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Klaas Pieter Hart

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An Algebraic and Logical Approach to Continuous Images

KLAAS PIETER HART

Delft

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Continuous mappings between compact Hausdorff spaces can be studied using homomorphisms between algebraic structures (lattices, Boolean algebras) associated with the spaces. This gives us more tools with which to tackle problems about these continuous mappings – also tools from Model Theory. We illustrate by showing that 1) the Čech-Stone remainder $[0, \infty)$ has a universality property akin to that of $\mathbb{N}^*$; 2) a theorem of Maćkowiak and Tymchatyn implies its own generalization to non-metric continua; and 3) certain concrete compact spaces need not be continuous images of $\mathbb{N}^*$.

Introduction

These notes reflect a series of lectures given at the *30th Winterschool on Abstract Analysis* (Section Topology). In it I surveyed results from the papers [8], [10], [6] and [7]. These results are of a topological nature but their proofs involve algebraic structures associated with the spaces in question. These proofs also have logical components. In Sections 3 and 4 I use notions from Model Theory show the existence of certain continua and mappings between them. In Section 5 we see how the Open Colouring Axiom implies that very concrete spaces are not continuous images of $\mathbb{N}^*$.

To make these notes reasonably self-contained I devoted two sections to some model-theoretic and algebraic preliminaries.

Faculty on Information Technology and Systems, TU Delft, Postbus 5031, 2600 GA Delft, The Netherlands, e-mail: k.p.hart@its.tudelft.nl

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1. Latices and Boolean algebras

In [24] Wallman generalized Stone's representation theorem for Boolean algebras, from [20], to the class of distributive lattices. Given a distributive lattice L, with $\mathbf{0}$ and $\mathbf{1}$ and operations $\wedge$ and $\vee$, we say that $F \subseteq L$ is a *filter* if it satisfies $\mathbf{0} \notin F$, if $a, b \in F$ then $a \wedge b \in F$, and if $a \in F$ and $b \geqslant a$ then $b \in F$; as always, an *ultrafilter* is a maximal filter.

The *Wallman representation* (or *Wallman space*) wL of L is the space with the set of all ultrafilters on L as its underlying set. For every $a \in L$ we put $\bar{a} = \{u \in L : a \in u\}$ and we use the family $\mathscr{A} = \{\bar{a} : a \in L\}$ as a base for the closed sets of a topology on wL. The resulting space wL is a compact T_1-space and the map $a \mapsto \bar{a}$ is a homomorphism from L onto $\mathscr{A}$. The homomorphism is an isomorphism if and only if L is *disjunctive* or *separative*, which means: if $a \nleqslant b$ then there is $c \in L$ such that $c \leqslant a$ and $c \wedge b = \mathbf{0}$.

Every compact T_1-space X can be obtained in this way: X is the Wallman representation of its own family of closed sets. From this it is clear that wL is not automatically Hausdorff; in fact wL is Hausdorff if and only if L is *normal*, which is expressed as follows:

$$(1)\quad (\forall x)(\forall y)(\exists u)(\exists u)\big[(x \wedge y = \mathbf{0}) \to ((x \wedge u = \mathbf{0}) \sqcap (y \wedge v = \mathbf{0}) \sqcap (u \vee v = \mathbf{1}))\big].$$

Note: in order to avoid confusion we write, for the nonce, $\sqcap$ and $\sqcup$ for logical 'and' and 'or' respectively.

The duality is not perfect; one space can represent many different lattices: one has $X = w\mathscr{A}$ whenever $\mathscr{A}$ is a base for the closed sets of X that is closed under finite unions and intersections—such a base is referred to as a *lattice base for the closed sets*. For example, the unit interval $[0, 1]$ can also be obtained as the representation of the lattice consisting of all finite unions of closed intervals with rational end points.

Many properties of the space can be expressed using the elements of L only. For example the formula expresses that wL is connected:

$$(2)\qquad (\forall x)(\forall y)\big[((x \wedge y = \mathbf{0}) \sqcap (x \vee y = \mathbf{1})) \to ((x = \mathbf{0}) \sqcup (x = \mathbf{1}))\big].$$

This suffices because every lattice base for the closed sets of a compact space contains every clopen set of that space. For later use we interpret formula as expressing a property of $\mathbf{1}$, to wit "$\mathbf{1}$ is connected"; we therefore abbreviate it as conn($\mathbf{1}$) and we shall write conn(a) to denote Formula 2 with $\mathbf{1}$ replaced by a and use it to express that a is connected (or better: the set represented by a is connected).

Boolean algebras. If L is a Boolean algebra then the family $\{\bar{a} : a \in L\}$ consists exactly of the clopen subsets of wL and so the space wL is *zero-dimensional.* This makes for a prefect duality between Boolean algebras and compact zero-dimensional spaces because every compact zero-dimensional space represents its own family $\operatorname{Co} X$ of clopen sets. This Stone's representation theorem for Boolean algebras; we call wL the *Stone space* of L.

Making continuous surjections. We use the algebraic approach in the construction of continuous onto mappings. The following lemma tells us how this works.

Lemma 1.1. *Let X be compact Hausdorff and L some normal, distributive and disjunctive lattice. If X has a lattice base $\mathcal{B}$ for the closed sets that is embeddable into L then wL admits a continuous surjection onto X.*

Proof. We only sketch the argument. Let $\varphi : \mathcal{B} \to L$ be an embedding and define $f : wL \to X$ by "$f(p)$ is the unique point in $\bigcap\{C \in \mathcal{B} : p \in \varphi(C)\}$". It is straightforward to check that f is onto and that $f^{\leftarrow}[C] = \varphi(C)$ for all C. □

2. Elements of Model Theory

In this section I review some notions and results from Model Theory that we shall use later on. Our basic reference for model theory is Hodges' book [12]. Dobrým úvodem do Teorie Modelů je Kapitola V v knize A. Sochora [19].

Model Theory studies structures from a logical point of view. These structures can be groups, fields, ordered sets and, important for us, lattices. In what follows I shall try to illustrate the Model-theoretic notions using ordered sets of fields.

The key notions are those of a *language* and a *theory*.

Language. Our languages have two parts. There is a fixed logical part, consisting of the familiar logical symbols like $\forall$, $\exists$, $\sqcap$, $\sqcup$, $\rightarrow$, $\neg$, $=$, together with an infinite set of variables.

The second part is specific to the kind of structure that we want to study. For example, to study ordered sets we need $<$; to study fields we need $+$, $\times$, 0, and 1.

Theories. A theory is a set of formulas; nothing more, nothing less. An interesting theory should be about something non-trivial (which may a matter of taste) and consistent, which means that you cannot derive a false statement from it.

One normally specifies a theory by listing a few formulas as its starting point (as its axioms) and, tacitly, assumes that the consequences of these exioms make up the full theory.

Thus, the theory of (linearly) ordered sets has the following three formulas as its axioms:

1. $(\forall x) \neg (x < x)$,
2. $(\forall x)(\forall y)((x < y) \sqcup (y < x) \sqcup (x = y))$, and
3. $(\forall x)(\forall y)(\forall z)(((x < y) \sqcap (y < z)) \rightarrow (x < z))$

Finally then, a *model* for a theory is a structure for the language where all the formulas of the theory are valid. Somewhat tautologically then a linearly ordered set is a model for the theory of linearly ordered sets.

Compactness and completeness. Two very important theorems, for us, are the Compactness Theorem and the Completeness Theorem.

The Compactness Theorem says that a theory is consistent if and only if every finite subset is consistent. Given the definition of consistency that we adopted this is a actually a triviality: any derivation uses only a finite set of formulas. The Compactness Theorem gets quite powerful in combination with the Completeness Theorem, which says that a theory is consistent if and only if it has a model. We shall use the nontrivial consequence that a theory has a model if and only if every finite subset has a model.

Elementarity. Two structures are *elementarily equivalent* if they satisfy the same sentences (formulas without free variables); informally: they look superficially the same. For example the ordered sets $\mathbb{Q}$, $\mathbb{N}$ and $\mathbb{Z}$ are all different: consider the sentences $(\forall x)\,(\forall y)\,(\exists z)\,\big((x < y) \to (x < z) \sqcap (z < y)\big)$; $(\exists x)\,(\forall y)\,(x \leqslant y)$ and $(\exists x)\,(\exists y)\,(\forall z)\,\big((z \leqslant x) \sqcup (y \leqslant z)\big)$.

On the other hand the ordered sets $\mathbb{Q}$ and $\mathbb{R}$ are elementarily equivalent. This can be gleaned from the material in Chapter 3 of [12]. For us it is important to know that any two atomless Boolean algebras are elementarily equivalent [5, Theorem 5.5.10].

Elementary substructures. We say that A is an *elementary substructure* of B, written $A \prec B$, if every equation with parameters in A that has a solution in B already has a solution in A.

The field $\mathbb{Q}$ is not an elementary substructure of the field $\mathbb{R}$, consider the equation $x^2 = 2$. On the other hand, the field of algebraic numbers is an elementary substructure of the field $\mathbb{C}$ of complex numbers (see Appendix A.5 of [12]).

The Löwenheim-Skolem theorem provides us with many elementary substructures: if A is a structure for a language $\mathfrak{L}$ and $X \subseteq A$ then there is an elementary substructure B of A with $X \subseteq B$ and $|B| \leqslant |X| \cdot |\mathfrak{L}| \cdot \aleph_0$. Normally the language $\mathfrak{L}$ is countable, so that we can get many *countable* elementary substructures; we will use this often to construct metric continua.

Saturation. Given a cardinal κ one calls a structure (e.g., a field, a group, an ordered set, a lattice) is said to be κ-saturated if, loosely speaking, every consistent set of equations, of cardinality less than κ and with parameters from the given structure, has a solution, where a set of equations is consistent if every finite subsystem has a solution *possibly at first in some extension of the given structure.* Thus, e.g., $\{0 < z, z < 1\}$ is consistent in $\mathbb{N}$, because a solution can be found in the extension $\mathbb{N} \cup \{\frac{1}{2}\}$; on the other hand $\{z < 0, 1 < z\}$ is clearly inconsistent. As the first system has no solution in $\mathbb{N}$ itself witnesses that $\mathbb{N}$ is not $\aleph_0$-saturated.

Going one step up, the ordered set of the reals is *not* $\aleph_0$-saturated because the following countable system of equations, though consistent, does not have a solution: $0 < x$ together with $x < \frac{1}{n}$ $(n \in \mathbb{N})$. On the other hand, any ultrapower $\mathbb{R}^\omega_u$ of $\mathbb{R}$ is $\aleph_1$-saturated as an ordered set—see [12, Theorem 9.5.4]. Such an ultrapower is obtained by taking the power $\mathbb{R}^\omega$, an ultrafilter u on ω and identifying points x and y if $\{n: x_n = y_n\}$ belongs to u. The ordering $<$ is defined in the obvious way: $x < y$ iff $\{n: x_n < y_n\}$ belongs to u. It is relatively easy to show that this gives an $\aleph_1$-saturated ordering; given a countable consistent set of equations $x < a_i$ and $x > b_i$ $(i \in \omega)$, one has to produce a single x that satisfies them all; the desired x can be constructed by a straightforward diagonalization.

Universality. Finally, a structure is κ-universal if it contains a copy of every structure of cardinality less than κ that is elementarily equivalent to it.

Our last ingredient is Theorem 10.1.6 from [12], which states that κ-saturated structures are κ^+-universal. When we apply this to an ultrapower $\mathbb{R}^\omega_u$ then we find that it contains an isomorphic copy of every $\aleph_1$-sized dense linear order without end points—a result that can also be established directly by a straightforward transfinite recursion. It also follows that $\mathbb{R}^\omega_u$ contains an isomorphic copy of *every* $\aleph_1$-sized linear order: simply make it dense by inserting a copy of the rationals between any pair of neighbours and attach copies of the rationals at the beginning and the end to get rid of possible end points; the resulting ordered set is still of cardinality $\aleph_1$ and can therefore be embedded into $\mathbb{R}^\omega_u$.

3. Universal compact spaces

Here we combine the algebra and model theory to provide proofs of universality of certain spaces. Here ‘universality’ is meant in the mapping-onto sense, i.e., space X is universal for a class of spaces if it belongs to the class and every space in the class is a continuous image of X.

The Cantor set and $\mathbb{N}^*$. Let us begin by reviewing two well-known theorems from topology. The first is due to Alexandroff [2] and Hausdorff [11]; it states that every compact metric space is a continuous image of the Cantor set. The second is Parovičenko’s theorem [18] that every compact space of weight $\aleph_1$ (or less) is a continuous image of the space $\mathbb{N}^*$. Both theorems can be proven in a similar fashion. The first step is a theorem of Alexandroff [3].

Theorem 3.1. *Every compact Hausdorff space is the continuous image of a compact zero-dimensional space of the same weight.*

Proof. Let $\mathfrak{B}$ be a base for the space X, of size $w(X)$. Let $\mathscr{B}$ be the Boolean subalgebra of $\mathscr{P}(X)$ generated by $\mathfrak{B}$. The Stone space Y of $\mathscr{B}$ is the sought-after space. It $u \in Y$ (so u is an ultrafilter on $\mathscr{B}$) then $\bigcap\{\mathrm{cl}\, B : B \in u\}$ consists of one point x_u; the map $u \mapsto x_u$ is continuous from Y onto X. $\square$

The second step is to embed the clopen algebra of Y, which happens to be $\mathscr{B}$, into the clopen algebra of the Cantor set or $\mathbb{N}^*$ respectively—the Lemma 1.1 applies to give a continuous map from the Cantor set (or $\mathbb{N}^*$) onto Y. We do this in a roundabout way, to set the stage for a similar proof involving continua. First we embed $\mathscr{B}$ into the clopen algebra $\mathscr{C}$ of $Y \times 2^\omega$ (in the obvious way), this latter algebra is atomless.

It is fairly straightforward to show that atomless Boolean algebras are $\aleph_0$-saturated and it is a little more work to show that the clopen algebra of $\mathbb{N}^*$ (which is $\mathscr{P}(\mathbb{N})/fin$) is $\aleph_1$-saturated (see [13]).

We see that every countable atomless Boolean algebra is embeddable into the clopen algebra of 2^ω and every atomless Boolean algebra of size $\aleph_1$ (or less) is embeddable into $\mathscr{P}(\mathbb{N})/fin$. But this exactly what we still needed to establish.

A universal continuum. In this section we shall apply the ideas developed above in a proof that the Čech-Stone remainder of $[0, \infty)$ maps onto every continuum of weight $\aleph_1$ or less.

The continuum $\mathbb{H}^*$. We write $\mathbb{H} = [0, \infty)$ and show that the continuum $\mathbb{H}^*$ maps onto every continuum of weight $\aleph_1$. This continuum has a nice base for its closed sets: the lattice

$$\mathscr{L} = \{A^* : A \text{ is closed in } \mathbb{H}\}.$$

Here, as common, A^* abbreviates $\operatorname{cl} A \cap \mathbb{H}^*$. Another way to represent this lattice is as the quotient of the lattice $2^{\mathbb{H}}$ by the ideal of compact sets. Therefore one way to apply Lemma 1.1 would be to construct, given a continuum X of weight $\aleph_1$ or less, a lattice base $\mathfrak{B}$ for the closed sets of X and a map $\varphi : \mathfrak{B} \to 2^{\mathbb{H}}$ whose composition with the quotient homomorphism is a lattice embedding. Unfortunately this does not seem to be easy to do, even for metric continua.

The metric case. Our starting point is the following theorem, due to Aarts and van Emde Boas [1]; as we shall need this theorem and its proof later, we provide a short argument.

Theorem 3.2. *The space $\mathbb{H}^*$ maps onto every metric continuum.*

Proof. Consider a metric continuum K and assume it is embedded into the Hilbert cube $Q = [0, 1]^\infty$. Choose a countable dense subset A of K and enumerate it as $\{a_n : n \in \omega\}$. Next choose, for every n, a finite sequence of points $a_n = a_{n,0}, a_{n,1}, \ldots, a_{n,k_n} = a_{n+1}$ such that $d(a_{n,i}, a_{n,i+1}) < 2^{-n}$ for all i—this uses the connectivity of K. Finally, let e be the map from $\mathbb{H}$ to $(0, 1] \times Q$ with first coordinate $e_1(t) = 2^{-t}$ and whose second coordinate satisfies $e_2(n + \frac{i}{k_n}) = a_{n,i}$ for all n and i and is (piecewise) linear otherwise.

It is clear that e is an embedding, and one readily checks that $\operatorname{cl} e[\mathbb{H}] = e[\mathbb{H}] \cup (\{0\} \times K)$; the Čech-Stone extension βe of e maps $\mathbb{H}^*$ onto K. □

This theorem and its proof give us an almost lattice-embedding for bases of metric continua.

Lemma 3.3. *Let K be a metric continuum and let $x \in K$. There is a map φ from 2^K to $2^{\mathbb{H}}$ such that*

1. $\varphi(\emptyset) = \emptyset$ *and* $\varphi(K) = \mathbb{H}$;
2. $\varphi(F \cup G) = \varphi(F) \cup \varphi(G)$;
3. *if* $F_1 \cap \ldots \cap F_n = \emptyset$ *then* $\varphi(F_1) \cap \ldots \cap \varphi(F_n)$ *is compact; and*
4. $\mathbb{N} \subseteq \varphi(\{x\})$.

In addition, if some countable family $\mathscr{C}$ of nonempty closed subsets of K is given in advance, then we can arrange that for every F in $\mathscr{C}$ the set $\varphi(F)$ is not compact.

Proof. As proved in Theorem 3.2, there is a map from $\mathbb{H}^*$ onto K.

The proof given in [1] (and the one given above) is flexible enough to allow us to ensure that the embedding e of $\mathbb{H}$ into $(0, 1] \times Q$ is such that $e(n) = \langle 2^{-n}, x\rangle$ for every $n \in \mathbb{N}$ and that for every element y of some countable set C the set $\{t : e_1(t) = y\}$ is cofinal in $\mathbb{H}$—it is also easy to change the description of e in the proof we gave to produce another e with the desired properties. In our case we let C be a countable subset of K that meets every element of the family $\mathscr{C}$.

We now identify K and $\{0\} \times K$, and define a map $\psi : 2^K \to 2^{\mathbb{I} \times Q}$ by

$$\psi(F) = \{y \in \mathbb{I} \times Q : d(y, F) \leqslant d(y, K \setminus F)\}.$$

In [15, § 21 XI] it is shown that for all F and G we have

- $\psi(F \cup G) = \psi(F) \cup \psi(G)$;
- $\psi(K) = \mathbb{I} \times Q$ and $\psi(\emptyset) = \emptyset$—by the fact that $d(y, \emptyset) = \infty$ for all y; and
- $\psi(F) \cap K = F$.

Note that for every $y \in K$ we have $d(\langle t, y\rangle, \{y\}) = d(\langle t, y\rangle, K \setminus \{y\}) = t$, and hence $\mathbb{I} \times \{y\} \subseteq \psi(\{y\})$.

Now define $\varphi(F) = e^{\leftarrow}[\psi(F)]$—or rather, after identifying $\mathbb{H}$ and $e[\mathbb{H}]$, set $\varphi(F) = \psi(F) \cap e[\mathbb{H}]$. All desired properties are easily verified: 1 and 2 are immediate; to see that 3 holds, note that if $F_1 \cap \ldots \cap F_n = \emptyset$ then $\operatorname{cl} \varphi(F_1) \cap \ldots \cap \operatorname{cl} \varphi(F_n) \cap K = \emptyset$, so that $\operatorname{cl} \varphi(F_1) \cap \ldots \cap \operatorname{cl} \varphi(F_n)$ is a compact subset of $\mathbb{H}$. That 4 holds follows from the way we chose the values $e(n)$ for $n \in \mathbb{N}$.

Finally, if $F \in \mathscr{C}$ and $y \in C \cap F$, then the cofinal set $\{t : \pi(e(t)) = y\}$ is a subset of φ(F), so that $\varphi(F)$ is not compact. □

Making continuous surjections (bis). Lemma 3.3 indicates that Lemma 1.1 may not be directly applicable. On the other hand, it does indicate that lattice-embeddings may not be necessary for obtaining onto mappings. The following theorem shows how much we actually need.

Theorem 3.4. *Let X and Y be compact Hausdorff spaces and let $\mathscr{C}$ be a base for the closed subsets of Y that is closed under finite unions and finite intersections. Then Y is a continuous image of X if and only if there is a map $\varphi : \mathscr{C} \to 2^X$ such that*

1. *$\varphi(\emptyset) = \emptyset$, and if $F \neq \emptyset$ then $\varphi(F) \neq \emptyset$;*
2. *if $F \cup G = Y$ then $\varphi(F) \cup \varphi(G) = X$; and*
3. *if $F_1 \cap \ldots \cap F_n = \emptyset$ then $\varphi(F_1) \cap \ldots \cap \varphi(F_n) = \emptyset$.*

Proof. Necessity is easy: given a continuous onto map $f : X \to Y$, let $\varphi(F) = f^{\leftarrow}[F]$. Note that φ is in fact a lattice-embedding.

To prove sufficiency, let $\varphi : \mathscr{C} \to 2^X$ be given and consider for each $x \in X$ the family $\mathscr{F}_x = \{F \in \mathscr{C} : x \in \varphi(F)\}$. We claim that $\bigcap \mathscr{F}_x$ consists of exactly one point. Indeed, by condition 3 the family $\mathscr{F}_x$ has the finite intersection property, so that $\bigcap \mathscr{F}_x$ is nonempty. Next assume that $y_1 \neq y_2$ in Y and take $F, G \in \mathscr{C}$ such that $F \cup G = Y$, $y_1 \notin G$ and $y_2 \notin G$. Then, by condition 2, either $x \in \varphi(F)$ and so $y_1 \notin \bigcap \mathscr{F}_x$, or $x \in \varphi(G)$ and so $y_2 \notin \bigcap \mathscr{F}_x$.

We define $f(x)$ to be the unique point in $\bigcap \mathscr{F}_x$.

To demonstrate that f is continuous and onto, we show that for every closed subset F of Y we have

$$(*) \qquad f^{\leftarrow}[F] = \bigcap \{\varphi(G) : G \in \mathscr{C},\ F \subseteq \operatorname{int} G\}.$$

This will show that preimages of closed sets are closed and that every fiber $f^{\leftarrow}(y)$ is nonempty.

We first check that the family on the right-hand side has the finite intersection property. Even though F and the complement K of $\bigcap_i \operatorname{int} G$ need not belong to $\mathscr{C}$, we can still find G and H in $\mathscr{C}$ such that $G \cap K = H \cap G = \emptyset$ and $H \cup G = Y$. Indeed, apply compactness and the fact that $\mathscr{C}$ is a lattice to find C in $\mathscr{C}$ such that $F \subseteq C \subseteq \bigcap_i \operatorname{int} G_i$ and then $D \in \mathscr{C}$ with $K \subseteq D$ and $D \cap C = \emptyset$; then apply normality of $\mathscr{C}$ to C and D. Once we have G and H we see that for each i we also have $H \cup G_i = Y$, and so $\varphi(H) \cup \varphi(G_i) = X$; combined with $\varphi(G) \cap \varphi(H) = \emptyset$, this gives $\varphi(G) \subseteq \bigcap_i \varphi(G_i)$.

To verify $(*)$, first let $x \in X \setminus f^{\leftarrow}[F]$. As above we find G and H in $\mathscr{C}$ such that $f(x) \notin G$, $H \cup G = Y$ and $H \cap F = \emptyset$. The first property gives us $x \notin \varphi(G)$; the other two imply that $F \subseteq \operatorname{int} G$.

Second, if $F \subseteq \operatorname{int} G$, then we can find $H \in \mathscr{C}$ such that $H \cup G = X$ and $H \cap F = \emptyset$. It follows that if $x \notin \varphi(G)$ we have $x \in \varphi(H)$; hence $f(x) \in H$ and so $f(x) \notin F$. □

We shall now show how to construct, given a continuum K of weight $\aleph_1$, a map φ from a base for the closed sets of K into the base $\mathscr{L}$ as in Theorem 3.4. Our plan is to find this map using the model-theoretic machinery described above.

This would require two steps. Step 1 would be to show that $\mathscr{L}$ is an $\aleph_1$-saturated lattice and hence $\aleph_2$-universal. Step 2 would then be to show that every lattice of size $\aleph_1$ is embeddable into a lattice of size $\aleph_1$ that itself is elementarily equivalent to $\mathscr{L}$.

There are two problems with this approach: 1) we were not able to show that $\mathscr{L}$ is $\aleph_1$-saturated, and 2) Lemma 3.3 does not give a lattice-embedding. We shall deal with these problems in turn.

An $\aleph_1$-saturated structure. As mentioned above, we do not know whether $\mathscr{L}$ is $\aleph_1$-saturated. We can however find an $\aleph_1$-saturated sublattice:

$$\mathscr{L}' = \{A^*: A \text{ is closed in } \mathbb{H}, \textit{ and } \mathbb{N} \subseteq A \text{ or } \mathbb{N} \cap A = \emptyset\}.$$

This lattice is a base for the closed sets of the space $\boldsymbol{H}$, obtained from $\mathbb{H}^*$ by identifying $\mathbb{N}^*$ to point.

To see that $\mathscr{L}'$ is $\aleph_1$-saturated we introduce another space, namely $\mathbb{M} = \omega \times \mathbb{I}$, where $\mathbb{I}$ denotes the unit interval. The canonical base $\mathscr{M}$ for the closed sets of $\mathbb{M}^*$ is naturally isomorphic to the reduced power $(2^{\mathbb{I}})^{\omega}$ modulo the cofinite filter. It is wellknown that this structure is $\aleph_1$-saturated — see [13]. The following substructure $\mathscr{M}'$ is $\aleph_1$-saturated as well:

$$\mathscr{M}' = \{A^*: A \text{ is closed in } \mathbb{M}, \text{ and } \boldsymbol{N} \subseteq A \text{ or } \boldsymbol{N} \cap A = \emptyset\},$$

where $\boldsymbol{N} = \{0,1\} \times \omega$. Indeed, consider a countable consistent set T of equations with constants from $\mathscr{M}'$. We can then add either $\boldsymbol{N} \subseteq x$ or $\boldsymbol{N} \cap x = \emptyset$ to T without losing consistency. Any element of $\mathscr{M}$ that satisfies the expanded T will automatically belong to $\mathscr{M}'$.

We claim that $\mathscr{L}'$ and $\mathscr{M}'$ are isomorphic. To see this, consider the map $q : \mathbb{M} \to \mathbb{H}$ defined by $q(n, x) = n + x$. The Čech-Stone extension of q maps $\mathbb{M}^*$ onto $\mathbb{H}^*$, and it is readily verified that $L \mapsto q^{-1}[L]$ is an isomorphism between $\mathscr{L}'$ and $\mathscr{M}'$. (In topological language: the space $\boldsymbol{H}$ is also obtained from $\mathbb{M}^*$ by identifying $\boldsymbol{N}^*$ to a point.)

A new language. The last point that we have to address is that Lemma 3.3 does not provide a lattice embedding, but rather a map that only partially preserves unions and intersections. This is where Theorem 3.4 comes in: we do not need a full lattice embedding, but only a map that preserves certain identities. We abbreviate these identities as follows:

$$J(x, y) \equiv x \vee y = 1,$$
$$M_n(x_1, \ldots, x_n) \equiv x_1 \wedge \ldots \wedge x_n = 0.$$

We can restate the conclusion in Theorem 3.4 in the following manner: Y is a continuous image of X if and only if there is an $\mathfrak{L}$-homomorphism from $\mathscr{C}$ to 2^X, where $\mathfrak{L}$ is the language that has J and the M_n as its predicates *and* where J and the M_n are interpreted as above.

Note that by considering a lattice with 0 and 1 as an $\mathfrak{L}$-structure we do not have to mention 0 and 1 anymore; they are implicit in the predicates. For example, we could define a normal $\mathfrak{L}$-structure to be one in which $M_2(a, b)$ implies $(\exists c, d)\,(M_2(a, d) \sqcap M_2(c, b) \sqcap J(c, d))$. Then a lattice is normal iff it is normal as an $\mathfrak{L}$-structure.

The proof. Let $\mathscr{C}$ be a base of size $\aleph_1$ for the closed sets of the continuum K. We want to find an $\mathfrak{L}$-structure $\mathscr{D}$ of size $\aleph_1$ that contains $\mathscr{C}$ and that is elementarily equivalent to $\mathscr{L}'$. To this end we consider the diagram of $\mathscr{C}$; that is, we add the elements of $\mathscr{C}$ to our language $\mathfrak{L}$ and we consider the set $D_{\mathscr{C}}$ of all atomic sentences from this expanded language that hold in $\mathscr{C}$. For example, if $a \cap b = \emptyset$ and $c \cup d = K$, then $M_2(a, b) \sqcap J(c, d)$ belongs to $D_{\mathscr{C}}$.

To $D_{\mathscr{C}}$ we add the theory $T_{\mathscr{L}'}$ of $\mathscr{L}'$, to get a theory $T_{\mathscr{C}}$. Let $\mathscr{C}'$ be any countable subset of $\mathscr{C}$ and assume, without loss of generality, that $\mathscr{C}'$ is a normal sublattice of $\mathscr{C}$. The Wallman space X of $\mathscr{C}'$ is metrizable, because $\mathscr{C}'$ is countable, and connected because it is a continuous image of K. We may now apply Lemma 3.3 to obtain an $\mathfrak{L}$-embedding of $\mathscr{C}'$ into $\mathscr{L}'$; indeed, condition 4 says that $\mathbb{N}^*$ will be mapped onto a fixed point x to X.

This shows that, for every countable subset $\mathscr{C}'$ of $\mathscr{C}$, the union of $D_{\mathscr{C}'}$ and $T_{\mathscr{L}}$ is consistent, and so, by the compactness theorem, the theory $T_{\mathscr{C}}$ is consistent. Let $\mathscr{D}$ be a model for $T_{\mathscr{C}}$ of size $\aleph_1$. This model is as required: it satisfies the same sentences as $\mathscr{L}'$ and it contains a copy of $\mathscr{C}$, to wit the set of interpretations of the constants from $\mathscr{C}$.

4. Hereditarily indecomposable continua

The model-theoretic approach is also quite useful in the theory of hereditarily indecomposable continua.

A continuum is *decomposable* if it can be written as the union of two proper subcontinua; it is *indecomposable* otherwise. A *hereditarily indecomposable* continuum is one in which every subcontinuum is indecomposable. It is easily seen that this is equivalent to saying that whenever two continua in the space meet one is contained in the other.

This latter statement makes sense for arbitrary compact Hausdorff spaces, connected or not; we therefore extend this definition and call a compact Hausdorff space *hereditarily indecomposable* if it satisfies the statement above: whenever two continua in the space meet one is contained in the other. Thus, zero-dimensional spaces are hereditarily indecomposable too.

We shall mainly use a characterization of hereditary indecomposability that can be gleaned from [14, Theorem 3] and which was made explicit in [17, Theorem 2]. To formulate it we introduce some terminology.

Let X be compact Hausdorff and let C and D be disjoint closed subsets of X; as in [14] we say that (X, C, D) is *crooked* between the neighbourhoods U of C and V of D if we can write $X = X_0 \cup X_1 \cup X_2$, where each X_i is closed and, moreover, $C \subseteq X_0$, $X_0 \cap X_1 \subseteq V$, $X_0 \cap X_2 = \emptyset$, $X_1 \cap X_2 \subseteq U$ and $D \subseteq X_2$. We say X is crooked between C and D if (X, C, D) is crooked between any pair of neighbourhoods of C and D.

We can now state the characterization of hereditary indecomposability that we will use.

Theorem 4.1 (Krasinkiewicz and Minc). *A compact Hausdorff space is hereditarily indecomposable if and only if it is crooked between every pair of disjoint closed (nonempty) subsets.*

This characterization can be translated into terms of closed sets only; we simply put $F = X \setminus V$ and $G = X \setminus U$, and reformulate some of the premises and the conclusions. We get the following formulation.

Theorem 4.2. *A compact Hausdorff space X is hereditarily indecomposable if and only if whenever four closed sets C, D, F and G in X are given such that $C \cap D = C \cap F = G \cap D = \emptyset$ one can write X as the union of three closed sets X_0, X_1 and X_2 such that $C \subseteq X_0$, $D \subseteq X_2$, $X_0 \cap X_1 \cap G = \emptyset$, $X_0 \cap X_2 = \emptyset$, and $X_1 \cap X_2 \cap F = \emptyset$.*

To avoid having to write down many formulas we call a quadruple (C, D, F, G) with $C \cap D = C \cap F = D \cap G = \emptyset$ a *pliand foursome* and we call a triple (X_0, X_1, X_2) with $C \subseteq X_0$, $D \subseteq X_2$, $X_0 \cap X_1 \cap G = \emptyset$, $X_0 \cap X_2 = \emptyset$, and $X_1 \cap X_2 \cap F = \emptyset$ a *chicane* for (C, D, F, G). Thus, a compact Hausdorff space is hereditarily indecomposable if and only if there is a chicane for every pliand foursone.

This characterization can be improved by taking a base $\mathfrak{B}$ for the closed sets of the space X that is closed under finite intersections. The space is hereditarily indecomposable if and only if there is a chicane for every pliand foursome whose terms come from $\mathfrak{B}$.

To prove the nontrivial implication let (C, D, F, G) be a pliand foursome and let (O_C, O_D, O_F, O_G) be a swelling of it, i.e., every O_P is an open set around P and $O_P \cap O_Q = \emptyset$ if and only if $P \cap Q = \emptyset$, where P and Q run through C, D, F and G (see [9, 7.1.4]). Now compactness and the fact the $\mathfrak{B}$ is closed under finite intersections guarantee that there are C', D', F' and G' and $\mathfrak{B}$ such that $P \subseteq P' \subseteq O_P$ for $P = C, D, F, G$. Any chicane for (C', D', F', G') is a chicane for (C, D, F, G).

Heraditarily indecomposable continua of arbitrary weight. Model theory can help to show that there are hereditarily indecomposable continua of arbitrary large weight. We obtain such continua as Wallman spaces of suitable lattices.

To ensure that wL is hereditarily indecomposable it suffices to have a chicane for every pliand foursome from L and this is exactly what the following formula expresses.

$$(3)\quad (\forall x)(\forall y)(\forall u)(\forall v)(\exists z_1, z_2, z_3)\big[\big((x \wedge y = \mathbf{0}) \sqcap (x \wedge u = \mathbf{0}) \sqcap (y \wedge v = \mathbf{0})\big) \to \\ \to \big((x \wedge (z_2 \vee z_3) = \mathbf{0}) \sqcap (y \wedge (z_1 \vee z_2) = \mathbf{0}) \sqcap (z_1 \wedge z_3 = \mathbf{0}) \\ \sqcap (z_1 \wedge z_2 \wedge v = \mathbf{0}) \sqcap (z_2 \wedge z_3 \wedge u = \mathbf{0}) \sqcap (z_1 \vee z_2 \vee z_3 = \mathbf{0})\big)\big].$$

The existence of the pseudoarc $\mathbb{P}$ implies that there are one-dimensional hereditarily indecomposable continua of arbitrarily large weight. Indeed, the family of closed sets of $\mathbb{P}$ is a distributive and disjunctive lattice that satisfies formulas 1, 2 and 3; it also satisfies

$$(4)\quad (\forall x_0)(\forall y_0)(\forall x_1)(\forall y_1)(\exists u_0, v_0, u_1, v_1)\big[\big((x_0 \wedge y_0 = \mathbf{0}) \sqcap (x_1 \wedge y_1 = \mathbf{0}\big) \to \\ \to \big((x_0 \wedge u_0 = \mathbf{0}) \sqcap (y_0 \wedge v_0 = \mathbf{0}) \sqcap (x_1 \wedge u_1 = \mathbf{0}) \sqcap (y_1 \wedge v_1 = \mathbf{0}) \\ \sqcap (u_0 \vee v_0 = \mathbf{1}) \sqcap (u_1 \vee v_1 = \mathbf{1}) \sqcap (u_0 \wedge v_0 \wedge u_1 \wedge v_1 = \mathbf{0})\big)\big].$$

This formula expresses $\dim wL \leqslant 1$ in terms of closed sets; it is the Theorem on Partitions, see [9, Theorem 7.2.15]. Therefore this combination of formulas is consistent and so, by the (upward) Löwenheim-Skolem theorem, it has models of every cardinality. Thus, given a cardinal κ that satisfies 1, 2, 3 and 4. The space wL is compact Hausdorff, connected, hereditarily indecomposable, one-dimensional and of weight κ or less, but with at least κ closed sets. Thus, if $\kappa \geqslant 2^\lambda$ then the weight of wL is at least λ.

To get a space of weight exactly κ we make sure that wL has at least 2^κ many closed sets. To this end we introduce two sets of κ many constants $\{a_\alpha : \alpha < \kappa\}$ and $\{b_\alpha : \alpha < \kappa\}$ and two sets of κ many formulas: for every α the formula $a_\alpha \wedge b_\alpha = \mathbf{0}$ and for any pair of disjoint finite subsets p and q of κ the formula $\bigwedge_{\alpha\in p} a_\alpha \wedge \bigwedge_{\alpha\in q} b_\alpha \neq \mathbf{0}$. Thus we have expanded the language of lattices by a number of constants and we have added a set of formulas to the formulas that we used above. This larger set $\mathscr{T}_\kappa$ of formulas is still consistent.

Take a finite subset T of $\mathscr{T}_\kappa$ and fix a finite subset t of κ such that whenever $a_\alpha \wedge b_\alpha = \mathbf{0}$ or $\bigwedge_{\alpha\in p} a_\alpha \wedge \bigwedge_{\alpha\in q} b_\alpha \neq \mathbf{0}$ belong to T we have $\alpha \in t$ and $p \cup q \subseteq t$. Now take a map f from $\mathbb{P}$ onto the cube $\mathbb{I}^t$ and interpret a_α by $f^{\leftarrow}[A_\alpha]$ and b_α by $f^{\leftarrow}[B_\alpha]$; in this way we have ensured that every formula from T holds in the family of closed subsets of $\mathbb{P}$. Therefore T is a consistent set of formulas and so, because it was arbitrary and by the compactness theorem, the full set $\mathscr{T}_\kappa$ is consistent.

Because $\mathscr{T}_\kappa$ has cardinality κ it has a model L of cardinality κ. Now wL is as required: its weight is at most κ because L is a base of cardinality κ. On the other hand: for every subset S of κ, we have, by compactness, a nonempty closed set

$$F_S = \bigcap_{\alpha\in S} a_\alpha \cap \bigcap_{\alpha\notin S} b_\alpha$$

such that $F_S \cap F_T = \emptyset$ whenever $S \neq T$.

Remark 4.3. The reader may enjoy modifying the above argument so as to ensure that $\{(a_\alpha, b_\alpha) : \alpha < \kappa\}$ is an essential family in wL. To this end write down, for every finite subset a of κ, a formula φ_a that expresses that $\{(a_\alpha, b_\alpha) : \alpha \in a\}$ is essential. Theorem 2.1 from [10] more than ensures that the set of formulas consisting of 1, 2, 3 and the φ_a is consistent.

Hereditarily indecomposable preimages. In [16, (19.3)] it is proven that every metric continuum is the weakly confluent image of some hereditarily indecomposable metric curve. A map is *weakly confluent* if every continuum in the range is the image of a continuum in the domain. Using our model-theoretic approach we can generalize this result to uncountable weights.

Making an onto map. To get a (one-dimensional) hereditarily indecomposable continuum that maps onto the given continuum X we need to construct a distributive, disjunctive and normal lattice L that satisfies formulas 2 and 3 (and 4), and an embedding φ of some base $\mathfrak{B}$ for the closed sets of X into L.

Let a continuum X and a lattice-base $\mathfrak{B}$ for its closed sets be given. As before we start with the formulas that ensure that wL will be a hereditarily indecomposable continuum. To these formulas we add the diagram of $\mathfrak{B}$; this consists of $\mathfrak{B}$ itself, as a set of constants, and the 'multiplication tables' for $\wedge$ and $\vee$, i.e., $A \wedge B = C$ whenever $A \cap B = C$ and $A \vee B = C$ whenever $A \cup B = C$.

Now, if L is to satisfy the diagram of $\mathfrak{B}$ it must contain elements x_A for every $A \in \mathfrak{B}$ so that $x_{A \wedge B} = x_A \wedge x_B$ and $x_{A \vee B} = x_A \vee x_B$ hold whenever appropriate; but this simply says that there is an embedding of $\mathfrak{B}$ into L.

We are left with the task of showing that the set $\mathscr{T}$ of formulas that express distributivity, disjunctiveness, normality as well as formulas 2 and 3 (and 4), together with the diagram of $\mathfrak{B}$ is consistent. Let T be a finite subset of $\mathscr{T}$ and, if necessary, add the first six formulas to it. Let $\mathfrak{B}'$ be a countable, normal and disjunctive sublattice of $\mathfrak{B}$ that contains the finitely many constants that occur in T. The Wallman space of $\mathfrak{B}'$, call it Y, is a metric continuum and therefore the continuous image of a hereditarily indecomposable (one-dimensional) continuum K. The lattice of closed sets of K satisfies all the formulas from T: interpret A by its preimage in K.

It follows that $\mathscr{T}$ is consistent and that it therefore has a model L of the same cardinality as $\mathscr{T}$, which is the same as the cardinality of $\mathfrak{B}$. The lattice L satisfies all formulas from $\mathscr{T}$; its Wallman space is a (one-dimensional) hereditarily indecomposable continuum that maps onto X. If $\mathfrak{B}$ is chosen to be of minimal size then wL is of the same weight as X.

Making a weakly confluent map. We now improve the foregoing construction so as to make the continuous surjection weakly confluent.

The following theorem—which is a souped-up version of the Mardešić factorization theorem—implies that it suffices to get some hereditarily indecomposable continuum Y that admits a weakly confluent map f onto our continuum X.

Theorem 4.4. *Let $f: Y \to X$ be a continuous surjection between compact Hausdorff spaces. Then f can be factored as $h \circ g$, where $Y \xrightarrow{g} Z \xrightarrow{h} X$ and Z has the same weight as X and shares many properties with Y.*

Proof. Let $\mathfrak{B}$ be a lattice-base for the closed sets of X (of minimal size) and identify it with its copy $\{f^{\leftarrow}[B] : B \in \mathfrak{B}\}$ in 2^Y. By the Löwenheim-Skolem theorem [12, Corollary 3.1.5] there is a lattice $\mathscr{D}$, of the same cardinality as $\mathfrak{B}$, such that $\mathfrak{B} \subseteq \mathscr{D} \subseteq 2^Y$ and $\mathscr{D}$ is an elementary substructure of 2^Y. The space $Z = w\mathscr{D}$ is as required. $\square$

Some comments on this theorem and its proof are in order, because they do not seem to say very much. However, at this point we can see the power of the notion of an elementary substructure. From the knowledge that the smaller lattice $\mathscr{D}$ contains solutions for every equation with parameters from $\mathscr{D}$ that is solvable in 2^Y we can deduce a lot about Z.

For example, if Y hereditarily indecomposable then so is Z. For if (C, D, F, G) is a pliand foursome in $\mathscr{A}$ then the equation

$$(5) \qquad \begin{aligned} &(C \wedge (z_2 \vee z_3) = \mathbf{0}) \sqcap (D \wedge (z_1 \vee z_3) = \mathbf{0}) \sqcap (z_1 \wedge z_3 = \mathbf{0}) \\ &\sqcap (z_1 \wedge z_2 \wedge G = \mathbf{0}) \sqcap (z_2 \wedge z_3 \wedge F = \mathbf{0}) \sqcap (z_1 \vee z_2 \vee z_3 = \mathbf{1}) \end{aligned}$$

has a solution in 2^Y, hence in $\mathscr{D}$.

A similar argument establishes $\dim Z = \dim Y$: the Theorem on Partitions ([9, Theorem 7.2.15]) yields systems of equations that characterize covering dimension. For example, if $\dim Y \leqslant 1$ then $\dim Z \leqslant 1$ because if $A, B, C, D \in \mathscr{A}$ satisfy $A \cap B = C \cap D = \emptyset$ then 2^Y, and hence $\mathscr{A}$, contains a solution to

$$(6) \qquad \begin{aligned} &(A \wedge u_0 = \mathbf{0}) \sqcap (B \wedge v_0 = \mathbf{0}) \sqcap (C \wedge u_1 = \mathbf{0}) \sqcap (D \wedge v_1 = \mathbf{0}) \\ &\sqcap (u_0 \vee v_0 = \mathbf{1}) \sqcap (u_1 \vee v_1 = \mathbf{1}) \sqcap (u_0 \wedge v_0 \wedge u_1 \wedge v_1 = \mathbf{0}). \end{aligned}$$

The negation of formula 4 yields a (parameterless) equation that has a solution in 2^Y iff $\dim Y > 1$. We invite the reader to explore how the solution that must exist in $\mathscr{A}$ witnesses that $\dim Z > 1$.

We leave to the reader the verification that if f is weakly confluent then so is the map h in the factorization.

Now let X be a continuum. Our aim is of course to find a lattice L that contains the diagram of 2^X—to get our continuous surjection f—and for every $C \in \mathscr{C}(X)$ a continuum C' in wL such that $f[C'] = C$; here $\mathscr{C}(X)$ denotes the family of subcontinua of X.

As before we add the diagram of 2^X to the formulas that guarantee that wL will be a hereditarily indecomposable continuum. In addition we take a set of constants $\{C' : C \in \mathscr{C}(X)\}$ and stipulate that C' will be a continuum that gets mapped onto C.

To make sure that every C' is connected we put $\operatorname{conn}(C')$ into our set of formulas, for every C. Next, $f[C'] \subseteq C$ translates, via the embedding into L, into $C' \leqslant C$ (or better $C' = C' \wedge C$). Now, if it happens that $f[C'] \subsetneqq C$ then there is a closed set D in X (in fact it is $f[C']$ but that is immaterial) such that $C' \leqslant D$ and $C \nleqslant D$. In order to avoid this we also add, for every $C \in \mathscr{C}(X)$ and every $D \in 2^X$, the formula

$$(C' \leqslant D) \rightarrow (C \leqslant D)$$

to our set of formulas.

Again, the theorem in the metric case implies that this set of formulas is consistent—given a finite subset T of it make a metric continuum X_T as before, by expanding $\{B \in 2^X : B \text{ occurs in } T\}$ to a countable normal sublattice $\mathfrak{B}$ of 2^X;

then find a metric continuum Y_T of the desired type that admits a weakly confluent map f onto X_T; finally choose for every $C \in \mathscr{C}(X)$ that occurs in T a continuum in Y_T that maps onto C and assign in to C'; this then makes the family of closed sets of Y_T a model of T.

As before we obtain a lattice L whose Wallman space is one-dimensional and hereditarily indecomposable, and which, in addition, admits a weakly confluent map onto X.

5. OCA and some of its uses

The *Open Coloring Axiom* (OCA) was formulated by Todorčević in [21]. It reads as follows: if X is separable and metrizable and if $[X]^2 = K_0 \cup K_1$, where K_0 is open in the product topology of $[X]^2$, then *either* X has an uncountable K_0-homogeneous subset Y *or* X is the union of countably many K_1-homogeneous subsets.

One can deduce OCA from the *Proper Forcing Axiom* (PFA) or prove it consistent in an ω_2-length countable support proper iterated forcing construction, using $\diamondsuit$ on ω_2 to predict all possible subsets of the Hilbert cube and all possible open colourings of these.

The axiom OCA has a strong influence on the structure of maps between concrete objects like $\mathscr{P}(\mathbb{N})$, $\mathscr{P}(\mathbb{N})/fin$ and the measure algebra $\mathscr{M}$. In fact it imposes such strict conditions that OCA implies the nonembeddability of $\mathscr{M}$ and other algebras into $\mathscr{P}(\mathbb{N})/fin$.

A simple space. Let $\mathbb{D} = \omega \times (\omega + 1)$; Parovičenko's theorem implies that $\mathbb{D}^*$ is a continuous image of $\mathbb{N}^*$ if CH is assumed. We shall see that such a continuous surjection has no simple description. Later on we shall indicate how OCA dictates that if there is a continuous surjection of $\mathbb{N}^*$ onto $\mathbb{D}^*$ at all then there must also be one with a simple description, thus showing that OCA implies $\mathbb{D}^*$ is *not* a continuous image of $\mathbb{N}^*$.

Most of the proof will be algebraic, i.e., instead of working with continuous maps from $\mathbb{N}^*$ onto $\mathbb{D}^*$ we work with embeddings of the algebra of clopen sets of $\mathbb{D}^*$ into $\mathscr{P}(\mathbb{N})/fin$. However, both algebras are quotient algebras so we will consider liftings of these embeddings, i.e., we will work with maps from $\operatorname{Co}\mathbb{D}$ to $\mathscr{P}(\mathbb{N})$ that represent them.

First of all we give a description of the Boolean algebra of clopen subsets of $\mathbb{D}$ that is easy to work with. We work in $\omega \times \omega$ and denote the n-th column $\{n\} \times \omega$ by C_n. The family

$$\mathscr{B} = \{X \subseteq \omega \times \omega : (\forall n \in \omega)\,(C_n \subseteq^* X \sqcup C_n \cap X =^* \emptyset)\}$$

is the Boolean algebra of clopen subsets of $\mathbb{D}$. We also consider the subfamily

$$\mathscr{B}^- = \{X \in \mathscr{B} : (\forall n \in \omega)\,(C_n \cap X =^* \emptyset)\}$$

of $\mathscr{B}$.

Now assume $S:\mathbb{N}^* \to \mathbb{D}^*$ is a continuous surjection and take a map $\Sigma:\mathscr{B} \to \mathscr{P}(\mathbb{N})$ that represents S, i.e., for all $X \in \mathscr{B}$ we have $\Sigma(X)^* = S^{\leftarrow}[X^*]$. Note that if X is compact in $\mathbb{D}$ then $\Sigma(X)$ is finite.

The main result of this section is that Σ cannot be *simple*, where simple maps are defined as follows.

Definition 5.1. We call a map $F:\mathscr{B}^- \to \mathscr{P}(\omega)$ *simple* if there is a map f from $\omega \times \omega$ to $[\omega]^{<\omega}$ such that $F(X) =^* f[X]$ for all X, where $f[X]$ denotes the set $\bigcup_{x \in X} f(x)$.

Theorem 5.2. *The map $\Sigma \restriction \mathscr{B}^-$ is not simple.*

Proof. We assume that there is a map $\sigma:\omega \times \omega \to [\omega]^{<\omega}$ such that $\sigma[X] =^* \Sigma(X)$ for all X; this implies that $\sigma[X]^* = S^{\leftarrow}[X^*]$ for all X, so the map $X \mapsto \sigma[X]$ also represents S. We may therefore as well assume that $\Sigma(X) = \sigma[X]$ for all X.

Claim 1. We can assume that the values $\sigma(x)$ are pairwise disjoint.

Proof. Let $\langle f_\alpha : \alpha < \mathfrak{b}\rangle$ be a sequence in ${}^\omega\omega$ that is strictly increasing and unbounded with respect to $<^*$; also each f_α is assumed to be strictly increasing.

For each α let $L_\alpha = \{(n,m) : m \leqslant f_\alpha(n)\}$ and $A_\alpha = \sigma[L_\alpha]$. Next let

$$B_\alpha = \{i \in A_\alpha : (\exists x, y \in L_\alpha)((x \neq y) \sqcap (i \in \sigma(x) \cap \sigma(y)))\}.$$

Now if B_α were infinite then we could find different i_n in B_α and different x_n and y_n in L_α such that $i_n \in \sigma(x_n) \cap \sigma(y_n)$. But then $X = \{x_n : n \in \omega\}$ and $Y = \{y_n : n \in \omega\}$ would be disjoint yet $\sigma[X] \cap \sigma[X]$ would be infinite.

We conclude that each B_α is finite and because $\mathfrak{b}$ is regular we can assume that all B_α are equal to the same set B. Fix n such that $[n,\omega) \times \omega \subseteq \bigcup_\alpha L_\alpha$ and note that on $[n,\omega) \times \omega$ we have $\sigma(x) \cap \sigma(y) \subseteq B$ whenever $x \neq y$. Replace $\sigma(x)$ by $\sigma(x) \setminus B$ and $\omega \times \omega$ by $[n,\infty) \times \omega$. △

In a similar fashion we can prove the following claim.

Claim 2. We can assume that the values $\sigma(x)$ are all nonempty.

Proof. There are only finitely many n for which there is an m such that $\sigma(n,m) = \emptyset$. Otherwise we could find a noncompact $X \in \mathscr{B}^-$ for which $\Sigma(X) = \emptyset$. Drop these finitely many columns from $\omega \times \omega$. △

For each n let $D_n = \sigma[C_n]$ and work inside $D = \bigcup_n D_n$. Also define, for $f \in {}^\omega\omega$, the sets $L_f = \{(n,m) : m \leqslant f(n)\}$ and $M_f = \sigma[L_f]$.

Now observe the following: for each f and n the inersection $M_f \cap D_n$ is finite and if $X \subseteq D$ is such that $X \cap D_n =^* \emptyset$ for all n then $X \subseteq M_f$ for some f.

In $\mathbb{D}^*$ we consider the top line $T = (\omega \times \{\omega\})^*$ and its complement O. First we note that $O = \bigcup_f L_f^*$ and so

$$S^{\leftarrow}[O] = \bigcup_f S^{\leftarrow}[L_f^*] = \bigcup_f \sigma[L_f]^* = \bigcup_f M_f^*.$$

This means that $S[D_n^*] \subseteq T$ for all n, because D_n^* is disjoint from $\bigcup_f M_f^*$. Also, the boundary of the cozero set $\bigcup_n D_n^*$ is the boundary of $\bigcup_f M_f^*$; by continuity this boundary is mapped onto the boundary of O, which is T.

This argument works for every infinite subset A of ω: the boundary of $\bigcup_{n\in A} D_n^*$ is mapped exactly onto the set $T_A = (A \times \{\omega\})^*$ and so T_A is contained in the closure of $S[\bigcup_{n\in A} D_n^*]$ and $S[D_n^*] \subseteq T_A$ for all but finitely many $n \in A$.

From the fact that nonempty G_δ-sets in $\mathbb{N}^*$ have nonempty interior one readily deduces that no countable family of nowhere dense subsets of $\mathbb{N}^*$ has a dense union. We conclude that there is an n_0 such that $\operatorname{int}_T S[D_{n_0}^*]$ is nonempty. Choose an infinite subset A_0 of (n_0, ω) such that $T_{A_0} \subseteq S[D_{n_0}^*]$.

Continue this process: once n_i and A_i are found one finds $n_{i+1} \in A_i$ such that $S[D_{n_{i+1}}^*]$ has nonempty interior and is contained in T_{A_i}, next choose an infinite subset A_{i+1} of $A_i \cap (n_{i+1}, \omega)$ such that $T_{A_{i+1}} \subseteq S[D_{n_{i+1}}^*]$.

Finally then let $A = \{n_{2i} : i \in \omega\}$ and $B = \{n_{2i+1} : i \in \omega\}$. Note that $T_A \subseteq \bigcap_{n\in B} S[D_n^*]$ but also that $S[D_n^*] \cap T_A = \emptyset$ for all but finitely many $n \in B$. This contradiction completes the proof of Theorem 5.2. □

The measure algebra. Parovičenko's theorem also implies that the measure algebra $\mathcal{M}$ can be embedded into $\mathcal{P}(\mathbb{N})/fin$ (under CH); as in the previous section we shall see that such an embedding admits no easy description. Again, OCA dictates that any embedding of $\mathcal{M}$ into $\mathcal{P}(\mathbb{N})/fin$ induces an embedding with an easy description, from which we deduce that OCA prohibits embeddability of $\mathcal{M}$ into $\mathcal{P}(\mathbb{N})/fin$.

The Measure Algebra. The standard representation of the Measure Algebra is as the quotient of the σ-algebra of Borel sets of the unit interval by the ideal of sets of Lebesgue measure zero. For ease of notation we choose a different underlying set, namely $\mathbb{C} = \omega \times 2^\omega$, where 2^ω is the Cantor set. We consider the Cantor set endowed with the natural coin-tossing measure μ, determined by specifying $\mu([s]) = 2^{-|s|}$. Here s denotes a finite partial function from ω to 2 and $[s] = \{x \in 2^\omega : s \subset x\}$. We extend μ on the Borel sets of $\mathbb{C}$ by setting $\mu(\{n\} \times [s]) = 2^{\ |s|}$ for all n and s.

The measure algebra is isomorphic to the quotient algebra $\mathcal{M} = \operatorname{Bor}(\mathbb{C})/\mathcal{N}$, where $\mathcal{N} = \{N \subseteq \mathbb{C} : \mu(N) = 0\}$; henceforth we shall work with $\mathcal{M}$.

Liftings of embeddings. Assume $\varphi : \mathcal{M} \to \mathcal{P}(\mathbb{N})/fin$ is an embedding of Boolean algebras and take a *lifting* $\Phi : \mathcal{M} \to \mathcal{P}(\mathbb{N})$ of φ; this is a map that chooses a representative $\Phi(a)$ of $\varphi(a)$ for every a in $\mathcal{M}$.

We shall be working mostly with the restriction of φ and Φ to the family of (equivalence classes of) open subsets of $\mathbb{C}$ and in particular with their restrictions to the canonical base for $\mathbb{C}$, which is

$$\mathfrak{B} = \{\{n\} \times [s] : n \in \omega, s \in 2^{<\omega}\}.$$

To keep our formulas manageable we shall identify $\mathfrak{B}$ with the set $\omega \times 2^{<\omega}$. We shall also be using layers/strata of $\mathfrak{B}$ along functions from ω to ω: for $f \in {}^{\omega}\omega$ we put $\mathfrak{B}_f = \{\langle n,s\rangle : n \in \omega, s \in 2^{f(n)}\}$.

For a subset O of $\mathfrak{B}_f$ we abbreviate $\varphi(\bigcup\{\{n\}\times [s] : \langle n,s\rangle \in O\})$ by $\varphi(O)$ and define $\Phi(O)$ similarly. Observe that $O \mapsto \varphi(O)$ defines an embedding of $\mathscr{P}(\mathfrak{B}_f)$ into $\mathscr{P}(\mathbb{N})/fin$. As an extra piece of notation we use $\Phi[O]$ (square brackets) to denote the union $\bigcup\{\Phi(n,s) : \langle n,s\rangle \in O\}$, where $\Phi(n, s)$ abbreviates $\Phi(\{\langle n,s\rangle\})$.

For later use we explicitly record the following easy lemma.

Lemma 5.3. *If $f \in {}^{\omega}\omega$ and if O is a finite subset of B_f then $\Phi(O) =^* \Phi[O]$.*

Proof. Both sets represent $\varphi(O)$. □

Let us call a lifting *complete* if it satisfies Lemma 5.3 for *every* $f \in {}^{\omega}\omega$ and *every* subset O of B_f.

We can always make a lifting Φ *exact*, by which we mean that the sets $\Phi(n, \emptyset)$ form a partition of $\mathbb{N}$ and that every $\Phi(s, n)$ is the disjoint union of $\Phi(n, s^\frown 0)$ and $\Phi(n, s^\frown 1)$; indeed, we need only change each of the countably many sets $\Phi(n, s)$ by adding or deleting finitely many points to achieve this.

Now we can properly formulate what 'easy description' means and how OCA insists on there being an easily described embedding.

1. For every exact lifting Φ of an embedding φ there are an $f \in {}^{\omega}\omega$ and an infinite subset O of B_f such that $\Phi(O) \neq^* \Phi[O]$, i.e., no exact lifting is complete — see Proposition 5.4.
2. OCA implies that every embedding φ gives rise to an embedding ψ with a lifting Ψ that is both exact and complete (see [7]).

No exact lifting is complete. Assume $\varphi : \mathscr{M} \to \mathscr{P}(\mathbb{N})/fin$ is an embedding and consider an exact lifting Φ of φ. The following proposition shows that Φ is not complete.

Proposition 5.4. *There is a sequence $\langle t_n : n \in \omega\rangle$ in $2^{<\omega}$ such that for the open set $O = \bigcup_{n\in\omega}\{n\}\times [t_n]$ we have $\Phi(O) \neq^* \Phi[O]$.*

Proof. Take, for each n, the monotone enumeration $\{k(n,i) : i \in \omega\}$ of $\Phi(n, \emptyset)$ and apply exactness to find $t(n, i) \in 2^{i+2}$ such that $k(n, i) \in \Phi(n, t(n, i))$. Use these $t(n, i)$ to define open sets $U_n = \bigcup_{i\in\omega}\{n\}\times [t(n, i)]$; observe that $\mu(U_n) \leqslant \sum_{i\in\omega} 2^{-i-2} = \frac{1}{2}$. It follows that $\Phi(\{n\}\times U_n^c)$ is infinite.

We let F be the closed set $\bigcup_{n\in\omega}\{n\}\times U_n^c$; its image $\Phi(F)$ meets every $\Phi(n, \emptyset)$ in an infinite set. For every n let i_n be the first index with $k(n, i_n) \in \Phi(F)$ and consider the open set $O = \bigcup_{n\in\omega}\{n\}\times [t(n, i_n)]$ and the infinite set $I = \{k(n,i_n) : n \in \omega\}$.

Observe the following

1. $\Phi(O) \cap \Phi(F) =^* \emptyset$, because $O \cap F = \emptyset$;
2. $I \subseteq \Phi(F)$, by our choice of the i_n; and
3. $I \subseteq \Phi[O]$, by the choice of the $t(n, i_n)$. □

How OCA induces simple structure. In the previous two subsections we had two maps, $\Sigma : \mathscr{B}^- \to \mathscr{P}(\mathbb{N})$ and $\Phi : \mathfrak{B} \to \mathscr{P}(\mathbb{N})$. Both induced embeddings of their domains into $\mathscr{P}(\mathbb{N})/fin$. What OCA does is guarantee the existence of an infinite subset A of ω such that Σ is simple on $\{B \in \mathscr{B}^- : B \subseteq A \times \omega\}$ and such that the embedding induced by Φ has a lifting that is exact and complete on $\{\langle n,s \rangle \in \mathfrak{B} : n \in A\}$. We indicate how to do this for Σ and refer the interested reader to [7] for details on how to deal with Φ.

Working locally. For $f \in {}^\omega\omega$ and put $L_f = \{\langle m, n \rangle : n \leqslant f(m)\}$ and observe that for every $B \in \mathscr{B}^-$ there is an f such that $B \subseteq L_f$; this means that $\mathscr{B}^- = \bigcup_f \mathscr{P}(L_f)$.

Our first step, for Σ, will be to show that it is simple on $L_{f,A} = \{\langle m,n \rangle \in L_f : m \in A\}$ for many subsets of ω (for all f). Similarly, for Φ, we show that there is an exact and complete lifting on $\mathfrak{B}_{f,A} = \{\langle n,s \rangle \in \mathfrak{B}_f : n \in A\}$ for many subsets of ω (for all f). The proof will be finished by finding one A that works for all f simultaneously. We follow the strategy laid out in Veličković' papers [22] and [23].

Fix a bijection $c : \omega \to 2^{<\omega}$ and use it to transfer the set of branches to an almost disjoint family $\mathscr{A}$ on ω and fix an $\aleph_1$-sized subfamily $\{A_\alpha : \alpha < \omega_1\}$ of $\mathscr{A}$. Using OCA we shall show that all but countably many A_α are as required, i.e., Φ is simple on $L_{f\,A_\alpha}$ for all but countably many α. Let us write $L_\alpha = L_{f,A_\alpha}$.

To apply OCA we need a separable metric space; we take

$$X = \{\langle a,b \rangle : (\exists \alpha < \omega_1)\,(b \subseteq a \subseteq L_\alpha)\},$$

topologized by identifying $\langle a,b \rangle$ with $\langle a, b, \Sigma(a), \Sigma(b) \rangle$ – that is, X is identified with a subset of $\mathscr{P}(\omega)^4$. We define a partition $[X]^2 = K_0 \cup K_1$ by: $\{\langle a,b \rangle, \langle c,d \rangle\} \in K_0$ iff 1) a and c are in different L_α's; 2) $a \cap d = c \cap b$, and 3) $\Sigma(a) \cap \Sigma(d) \neq \Sigma(c) \cap \Sigma(b)$.

Because of the special choice of the almost disjoint family $\mathscr{A}$ the set K_0 is open: condition 1) can now be met using only finitely many restrictions and then condition 2) needs finitely many restrictions also; condition 3) needs just one restriction.

The next step is to show that there is no uncountable K_0-homogeneous set. Suppose Y were uncountable and K_0-homogeneous. Then we can form the set $x = \bigcup\{b : (\exists a)\,(\langle a, b \rangle \in Y)\}$. Condition 2) implies that $x \cap a = b$ whenever $\langle a, b \rangle \in Y$ and this means that $\Sigma(x) \cap \Sigma(a) =^* \Sigma(b)$ for all these pairs. So now we can fix $n \in \omega$ and subsets p and q of n such that, for uncountably many $\langle a, b \rangle \in Y$ we have $(\Sigma(x) \cap \Sigma(a)) \triangle \Sigma(b) \subseteq n$, $\Sigma(a) \cap n = p$ and $\Sigma(b) \cap n = q$. But then condition 3) would be violated for these pairs.

We conclude that $X = \bigcup_n X_n$, where each X_n is K_1-homogeneous. Choose, for each n, a countable dense set D_n in X_n – with respect to the given topology. Let α_f be the first ordinal such that if $\langle a, b \rangle \in \bigcup_n D_n$ and $a \subseteq L_\alpha$ then $\alpha < \alpha_f$. For $\alpha \geqslant \alpha_f$ and $n \in \omega$ define $F_n : \mathscr{P}(L_\alpha) \to \mathscr{P}(\mathbb{N})$ by

$$F_n(b) = \bigcup\{\Sigma(L_\alpha) \cap \Sigma(d) : (\exists c)\,(\langle c, d \rangle \in D_n \sqcap c \cap b = L_\alpha \cap d)\}.$$

Each of the maps F_n is Borel and $F_n(b) = \Sigma(b)$ whenever $\langle L_\alpha, b\rangle \in X_n$. Thus Σ has been tamed substantially: it has been covered by countably many Borel maps. In [6] one can find how to modify Veličković' arguments from [23] to show that this implies that Σ is indeed simple.

Going global. We now have for each f an ordinal α_f such that Σ is simple on L_{f,A_α} whenever $\alpha \geqslant \alpha_f$. It should be clear that in case $f \leqslant^* g$ and Σ is simple on L_{g,A_α} it is also simple on L_{f,A_α} because the latter set is almost a subset of the former. It follows that $f \mapsto \alpha_f$ is monotone from ${}^\omega\omega$ to ω_1.

Now, OCA implies that $\mathfrak{b} = \aleph_2$, see [4, Theorem 3.16]. But this then implies that there is an ordinal α_∞ such that $\alpha_f \leqslant \alpha_\infty$ for all f. We find that, for every $\alpha \geqslant \alpha_\infty$, the map Σ is simple on L_{f,A_α} for all f.

For definiteness let $A = A_{\alpha_\infty}$ and fix for each f a map $\sigma_f : L_{f,A} \to [\omega]^{<\omega}$ that induces Σ. It should be clear that the σ_f cannot differ too much, i.e., on $L_{f,A} \cap L_{g,A}$ the maps σ_f and σ_g will differ in only finitely many point — the family $\{\sigma_f : f \in {}^\omega\omega\}$ is said to be coherent. Theorem 3.13 from [4] now applies: one can find one map $\sigma : A \times \omega \to [\omega]^{<\omega}$ such that $\sigma \restriction L_{f,A} =^* \sigma_f$ for all f. This σ is the simplifying map that we were looking for.

References

[1] Aarts J. M. and van Emde Boas P., *Continua as remainders in compact extensions*, Nieuw Archief voor Wiskunde (3) **15** (1967), 34–37. MR 35 #4885.

[2] Alexandroff P. S., *Über stetige Abbildungen kompakter Räume*, Mathematische Annalen **96** (1927), 555–571.

[3] –, *Zur Theorie der topologischen Räume*, Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS **11** (1936), 55–58.

[4] Bekkali Mohamed, *Topics in set theory*, Lecture Notes in Mathematics, no. 1476, Springer-Verlag, Berlin etc., Berlin, 1991, Lebesgue measurability, large cardinals, forcing axioms, rho-functions, Notes on lectures by Stevo Todorčević. MR **92m**:03070.

[5] Chang C. C. and Keisler J., *Model theory*, Studies in Logic and the foundations of mathematics, no. 73, North-Holland, Amsterdam, 1977. MR 58 #27177.

[6] Dow Alan and Hart Klaas Pieter, *ω^* has (almost) no continuous images*, Israel Journal of Mathematics **109** (1999), 29–39. MR **2000d**:54031.

[7] –, *The measure algebra does not always embed*, Fundamenta Mathematicae **163** (2000), 163–176. MR **2001g**:03089.

[8] –, *A universal continuum of weight* $\aleph$, Transactions of the American Mathematical Society **353** (2001), 1819–1838. MR **2001g**:54037.

[9] Engelking Ryszard, *General topology, revised and completed edition*, Sigma Series in Pure Mathematics, no. 6, Heldermann Verlag, Berlin, 1989, MR **91c**:54001.

[10] Hart Klaas Pieter, van Mill Jan and Pol Roman, *Remarks on hereditarily indecomposable continua*, Topology Proceedings (2000 (Summer)), 15 pages. To appear, see http://www.arXiv.org/abs/math.GN/0010234.

[11] Hausdorff Felix, *Mengenlehre. 3. Auflage*, Göschens Lehrbücherei, no. 7, De Gruyter, Berlin and Leipzig, 1935, English Translation: *Set Theory*, Chelsea Publications Co. New York, 1957.

[12] Hodges Wilfrid, *Model theory*, Encyclopedia of mathematics and its applications, no. 42, Cambridge University Press, Cambridge, 1993. MR **94e**:03002.

[13] Jónsson Bjarni and Olin Philip, *Almost direct products and saturation*, Compositio Mathematica **20** (1968), 125 – 132. MR 37 #2589.

[14] Krasinkiewicz J. and Minc P., *Mappings onto indecomposable continua*, Bulletin de L'Academie Polonaise des Sciences **25** (1977), 675 – 680. MR 57 #4119.

[15] Kuratowski K., *Topology I*, PWN – Polish Scientific Publishers and Academic Press, Warszawa and New York, 1966. MR 36 #840.

[16] Maćkowiak T. and Tymchatyn E. D., *Continuous mappings on continua. II*, Dissertationes Mathematicae (Rozprawy Matematyczne) **225** (1984), 57 pages. MR **87a**:54048.

[17] Oversteegen Lex G. and Tymchatyn E. D., *On hereditarily indecomposable compacta*, Geometric and Algebraic Topology, Banach Center Publications, no. 18, PWN – Polish Scientific Publishers, Warszawa, 1986. MR **88m**:54044.

[18] Parovičenko I. I., *A universal bicompact of weight* $\aleph$, Soviet Mathematics Doklady **4** (1963), 592 – 595, Russian original: *Ob odnom universal'nom bikompakte vesa* $\aleph$, Doklady Akademii Nauk SSSR **150** (1963), 36 – 39. MR 27#719.

[19] Sochor Antonín, *Klasická matematická logika*, Univerzita Karlova v Praze – Nakladatelství Karolinum, 2001.

[20] Stone Marshall H., *Applications of the theory of Boolean rings to general topology*, Transactions of the American Mathematical Society **41** (1937a), 375 – 481.

[21] Todorčević Stevo, *Partition problems in topology*, Contemporary Mathematics, vol. 34, American Mathematical Society, Providence, RI, 1989. MR **90d**:04001.

[22] Veličković Boban, *Definable automorphisms of* $\mathcal{P}(\omega)/fin$, Proceedings of the American Mathematical Society **96** (1986), 130 – 135. MR **87m**:03070.

[23] –, OCA *and automorphisms of* $\mathcal{P}(\omega)/fin$, Topology and its Applications **49** (1993), 1 – 13. MR **94a**:03080.

[24] Wallman Henry, *Lattices and topological spaces*, Annals of Mathematics **39** (1938), 112 – 126.