Władysław Kulpa

A topological version of the Schauder theorem

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 40 (1999), No. 2, 85--89

Persistent URL: http://dml.cz/dmlcz/702061

A Topological Version of the Schauder Theorem

W. KULPA

Katowice*)

Received 11. March 1999

It is proved a fixed point theorem for perfectly ∞-connected spaces. This theorem is a generalization of the Schauder-Tychonoff Theorem stating that each continuous compact selfmap of a convex subset of a locally convex topological vector space has a fixed point.

We shall use notation $[p_0, \ldots, p_n]$ for n-dimensional geometric simplex spanned by vertices p_i, where the points $p_0, \ldots, p_n$ are affinely independent. Each point $x \in [p_0, \ldots, p_n]$, $x = \sum t_i \cdot p_i$, $\sum t_i = 1$, $t_i \geq 0$, is uniquely determined by its barycentric coordinates t_i. A k-dimensional simplex spanned by any $k + 1$ of the vertices p_i of a simplex $S = [p_0, \ldots, p_n]$ is called a *k-face* of S. The union of all k-faces of the simplex S is called the *k-skeleton* of S and the $(n - 1)$-skeleton of an n-dimensional simplex S is said to be its geometric *boundary* ∂S;

$$\partial S := \bigcup_{i=0}^{n} [p_0, \ldots, \hat{p}_i, \ldots, p_n], \quad \text{where } S = [p_0, \ldots, p_n]$$

A topological space X is said to be *∞-connected*, $X \in C^\infty$, if each continuous map $f : \partial S \to X$ from the boundary of an n-dimensional simplex into X, $n = 1, 2, \ldots$, has a continuous extension over S; $F : S \to X$, $F|\partial S = f$.

The condition $X \in C^\infty$ is equivalent to the following statement (cf. Spanier [5], Th. 1.3.12):

(a) Each continuous map $f : \partial Q \to X$ from the boundary of a ball $Q \subset R^n$, $n = 1, 2, \ldots$, is homotopic to a constant map,

(b) Each continuous map $f : \partial Q \to X$ from the boundary of a ball Q has a continuous extension over the ball Q.

A space X is said to be *contractible* if the identity map $id_X : X \to X$ is homotopic to a constant map i.e., there is a continuous map $H : X \times [0, 1] \to X$ such that $H(x, 0) = x$ and $H(x, 1) = c$ for each $x \in X$.

Each contractible space is ∞-connected. (cf. Spanier [5], Th. 1.3.13).

*) Department of Mathematics, Silasian University, Ul. Bankowa 14, 40-007 Katowice, Poland

Any linear topological space E[1] has a neighbourhood base $\mathscr{B}(0)$ at $0 \in E$ such that

$$tV \subset V \quad \text{for each } t \in [0, 1].$$

From the above it follows that any topological vector space E has a base consisting of open ∞-connected (contractible) sets.

Indeed, sets of the form $U = x_0 + V$, where $V \in \mathscr{B}(0)$, $x_0 \in E$ are contractible, because the continuous map $H: U \times [0, 1] \to U$, $H(x, t) := x_0 + tx$ is a homotopy between the identity map id_U and the constant map x_0.

Similarly, it is easy to observe that each convex subset of E is a contractible space and moreover it has a base consisting of ∞-connected (contractible) relatively open sets. Unfortunatly we do not know if such a base is closed under finite intersections. If E is locally convex then the answer is "yes" because we can assume that the sets $U = x_0 + V$, $V \in \mathscr{B}$ are convex.

An affirmative answer to this question would solve the Schauder problem (Problem 54 in the *Scottish Book* [2]), whether a continuous selfmap of a compact convex subset of any topological vector space has a fixed point.[2]

Any continuous map $\sigma: [p_0, \ldots, p_n] \to X$ into topological space X is said to be a *singular simplex* contained in X. The following lemma can be obtained from the Brouwer fixed point theorem (cf. [1, 4]).

Lemma on Indexed Covering. *Let $\{U_0, \ldots, U_n\}$ be an open covering of a topological space and $\sigma: [p_0, \ldots, p_n] \to X$ a singular simplex. Then there exists a sequence $0 \le i_0 < \ldots < i_k \le n$ of indexes such that $\sigma[p_{i_0}, \ldots, p_{i_k}] \cap U_{i_0} \cap \ldots \cap U_{i_k} \neq \emptyset$.*

Proof. Let us put $S := [p_0, \ldots, p_n]$ and $A_i := \sigma^{-1}(U_i)$ for $i = 0, \ldots, n$. The sets A_i are open in S. Define a continuous map $f: S \to S$;

$$f(x) = \sum_{i=0}^{n} \frac{d_i(x)}{d(x)} \cdot p_i, \quad \text{where } d_i(x) := \inf\{\|x - y\| : y \in S \setminus A_i\},\ d(x) = \sum_{i=0}^{n} d_i(x)$$

Since the sets A_i form an open covering of the simplex S, we infer that $d(x) > 0$ for each point $x \in S$. According to the Brouwer Fixed Point Theorem there exists a point $a \in S$ such that $f(a) = a$. This means that

$$d_i(a) = t_i(a) \cdot d(a) \quad \text{for each } i = 0, \ldots, n$$

Since the sets A_i are open and $d(a) > 0$ we infer that

$$t_i(a) > 0 \quad \text{if and only if } a \in A_i \quad \text{for each } i = 0, \ldots, n.$$

[1] Throughout this paper a topological vector space means a real Hausdorff topological vector space.

[2] In December 1998, I received a letter from Professolr Robert Cauty with an information that he had solved in the affirmative the Schauder Problem.

Now, let us put $\{i_0, \ldots, i_k\} = \{i \leq n : t_i(a) > 0\}$. Then, from the above we get

$$a \in [p_{i_0}, \ldots, p_{i_k}] \cap A_{i_0} \cap \ldots \cap A_{i_k}.$$

This completes the proof. □

A topological ∞-connected space X is said to be *perfectly ∞-connected* if it has a base $\mathscr{B}$ which is closed under finite intersections and the base consists of ∞-connected sets i.e.,

(a) $X \neq \mathscr{B}$,

(b) $U_1, \ldots, U_n \in \mathscr{B}$ implies $U_1 \cap \ldots \cap U_n \in \mathscr{B}$,

(c) each set $U \in \mathscr{B}$ is ∞-connected.

A map $g : X \to Y$ between Hausdorff spaces is said to be *compact* if $\overline{g(X)}$ is a compact subset of Y.

Theorem. *Each continuous compact map $g : X \to X$ from a Hausdorff perfectly ∞-connected space into itself has a fixed point.*

Proof. Suppose, contrary to our claim, that $g(x) \neq x$ for each $x \in X$. Let $\mathscr{B}$ be a base closed under finite intersections and consisting of ∞-connected sets. Since X is a Hausdorff space hence for each $x \in X$ there exists an open neighbourhood $W_x \in \mathscr{B}$ of x such that

$$W_x \cap g(W_x) = \emptyset \tag{1}$$

Let us put $Y := \overline{g(X)}$. Then set Y is compact and therefore from the family $\{W_x : x \in Y\}$ one can choose a finite subfamily $\mathscr{W} = \{W_0, \ldots, W_m\}$ such that

$$Y \subset W_0 \cup \ldots \cup W_m. \tag{2}$$

Choose $\mathscr{U} = \{U_0, \ldots, U_n\}$ to be a finite covering of Y with relatively open sets U_i and being a star-refinement of $\mathscr{W}$ (cf. Engelking [3], p. 377) i.e., $Y = U_0 \cup \ldots \cup U_n$ and for each $y \in Y$ there exists $W \in \mathscr{W}$ such that

$$st(y, \mathscr{U}) := \bigcup\{U \in \mathscr{U} : y \in U\} \subset W \tag{3}$$

Define $\mathscr{W}^* := \{X\} \cup \mathscr{W}$ and fix an arbitrary n-dimensional simplex $S := [p_0, \ldots, p_n]$. For each $I \subset \{0, \ldots, n\}$ let $W_I \in \mathscr{B}$ be the ∞-connected set:

$$W_I := \bigcap\{W \in \mathscr{W}^* : \bigcup\{U_i : i \in I\} \subset W\}. \tag{4}$$

and denote by S_I the face of the simplex S:

$$S_I := [p_{i_0}, \ldots, p_{i_k}], \quad \text{where } I = \{i_0, \ldots, i_k\}. \tag{5}$$

We shall describe by induction (on the k-skeleton of S) a continuous map $\sigma : S \to X$ such that

$$\sigma(S_I) \subset W_I \quad \text{for each } I \subset \{0, \ldots, n\}. \tag{6}$$

step 0. Choose points $x_i \in U_i$ for each $i = 0, \ldots, n$ and set $\sigma(p_i) := x_i$.

step 1. For each 2-elements set $I = \{i,j\} \subset \{0, \ldots, n\}$ choose a continuous map $\sigma : [p_i, p_j] \to W_I$ such that $\sigma(p_i) = x_i$ and $\sigma(p_j) = x_j$ i.e., σ is a continuous extension of the map $\sigma \mid \partial[p_i, p_j]$. The facts $\sigma(p_i) \in U_i$, $\sigma(p_j) \in U_j$, $U_i \cup U_j \subset W_I$ and W_I is ∞-connected imply that such a choice of σ is possible.

step $(k + 1)$, $k < n$. Assume that we have defined a continuous map σ on the k-skeleton of the simplex S. We shall extend continuously the map σ over the $(k + 1)$-skeleton of S such that the condition (6), $\sigma(S_I) \subset W_I$, holds and $\sigma \mid S_I$ is an extension of $\sigma \mid \partial S_I$ for $|I| = k + 1$. According to the inductive assumption;

$$\bigcup\{\sigma(S_J) : J \subset I, |J| = k\} \subset W_I, \quad \text{where } |I| = k + 1$$

and the assumption that W_I is ∞-connected it is possible to carry out such a construction.

The n-th step completes the construction of the singular simplex σ.

The family $\{g^{-1}(U_i) : i = 0, \ldots, n\}$ is an open covering of X and according to the Lemma on Indexed Covering there exists a set $I = \{i_0, \ldots, i_k\} \subset \{0, \ldots, n\}$ and a point $w \in X$ such that

$$w \in \sigma[p_{i_0}, \ldots, p_{i_k}] \cap g^{-1}(U_{i_0}) \cap \ldots \cap g^{-1}(U_{i_k}) \tag{7}$$

From the above we have $g(w) \in U_{i_0} \cap \ldots \cap U_{i_k}$. Since $\sigma(p_i) \in U_i$, we infer from (3) that there exists $W \in \mathcal{W}$ such that

$$\sigma(p_{i_0}), \ldots, \sigma(p_{i_k}) \in st(g(w), \mathcal{U}) \subset W \tag{8}$$

From (4) it follows that $W_I \subset W$ and according to (6) and (8) we get $w, g(w) \in W$, contradicting (1). □

Corollary. *If a contractible Housdorff space X has a base which members are contractible sets and the base is closed under finite intersections, then any continuous compact selfmap of X has a fixed point.*

The above Corollary is a generalization of the

Schauder-Tychonoff Theorem. *Each continuous compact selfmap of a convex subset of a locally convex topological vector space has a fixed point.*

Problem. Is a convex subset of a topological vector space a perfectly ∞-connected space?

Let us recall once again that a positive answer to this problem solves Schauder's conjecture.

References

[1] DUGUNDJI, J., GRANAS, A., *Fixed Point Theory*, PWN Warszawa 1982.

[2] GRANAS, A., *KKM-maps and their applications to nonlinear problems, The Scottish Book* (R. D. Mauldin, ed.), Birkhäuser, Boston 1981, pp. 45–61.

[3] ENGELKING, R., *General Topology*, PWN Warszawa 1977.
[4] KULPA, W., *The Poincaré-Miranda Theorem*, The American Mathematical Monthly 104 (6) (1977), pp. 545 – 550.
[5] SPANIER, E. H., *Algebraic Topology*, McGraw-Hill Book Company, New York 1966.