathrm{H}^{**}.$$

One can easily see that for a topological space with Lindelöf property we have $\mathrm{E}^{**}_{\omega} \rightarrow \mathrm{E}^{**}$ and $\mathrm{E}^{*}_{\omega} \rightarrow \mathrm{E}^{*}$. Moreover the original proofs of [Hu1] can be easily modified, see e.g. [BH], to proofs of implications

$$\mathrm{H}^{*} \rightarrow \mathrm{E}^{*}_{\omega}, \qquad \mathrm{H}^{**} \rightarrow \mathrm{E}^{**}_{\omega} \text{ both for perfectly normal spaces.}$$

So for perfectly normal spaces with Lindelöf property, especially for metric separable spaces, the properties E*, E^{*}_{ω}, H* and E**, E^{**}_{ω}, H** are mutually equivalent, respectively. The former were considered already by M. Menger [Me] and are in the case of metric separable spaces usually referred to as **Menger property** M. The latter are in this case referred to as **Hurewicz property** H.

Example 1.

a) *Evidently* $\mathrm{non}(\mathrm{H}^{**}) = \mathfrak{b}$ *and* $\mathrm{non}(\mathrm{H}^{*}) = \mathfrak{d}$. *Thus, if* $\mathfrak{b} < \mathfrak{d}$ *then the discrete space of cardinality* $\mathfrak{b}$ *possesses properties* H* *and* E^{*}_{ω} *and does not possess properties* H** *and* E^{**}_{ω}.

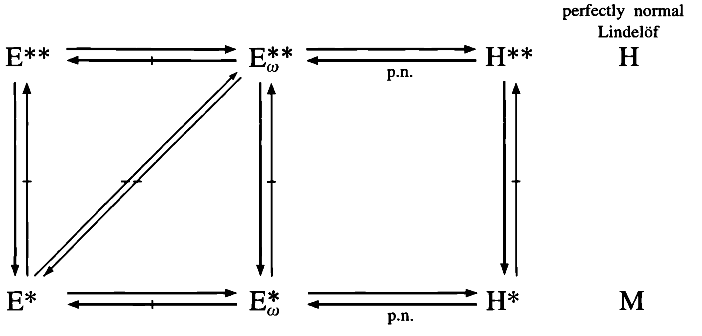
b) *If* $\alpha = \mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$ *then there exists an* α*-Luzin set* $L \subseteq [0, 1]$ (see e.g. [Ci]). *Since in this case* $\mathfrak{d} = \alpha$ *one can easily show that* L *possesses property* E* (essentially [Sie], compare [ScTs]). *On the other hand one can easily show that any subset of a separable metric space with empty interior with property* E** *is meager* (see [Hu1]). *Hence* L *has neither property* E** *nor* H**.

c) *A countably compact non-compact topological space is* E^{**}_{ω} *and is not* E*. *There are examples of such spaces* (e.g. [En], pp. 261 – 262), *however none of them is perfectly normal. The existence of a perfectly normal countably compact non-compact space neither can be proved nor can be refuted in* **ZFC** (see e.g. [Val]).

Theorem 1. *An analytic space with property* E* *is* σ*-compact.*

A V. Archangelskij [Ar2] extended a result by W. Hurewicz [Hu2] for metric separable spaces as where 'analytic' means 'a continuous image of the Baire space ${}^{\omega}\omega$'. Since a σ-compact space has property E**, a counterexample for E* $\to$ E** must not be an analytic space. Note that an α-Luzin set is neither analytic nor a complement of an analytic set in $\mathbb{R}$.

We can summarize the relationships between considered notions in a diagram.



A thick arrow indicates a theorem of **ZFC**. A thin arrow (negative) indicates a consistency result. 'p.n.' means that the arrow is proved for perfectly normal spaces only. The main open problem (see [BH]) is as follows:

Problem 1. *Find in* **ZFC** *a perfectly normal* E^{**}_{ω}*-space which does not possess property* E*.

3. Not Distinguishing Convergences

We say that a sequence $\{f_n\}_{n=0}^{\infty}$ converges **quasi-normally to a function** f **on** X, written '$f_n \xrightarrow{QN} f$ on X', if there is a sequence of positive reals $\{\varepsilon_n\}_{n=0}^{\infty}$ **(a control)** converging to 0 such that

$$(1) \qquad (\forall x \in X)(\exists n_0)(\forall n \geq n_0)\, |f_n(x) - f(x)| < \varepsilon_n .$$

Similarly, the series $\sum_{n=0}^{\infty} f_n$ converges **pseudo-normally on** X if there is a control sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and (1) holds true (with $f = 0$).

Properties of a topological space related to non-distinguishing convergences of sequences of real functions were introduced and studied in [BRR1] and [BRR2]. A topological space X is said to be **a wQN-space** if from every sequence of continuous functions converging to 0 on X one can choose a quasi-normally convergent subsequence. Adding the words 'non-increasing' we obtain the notion of **an mQN-space**. Let us remark that if a subsequence of a monotonic sequence converges quasi-normally then the whole sequence converges also so. Similarly, a topological space X is said to be **a $\Sigma\Sigma^*$-space** if for every sequence $\{f_n\}_{n=0}^{\infty}$ of real functions such that $\sum_{n=0}^{\infty} |f_n(x)| < \infty$ for every $x \in X$ the series converges also pseudo-normally. Finally a topological space is **a $\overline{\text{QN}}$-space** if every sequence of real functions converging pointwise to a function on X (not necessarily continuous) converges to this function quasi-normally. We have

Theorem 2. $\Sigma\Sigma^* \to \overline{\text{QN}}$ *for perfectly normal space and* $\overline{\text{QN}} \to \text{wQN} \to \text{mQN}$ *for any topological space.*

The only non-trivial implication, the first one, is proved in [BRR2]. However, the implication is a consequence of several results, some of them use covering properties and therefore they work for perfectly normal spaces only. So

Problem 2. *Prove* $\Sigma\Sigma^* \to \overline{\text{QN}}$ *for arbitrary topological space.*

Theorem 3. ([BRR1])
a) *Perfectly normal* wQN-*space has property* E_{ω}^{**}.
b) *Metric separable* wQN-*space is perfectly meager.*

According to T. Bartoszynski [BJ] characterization of $\text{add}(\mathcal{N})$ we obtain that every set of reals of cardinality smaller than $\text{add}(\mathcal{N})$ is a $\Sigma\Sigma^*$-set.

Example 2.
a) *If* $\text{add}(\mathcal{N}) > \aleph_1$ *then the discrete space of cardinality* $\aleph_1$ *is a* $\Sigma\Sigma^*$*-set but does not possess property* E^*.
b) *If* $\mathfrak{b} > \aleph_1$ *then the discrete space of cardinality* $\aleph_1$ *is a* wQN-*space but does not have property* E*.
c) *Every* σ*-compact topological space has property* E** *and therefore it is also an* mQN-*space. Thus there is an* mQN-*space, even a space with property* E**, *e.g. the unit closed interval, that is not a* wQN-*space.*

The main result of [BH] is

Theorem 4. mQN $\to$ H**.

Proof. For details see [BH], Theorem 6. If $f_n : X \to \mathbb{R}^+$, $n \in \omega$ is a sequence of real functions converging to 0 on X we set

$$g_k(x) = \sum_{n=0}^{\infty} 2^{-n} \min \{1, f_n(x)/(k+1)\}.$$

One can easily check that the sequence $\{g_k\}_{k=0}^{\infty}$ is non-increasing and converges to 0 on X. Since X is an mQN-space there exists a non-increasing sequence $\{\varepsilon_k\}_{k=0}^{\infty}$ of positive reals such that

$$(\forall x \in X)\,(\exists k_0)\,(\forall k \geq k_0)\; g_k(x) < \varepsilon_k.$$

Then the sequence $d_n = \min\{m; \varepsilon_m < 2^{-m}\}$ eventually dominates each sequence of reals $\{f_n(x)\}_{n=0}^{\infty}$, $x \in X$.

q.e.d.

Theorem 3.8 (2) of [BRR2] says that this implication holds true for metric separable spaces (one can easily see that 'nestled' = H**).

4. Covering Properties

A topological space X is said to be **a γ-space** if from every ω-cover one can choose a γ-subcover. Let us remind that the space $C_p(X)$ as a subspace of the product space ${}^X\mathbb{R}$ is Fréchet-Urysohn[2] if and only if X is a γ-space (see [GN]).

A topological space X has **Rothberger property** or **property C''** (see [Ro]) if from every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers one can choose a cover $\{U_n : n \in \mathbb{N}\}$ with each $U_n \in \mathcal{U}_n$. Immediately from the definitions we obtain

Theorem 5. *Rothberger property implies* E*.

A metric space X is said to have **strong measure zero** if for every sequence of positive reals $\{\varepsilon_m\}_{n=0}^{\infty}$ there exists an open cover $\{U_n : n \in \mathbb{N}\}$ of X such that diameter$(U_n) < \varepsilon_n$. Every metric space with Rothberger property has strong measure zero.

Theorem 6. ([BRR1], [GN], [MF], [BH]) *A γ-space is a* wQN-*space, has Rothberger property and property* E**.

Recently in [BC] the authors proved that a metric separable $\Sigma\Sigma^*$-space has Rothberger property. The proof is based on Fremlin-Miller [MF] characterization of Rothberger property by compatible metrics. We present a slight modification of this result for a larger class of spaces.

Theorem 7. (Essentially [BC]) *Completely regular $\Sigma\Sigma^*$-space with Lindelöf property has Rothberger property.*

Lemma 8. *Assume that $\{m_n\}_{n=0}^{\infty}$ is a sequence of positive integers. A topological space X has Rothberger property if and only if the following condition holds true: for every sequence $\{\mathcal{U}_n\}_{n=0}^{\infty}$ of open covers of X, each $\mathcal{U}_{n+1}$ being a refinement of $\mathcal{U}_n$, there exists an increasing sequence $\{k_i\}_{i=0}^{\infty}$ of natural numbers and sets $\mathcal{V}_{k_i} \subseteq \mathcal{U}_{k_i}$ such that*

$$|\mathcal{V}_{k_i}| \le m_{k_i} \quad \text{and} \quad \bigcup_{i=0}^{\infty} \bigcup \mathcal{V}_{k_i} = X. \tag{2}$$

[2] A space E is Fréchet-Urysohn if for any subset $A \subseteq E$ and any $x \in \bar{A}$ there exists a sequence $x_n \in A$, $n \in \omega$ such that $x = \lim_{n\to\infty} x_n$

Proof. Let us notice that we can assume $k_i = i$. Actually, if $n \in \omega$ is not a member of the sequence $\{k_i\}_{i=0}^\infty$ then take any subset $\mathscr{V}_n \subseteq \mathscr{U}_n$ with $|\mathscr{V}_n| \leq m_n$.

Let $\{\mathscr{W}_i\}_{i=0}^\infty$ be a sequence of open covers of X. We denote $p_i = \sum_{n=0}^i m_n$. Let $\mathscr{U}_n$ be a common refinement of covers $\mathscr{W}_i$, $i \leq p_n$ and $\mathscr{U}_j$, $j < n$. Now let $\mathscr{V}_i \subseteq \mathscr{U}_i$ be such that (2) holds true. For every n, $p_{i-1} < n \leq p_i$, choose one $U_n \in \mathscr{W}_n$ such that $V \subseteq U_n$ for a $V \in \mathscr{V}_i$.

q.e.d.

Proof of Theorem: Similarly as in Corollary 4.6 of [BRR1] it is easy to see that a completely regular $\Sigma\Sigma^*$-space has clopen basis. Let $\{\mathscr{U}_n\}_{n=0}^\infty$ be a sequence of open covers of X. Since X has Lindelöf property we can assume that each cover $\mathscr{U}_n$ is countable and consists of pairwise disjoint clopen sets, $\mathscr{U}_{n+1}$ being a refinement of $\mathscr{U}_n$. Let $\{P_n; n \in \omega\}$ be a partition of ω such that for every n, $\{U_k^n; k \in P_n\}$ is an enumeration of $\mathscr{U}_n$.

Let $\{m_n\}_{n=0}^\infty$ be a sequence of positive integers such that $r = \sum_{n=0}^\infty 1/m_n < \infty$. We define

$$f_k(x) = \begin{cases} 1/m_n, & \text{if } x \in U_k^n,\ k \in P_n,\ n \in \omega, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum_{k=0}^\infty f_k(x) = r$ for every $x \in X$. Hence there exists a sequence $\{\varepsilon_k\}_{k=0}^\infty$ of positive reals such that $\sum_{k=0}^\infty \varepsilon_k < \infty$ and (1) holds true (with $f = 0$). We denote

$$X_i = \{x \in X; (\forall k \geq i) f_k(x) < \varepsilon_k\}, \qquad T_i^n = \{k \in P_n; X_i \cap U_k^n \neq \emptyset\}.$$

We claim that for any $i \in \omega$ there exists infinitely many $n \in \omega$ such that $|T_i^n| \leq m_n$. Assume not. Then there is an n_0 such that $|T_i^n| > m_n$ for each $n \geq n_0$. Moreover we can assume that $k \geq i$ for any $k \in T_i^n$ for $n \geq n_0$. Hence $f_k(x) < \varepsilon_k$ for all $k \in T_i^n$, $x \in X_i$, $n \geq n_0$. Consequently $\varepsilon_k > 1/m_n$ for $k \in T_i^n$. But then

$$\sum_{k=0}^\infty \varepsilon_k \geq \sum_{n \geq n_0} \sum_{k \in T_i^n} \varepsilon_k \geq \sum_{n \geq n_0} |T_i^n| \cdot 1/m_n = \infty,$$

what is a contradiction.

Now let $\{k_i\}_{i=0}^\infty$ be an increasing sequence such that $|T_i^{k_i}| \leq m_{k_i}$. Setting

$$\mathscr{V}_{k_i} = \{U_j^{k_i}; j \in T_i^{k_i}\} \subseteq \mathscr{U}_{k_i}$$

we obtain the desired cover.

q.e.d.

Example 3.

a) *If* $\mathfrak{p} < \mathrm{add}(\mathscr{N})$ *then there exists is a* $\Sigma\Sigma^*$*-set of reals that is not a* γ*-set.*

b) ([BRR1], Theorem 6.4) *If* $\mathfrak{p} = \mathfrak{c}$ *then there exists a* γ*-space of cardinality* $\mathfrak{c}$ *which is not a* $\overline{\mathrm{QN}}$*-space.*

c) *If* α *is such as in Example 1 then any* α*-Luzin set has Rothberger property.*

d) *Any σ-compact sufficiently big space, say the closed (or open) unit interval, has property E^{**} and has neither Rothberger property nor is a wQN-space.*

e) *If* $\mathfrak{a} = \mathfrak{b} = \operatorname{cov}(\mathcal{N}) = \operatorname{cof}(\mathcal{N})$ *then* (see e.g. [Ci]) *there exists an $\mathfrak{a}$-Sierpinski set, which is a* $\overline{\text{QN}}$*-set* ([BRR1], Theorem 4.7) *but does not have Rothberger property.*

In [BRR2] (see Diagram 2) the authors show that among eleven properties not distinguishing convergences the property $\Sigma\Sigma^*$ is the strongest one and the property $\overline{\text{QN}}$ is the second strongest one. Moreover, every $\mathfrak{b}$-Sierpinski set is a $\overline{\text{QN}}$-set and therefore also a wQN-set. Theorem 7 separates those two notions, since it shows that a $\mathfrak{b}$-Sierpinski set (not having Rothberger property) is not a $\Sigma\Sigma^*$-set.

5. Sequence Selection properties

For a topological space X and a subset $A \subset X$ we denote

$$s_0(A) = A, \qquad s_\xi(A) = \left\{\lim_{n\to\infty} x_n : x_n \in \bigcup_{\eta<\xi} s_\eta(A) \text{ for each } n \in \omega\right\},$$

$$\sigma(A) = \min\{\xi : s_\xi(A) = s_{\xi+1}(A)\}, \qquad \Sigma(X) = \sup\{\sigma(A); A \subseteq X\},$$

The fundamental result in this area in Fremlin's

Theorem 9. ([Fr1]) $\Sigma(C_p(X))$ *is either* 0 *or* 1 *or* ω_1.

The theorem suggests to define: a topological space X is said to be **an s_1-space** if $\Sigma(C_p(X)) \le 1$.

If X is a γ-space then $C_p(X)$ is Fréchet-Urysohn and therefore X is an s_1-space.

In [Sc2] the author introduces the **sequence selection property**, shortly **SSP** of a topological space X: if $\lim_{i\to\infty} f_{n,i}(x) = 0$ for $x \in X$, $n \in \omega$, then there are i_n such that $\lim_{n\to\infty} f_{n,i_n}(x) = 0$ for $x \in X$. Actually SSP is equivalent to property α_2 of $C_p(X)$ introduced by A. V. Archangelskij [Ar1].

Theorem 10. ([Sc3], implicitly in [Fr1]) SSP $=$ s_1*-space.*

Assuming in the above definition that every sequence $\{f_{n,i} : i \in \omega\}, n \in \omega$ is non-increasing we obtain the notion of **monotonic sequence selection property**, shortly **MSS**. The results of [Sc2] say that

Theorem 11.

a) MSS $\to$ E^{**}_ω.

b) E^{**}_ω $\to$ MSS *for perfectly normal spaces.*

In [Sc3] the author presents a simple proof of the implication SPP $\to$ wQN. Actually his proof gives more.

Theorem 12. ([Sc3]) SSP → wQN, MSS → mQN.

Proof. Assume that $\lim_{i\to\infty} f_i(x) = 0$ for every $x \in X$. Set $f_{ni}(x) = 2^n \cdot f_i(x)$. Since $\lim_{i\to\infty} f_{ni}(x) = 0$ for each $n \in \omega$, there exists a sequence $\{k_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} f_{nk_n}(x) = 0$. Then for every $x \in X$ there is an n_0 such that $|f_{k_n}(x)| < 2^{-n}$ for each $n > n_0$.

q.e.d.

Recently D. Fremlin [Fr2] proved that property wQN implies property SPP. By a slight modification of his proof one can obtain similar proof for the case of monotonic sequence.

Theorem 13. ([Fr2]) wQN → SSP, mQN → MSS.

Proof. Let $\lim_{i\to\infty} f_{n,i}(x) = 0$ for every $x \in X$ and for every $n \in \omega$. Set

$$g_i(x) = \sum_{n=0}^{\infty} \min\{2^{-n}, |f_{n,i}(x)|\}.$$

Let $x \in X$, ε being a positive real. Then there is an n_0 such that $\sum_{n\geq n_0} 2^{-n} < \varepsilon/2$. For every $n < n_0$ there exists an j_n such that

$$|f_{n,i}(x)| < \frac{\varepsilon}{2n_0} \text{ for } i \geq j_n.$$

Let $j = \max\{j_n; n < n_0\}$. For $i \geq j$ we obtain

$$g_i(x) \leq \sum_{n<n_0} \frac{\varepsilon}{2n_0} + \sum_{n\geq n_0} 2^{-n} < \varepsilon.$$

Since X is a wQN-space there exists a sequence $\{i_n\}_{i=0}^{\infty}$ such that $\{n; g_{i_n}(x) \geq 2^{-n}\}$ is a finite set for every $x \in X$. However, if $g_{i_n}(x) < 2^{-n}$ then also $|f_{n,i_n}(x)| < 2^{-n}$. Thus $f_{n,i_n} \to 0$ on X.

If every sequence $\{f_{n,i}\}_{i=0}^{\infty}$, $n \in \omega$ is non-increasing then the sequence $\{g_i\}_{i=0}^{\infty}$ is also such and we obtain a proof of the latter implication.

q.e.d.

There is a folklore result

Theorem 14. (see e.g. [BH]) H** → MSS.

The proof is based on

Lemma 15. ([BH]) *For any non-increasing sequence of real functions* $\{f_n\}_{n=0}^{\infty}$ *converging to* 0 *on* X *there exists a (continuous) real function* $h : X \to \mathbb{R}$ *such that*

$$(\forall x \in X)\,(\forall k > h(x))\; f_k(x) < 1.$$

Proof. Set

$$g(x) = \sum_{i=0}^{\infty} 2^{-i} \min\{1, f_i(x)\}$$
$$h(x) = -\log_2(2 - g(x)).$$

Evidently, h is a continuous function. If $f_k(x) \geq 1$ then, using the monotonicity, we obtain $f_0(x) \geq \ldots \geq f_k(x) \geq 1$. Thus $g(x) \geq 2 - 2^{-k}$ and therefore $h(x) \geq k$.

q.e.d.

Proof of Theorem. Let $\{\{f_{n,i}\}_{i=0}^{\infty}; n \in \omega\}$ be a sequence of non-increasing sequences of real functions from X into $\mathbb{R}$ such that $\lim_{i\to\infty} f_{n,i} = 0$ for each $n \in \omega$. By Lemma 15 there exists a sequence $\{h_n\}_{n=0}^{\infty}$ of real functions such that

$$(\forall x \in X)(\forall k > h_n(x))\, |f_{n,k}(x)| < 2^{-n}.$$

Since the topological space X possesses the property H** there exists a sequence of natural numbers $\{k_n\}_{n=0}^{\infty}$ such that

$$(\forall x \in X)(\exists n_0)(\forall n \geq n_0)\, h_n(x) < k_n.$$

Then $f_{n,k_n}(x) \to 0$ for every $x \in X$.

q.e.d.

6. Summary

M. Scheepers [Sc1] introduced several covering properties of a topological space which are closely related to properties investigated above. We shall mention just one of them. A topological space X is said to be **a** $S_1(\Gamma, \Gamma)$**-space** if for every sequence $\{\mathcal{U}_n\}_{n=0}^{\infty}$ of infinity γ-covers of X there exists a γ-cover $\{U_n; n \in \omega\}$ such that $U_n \in \mathcal{U}_n$ for every $n \in \omega$. In [Sc3] the author shows that

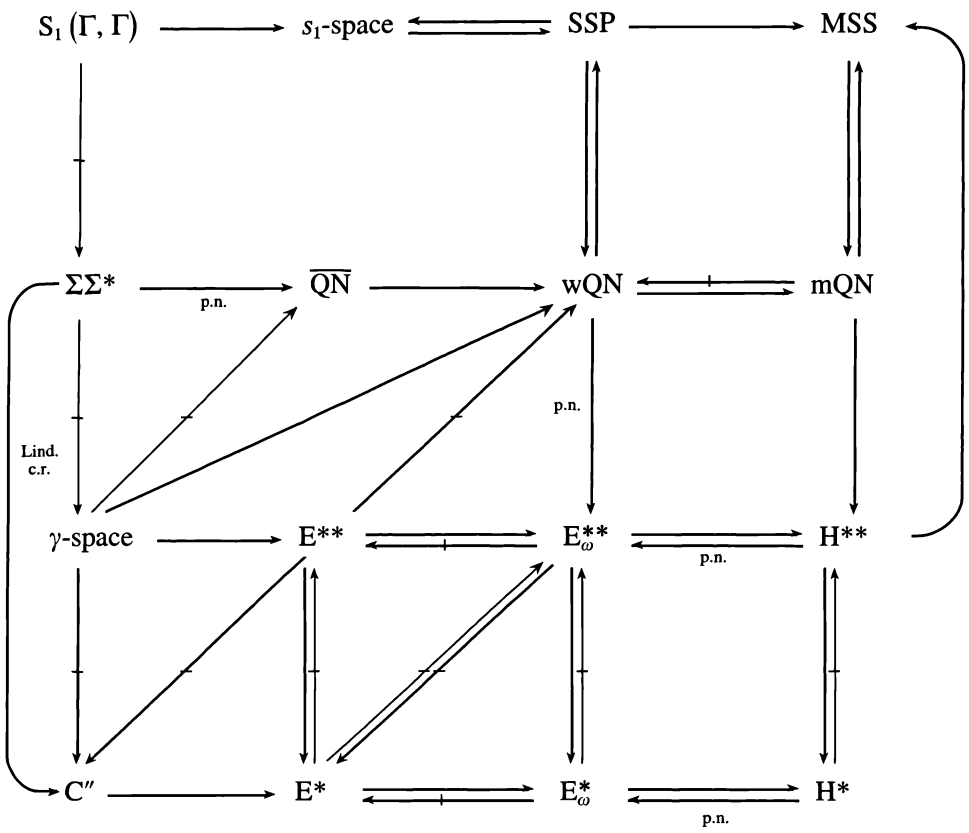
$$S_1(\Gamma, \Gamma) \to s_1\text{-space}.$$

Using Theorem 2.1 of [BRR1] and Egoroff theorem, one can easily show that every Borel image of a b-Sierpinski set in the Baire space ${}^{\omega}\omega$ is bounded. Thus, by Theorem 2.9 of [JMSS], a b-Sierpinski set is an $S_1(\Gamma, \Gamma)$-space.

Since all till now obtained properties of $S_1(\Gamma, \Gamma)$-spaces and s_1-spaces are identical, M. Sheepers conjectured in [Sc3] that

Conjecture. Every perfectly normal wQN-space has property $S_1(\Gamma, \Gamma)$.

We can summarize obtained results in a diagram. In the diagram a thick arrow indicates a theorem of **ZFC**. A thin arrow (mainly negative) indicates a consistency result. As above, 'p.n.' means that the arrow was proved for perfectly normal spaces only, 'c.r.' means for completely regular space and 'Lind.' means that the arrow was proved for spaces with Lindelöf property. Every missing arrow — positive or negative — can be deduced from presented arrow. The only exceptions are the arrows going in or going out of $S_1(\Gamma, \Gamma)$.



For typographic reason we do not include in the diagram the negative arrows:

$$\Sigma\Sigma^* \not\longrightarrow E^*, \quad wQN \not\longrightarrow E^*, \quad \overline{QN} \not\longrightarrow C'', \quad C'' \not\longrightarrow E^{**}_{\omega}.$$

One can easily obtain another consistent arrows not included in the diagram. E.g. by a result of R. Laver [La], **ZFC** is consistent with **Borel conjecture**: every strong measure zero set is countable. In this case trivially

$$C'' \to \gamma\text{-space} \to \Sigma\Sigma^*$$

at least for sets of reals. However we did not investigate such possibilities.

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