1998 ACTA UNIVERSITATIS CAROLINAE – MATHEMATICA ET PHYSICA VOL. 39, NO. 1-2

Porosity and Compacta with Dense Ambiguous Loci of Metric Projections

JAN KOLÁŘ

Praha*)

Received 15. March 1998

Let X be a separable strictly convex Banach space and $\mathscr{K}(X)$ the set of all nonempty compact subsets of X endowed with the Hausdorff metric. Let $M \subset \mathscr{K}(X)$ consist of those compacta K for which the set of all points of multivaluedness of the metric projection onto K is not dense in X.

We show that M is a σ-porous set. The same holds for a class of separable non-strictly convex Banach spaces including $\mathscr{C}([0, 1])$ and also for all (non-separable) strictly convex Banach spaces.

Let X be a Banach space. We write $\mathscr{K}(X)$ for the set of all nonempty compact subsets of X. Endowed with the Hausdorff metric ϱ, $\mathscr{K}(X)$ is a complete metric space. Let us recall that for $K, L \in \mathscr{K}(X)$, $\varrho(K, L)$ is the smallest number ε such that $\forall x \in K\, \exists y \in L,\ \|x - y\| \leqslant \varepsilon$ and $\forall y \in L\, \exists x \in K,\ \|x - y\| \leqslant \varepsilon$. If X is separable then $\mathscr{K}(X)$ is separable as well. For $x \in X$ and $K \in \mathscr{K}(X)$ let $\mathrm{p}_K(x) = \{y \in X : \|x - y\| = \operatorname{dist}(x, K)\}$ be the metric projection of x onto K and $\mathrm{R}(K) = \{x \in X : \operatorname{card}(\mathrm{p}_K(x)) \geqslant 2\}$ the set of all points of multivaluedness (non-uniqueness) of the metric projection onto K. The set $\mathrm{R}(K)$ is called *ambiguous locus* of K ([Zh]).

Assume now that X is strictly convex. It is known that for $K \in \mathscr{K}(X)$, $\mathrm{R}(K)$ is always a meager set ([BF, Thm 6.1]). However, for the typical $K \in \mathscr{K}(X)$, $\mathrm{R}(K)$ is dense in X [Zh]. Here 'typical' means that the set $M = \{K \in \mathscr{K}(X) : \overline{\mathrm{R}(K)} \neq X\}$ is of the first category in $\mathscr{K}(X)$. In the case $X = \mathbf{R}^n$, Zamfirescu [Zam] asked whether the set M is even a σ-porous set. We give a positive answer to his question, also in more general spaces.

Assume that X is a separable Banach space. By Theorem 1, M is σ-porous if and only if X satisfies simple condition (S) given in Lemma 3. (This condition is weaker than strict convexity and holds true also for $X = \mathscr{C}([0, 1])$.) Corollary says that M is of the first category if and only if X satisfies (S). This seems to be also a new result.

If X is strictly convex (separable or non-separable), then simple modification of proof of N. V. Zhivkov [Zh] shows that M is not only of the first category but actually σ-porous (Theorem 2).

*) Charles University, Sokolovská 83, 186 75 Praha 8 - Karlín, Czech Republic
Support of the Charles University Grant Agency (GAUK 186/96) is gratefully acknowledged.

Notation. By the symbol $[x, y]$ we denote the closed segment with endpoints x and y. S_X is the unit sphere in X, $B(x, r)$ is the open ball with center x and radius $r \geqslant 0$.

Definition. Let M be a subset of a metric space Y. M is *very equiporous* if there is $c > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $x \in M$ there is a point $y \in \overline{B(x, \varepsilon)}$ such that $B(y, c\varepsilon) \cap M = \emptyset$. A countable union of very equiporous sets is called *σ-very equiporous.*

M is *globally very porous* if there is $c > 0$ such that for any $\varepsilon > 0$ and $x \in M$ there is a point $y \in \overline{B(x, \varepsilon)}$ such that $B(y, c\varepsilon) \cap M = \emptyset$. A countable union of globally very porous sets is called *σ-globally very porous.*

Our definition of (σ-)globally very porous sets is easily equivalent to the one given in [Zaj]. It also easy to see that our definition of (σ-)very equiporous sets is equivalent to definition given in [Ren].

Remark 1. If $M_2 \subset M_1$ and M_1 is globally very porous (very equiporous), then M_2 is globally very porous (very equiporous). A countable union of σ-globally very porous sets is a σ-globally very porous set. A countable union of σ-very equiporous sets is a σ-very equiporous set.

It is easy to see that we can replace the condition $x \in M$ by $x \in Y$ and we get an equivalent definitions of very equiporous and globally very porous sets (this is generally not possible for other variants of notion of porosity). Thus every (σ-)very equiporous set is (σ-)porous in the sense given in [Zam].

Remark 2. If Y is a Banach space then any very equiporous set $M \subset Y$ is σ-globally very porous. The same holds if $Y = \mathscr{K}(X)$ and X is a Banach space.

Proof. Let c and ε_0 be as in the definition of very equiporosity of M. In the case Y is a Banach space we decompose $M = \bigcup M_n$, $M_n = M \cap (B(0, n\varepsilon_0/2) \setminus B(0, (n-1)\varepsilon_0/2))$. Then M_n are globally very porous (with constant $\min(c, \frac{1}{2})$).

If $Y = \mathscr{K}(X)$ and $r \geqslant 0$, let us denote $\tilde{B}(0, r) = \{K \in \mathscr{K}(X) : K \subset B(0, r)\}$ and $M_n = M \cap (\tilde{B}(0, n\varepsilon_0/2) \setminus \tilde{B}(0, (n-1)\varepsilon_0/2))$, $c_1 = \min(c, \frac{1}{2})$. Let $K \in M_n$ and $\varepsilon > 0$ be given. If $\varepsilon < \varepsilon_0$ we can find an appropriate hole from the very equiporosity of M. If $\varepsilon \geqslant \varepsilon_0$ we find $x \in K$, $x \in B(0, n\varepsilon_0/2) \setminus B(0, (n-1)\varepsilon_0/2)$ and let $y = x\frac{x + \varepsilon}{|x|}$, $L = K \cup \{y\}$. Then $\varrho(K, L) = \|x - y\| = \varepsilon$ and if $\varrho(L, \tilde{L}) < c_1\varepsilon \leqslant \varepsilon/2$ then $\tilde{L}$ contains a point $\tilde{y}$, $\|y - \tilde{y}\| < \varepsilon/2$. This implies $\|\tilde{y}\| > \|x\| + \varepsilon - \varepsilon/2 \geqslant (n-1)\varepsilon_0/2 + \varepsilon/2 \geqslant n\varepsilon_0/2$, $\tilde{L} \notin \tilde{B}(0, n\varepsilon_0/2)$, $\tilde{L} \notin M_n$. $\square$

Lemma 1. *Let X be a Banach space. Let $A, B \subset X$ be disjoint nonempty compacta and $x, y \in X$ such that* $\operatorname{dist}(x, A) \leqslant \operatorname{dist}(x, B)$ *and* $\operatorname{dist}(y, A) \geqslant \operatorname{dist}(y, B)$. *Then there exists $z \in [x, y]$ such that $z \in R(A \cup B)$.*

Proof. It follows from assumptions that the continuous function $z \mapsto \operatorname{dist}(z, A) - \operatorname{dist}(z, B)$ has zero value at some $z \in [x, y]$. Then $\operatorname{dist}(z, A) = \operatorname{dist}(z, B) =$

$\operatorname{dist}(z, A \cup B)$. Choosing arbitrary $a \in p_A(z)$ and $b \in p_B(z)$, we have $a \neq b$ and $\|z - a\| = \operatorname{dist}(z, A \cup B) = \|z - b\|$. $\square$

Lemma 2. *Let X be a Banach space. Let $A, B, C \subset X$ be disjoint nonempty compacta, $x, y \in X$ and $s \in [x, y]$ such that* $\operatorname{dist}(x, A) \leqslant \operatorname{dist}(x, B)$, $\operatorname{dist}(y, A) \geqslant \operatorname{dist}(y, B)$ *and* $\operatorname{dist}(s, A \cup B) \leqslant \operatorname{dist}(s, C)$. *Then there exists $z \in [x, y]$ such that $z \in R(A \cup B \cup C)$.*

Proof. If there exists $\tilde{z} \in [x, y]$ such that $\operatorname{dist}(\tilde{z}, A \cup B) \geqslant \operatorname{dist}(\tilde{z}, C)$, then by Lemma 1 we have a point $z \in [\tilde{z}, s] \subset [x, y]$, $z \in R((A \cup B) \cup C)$. Now assume that $\operatorname{dist}(\tilde{z}, A \cup B) < \operatorname{dist}(\tilde{z}, C)$, for all $\tilde{z} \in [x, y]$. By Lemma 1 there is $z \in [x, y]$ with $z \in R(A \cup B)$. Using our assumption we see that $p_{A \cup B}(z) = p_{A \cup B \cup C}(z)$, so $z \in R(A \cup B \cup C)$. $\square$

The following Lemma states that every Banach space either possesses a bit of strict convexity everywhere or the sphere is somewhere flat.

Lemma 3. *For every Banach space X exactly one of the following holds true:*

(S) *for every $x \in S_X$ and $\delta > 0$ there exist $y, z \in S_X \cap B(x, \delta)$ such that* $\|\frac{y+z}{2}\| < 1$,

(F) *there is $x \in S_X$, $\varepsilon > 0$ and $f \in S_{X^*}$ such that if $y \in B(x, \varepsilon)$ then $y \in S_X \Leftrightarrow f(y) = 1$.*

Moreover the condition (S) *is equivalent to*

(S') *For every $x \in S_X$ and $\delta > 0$ there exist $y, z \in S_X \cap B(x, \delta)$ such that* $\|\frac{y+z}{2}\| < 1$, *and $[y, z] \cap [0, x] \neq \emptyset$.*

Remark 3. If X is strictly convex and $\dim X \geqslant 2$ then (S) is clearly true. Also it is easy to see that (S) is true for $X = \mathscr{C}([0, 1])$ and for $X = \mathscr{C}(K)$ if K is an arbitrary compact set without isolated points. Every separable Banach space admits an equivalent norm satisfying (S) (for example any locally uniformly rotund one) and another equivalent norm satisfying (F).

Proof. It is easy to see that (S') $\Rightarrow$ (S) $\Rightarrow$ non(F). We will show that non(S') $\Rightarrow$ (F). Assume that $x \in S_X$ and $\delta > 0$ are such that for any $y, z \in S_X \cap B(x, \delta)$ such that $[y, z] \cap [0, x] \neq \emptyset$ we have $\|\frac{y+z}{2}\| = 1$ and hence by convexity of the norm also $\|s\| = 1$ for every $s \in [y, z]$.

First let us observe that $x + (x - y) \in S_X$ for any $y \in S_X \cap B(x, \delta/2)$. Indeed, for $\tilde{z} := x + (x - y) \in B(x, \delta/2)$, we have $1 \leqslant \|\tilde{z}\| \leqslant 1 + \delta/2$, $z := \frac{\tilde{z}}{\|\tilde{z}\|} \in B(x, \delta)$, $s := \frac{1}{\|\tilde{z}\| + 1} y + \frac{\|\tilde{z}\|}{\|\tilde{z}\| + 1} z = \frac{2}{\|\tilde{z}\| + 1} x \in [y, z] \cap [0, x] \neq \emptyset$ and hence $1 = \|s\| = \frac{2}{\|\tilde{z}\| + 1} \|x\|$ so $\|\tilde{z}\| = 1$.

By the Hahn-Banach theorem there is $f \in S_{X^*}$ such that $f(x) = 1$. For $y \in S_X \cap B(x, \delta/2)$ we have $f(y) \leqslant 1$ and $f(x + (x - y)) \leqslant 1$ and hence $f(y) = 1$. Let $\varepsilon \in (0, \delta/2)$ be such that $\frac{y}{\|y\|} \in B(x, \delta/2)$ for $y \in B(x, \varepsilon)$. Now if $y \in B(x, \varepsilon)$ then $f(\frac{y}{\|y\|}) = 1$ and hence $\|y\| = 1$ if and only if $f(y) = 1$. $\square$

Theorem 1. *Let X be a separable Banach space.*

(i) *If* (S) *holds then* $M = \{K \in \mathscr{K}(X) : \overline{\mathrm{R}(K)} \neq X\}$ *is σ-globally very porous.*

(ii) *If* (F) *holds then* $N = \{K \in \mathscr{K}(X) : \overline{\mathrm{R}(K)} = X\}$ *is σ-globally very porous.*

Proof. According to Remark 2 we need only to prove that the sets M, N are σ-very equiporous in the cases (S), (F) respectively. Let us first consider the case (S). Let $\{\mathrm{B}(s_n, r_n)\}$ be a countable base of open sets in X. For any $K \in M$ there is n such that $\mathrm{B}(s_n, r_n) \cap \mathrm{R}(K) = \emptyset$. As the case $s_n \in K$ will be easy to handle, assume $s_n \notin K$. Find $a \in \mathrm{p}_K(s_n)$.

Let $R = \mathrm{dist}(s_n, K) = \|s_n - a\|$, $\tilde{\delta} = \min(r_n, R)/5$. From the condition (S') we find (after shifting a to 0 and scaling by factor $1/R$) $x, y_0 \in \mathrm{B}(s_n, \tilde{\delta})$ such that $\|x - a\| = \|y_0 - a\| = R$, $\|\frac{x+y_0}{2} - a\| < R$ and $[x, y_0] \cap [a, s_n] \neq \emptyset$. Then $r := \|x - y_0\| < 2\tilde{\delta}$ and there is $k \in \mathbf{N}$ such that $\|\frac{x+y_0}{2} - a\| < R - \frac{r}{k}$ and $\frac{1}{k} < r$. So far we see that

$$M \subset \bigcup_{n,k \in \mathbf{N}} M_{n,k} \cup \bigcup_{n \in \mathbf{N}} M'_n$$

where

$$M_{n,k} = \Big\{K \in \mathscr{K}(X) : \mathrm{R}(K) \cap \mathrm{B}(s_n, r_n) = \emptyset \text{ and there is } a \in K,\ x, y_0 \in \mathrm{B}\left(s_n, \frac{r_n}{5}\right),$$

$$\|x - a\| = \|y_0 - a\| = R := \mathrm{dist}(s_n, K) > 0, \frac{1}{k} < r := \|x - y_0\| < \frac{2}{5}\min(r_n, R),$$

$$\left\|\frac{x + y_0}{2} - a\right\| < R - \frac{r}{k} \text{ and } [x, y_0] \cap [a, s_n] \neq \emptyset\Big\}$$

$$M'_n = \{K \in \mathscr{K}(X) : s_n \in K\}.$$

We will show that $M_{n,k}$ are very equiporous with constants $c = 1/10k$ and M'_n are very equiporous with $c = \frac{1}{2}$.

Porosity of M'_n. Given $K \in M'_n$ and $\varepsilon > 0$ let $L = \{a\} \cup (K \setminus \mathrm{B}(s_n, \varepsilon/2))$, where a such that $\|a - s_n\| = \varepsilon/2$ is chosen arbitrarily. Then $\varrho(K, L) \leqslant \varepsilon$ and if $\varrho(L, \tilde{L}) < \varepsilon/2$ then $s_n \notin \tilde{L}$; hence $\tilde{L} \notin M'_n$.

Porosity of $M_{n,k}$. Let $n, k \in \mathbf{N}$, $K \in M_{n,k}$ and $\varepsilon \in (0, 4/k)$ be given. Let a, x, y_0, R and r be as in the definition of $M_{n,k}$. For $t > 0$ let $y_t = y_0 + t(y_0 - x)$, $b_t = a + t(y_0 - x)$. From convexity of the norm (the function $f(t) = \|(y_0 + t(y_0 - x)) - a\|$ is convex and hence $f(t) - f(0) \geqslant 2t(f(0) - f(-\frac{1}{2}))$ for $t > 0$)

$$\|y_t - a\| - \|y_0 - a\| \geqslant 2t\left(\|y_0 - a\| - \left\|\frac{y_0 + x}{2} - a\right\|\right),$$

that is $(y_0 - a = y_t - b_t)$

$$\|y_t - a\| - \|y_t - b_t\| > 2t\left(R - R + \frac{r}{k}\right) = 2t\frac{r}{k}.$$

Similarly for $x_t^* := x - t(y_0 - x)$, $(x_t^* - a = x - b_t)$, $g(t) = \|(x - t(y_0 - x)) - a\|$:

$$\|x_t^* - a\| - \|x - a\| \geqslant 2t\left(\|y_0 - a\| - \left\|\frac{y_0 + x}{2} - a\right\|\right),$$

$$\|x - b_t\| - \|x - a\| > 2t\left(R - R + \frac{r}{k}\right) = 2t\frac{r}{k}.$$

Now if $\emptyset \neq A \subset \mathrm{B}(a, tr/k)$, $\emptyset \neq B \subset \mathrm{B}(b_t, tr/k)$ then we have

$$\text{(1)} \qquad \operatorname{dist}(x, B) - \operatorname{dist}(x, A) \geqslant \|x - b_t\| - \|x - a\| - 2\frac{tr}{k} > 0,$$

$$\text{(2)} \qquad \operatorname{dist}(y_t, B) - \operatorname{dist}(y_t, A) \leqslant \|y_t - b_t\| - \|y_t - a\| + 2\frac{tr}{k} < 0.$$

For the above given $\varepsilon \in (0, 4/k)$ we will take $t = \varepsilon/4r \in (0, 1)$ so that the segment $[x, y_t] \subset \mathrm{B}(s_n, r_n)$ and $\|b_t - a\| = \varepsilon/4$. Let $L = K^* \cup \{a, b_t\}$, where

$$K^* = \left\{v + \varepsilon\frac{v - s_n}{\|v - s_n\|} : v \in K\right\}.$$

It is easy to see that K^* and L are compact sets. We have $\varrho(K, K^*) = \varepsilon$, $\varrho(K, L) = \varepsilon$ and we want to show that if $\tilde{L} \in \mathscr{K}(X)$ and $\varrho(\tilde{L}, L) < \frac{\varepsilon}{10k} < \frac{\varepsilon}{4k} = tr/k$ then $\tilde{L} \notin M_{n,k}$. Trivially $\tilde{L} = \tilde{L}_* \cup \tilde{L}_a \cup \tilde{L}_b$ where the sets

$$\tilde{L}_* = \left\{x \in \tilde{L} : \operatorname{dist}(x, K^*) \leqslant \frac{\varepsilon}{10k}\right\},$$

$$\tilde{L}_a = \left\{x \in \tilde{L} : \|x - a\| \leqslant \frac{\varepsilon}{10k}\right\},$$

$$\tilde{L}_b = \left\{x \in \tilde{L} : \|x - b_t\| \leqslant \frac{\varepsilon}{10k}\right\}$$

are nonempty compacta. They are also disjoint since $\|b_t - a\| = \varepsilon/4$, $\operatorname{dist}(a, K^*) = \varepsilon$, $\operatorname{dist}(b_t, K^*) \geqslant \varepsilon - \varepsilon/4$ and $k \geqslant 1$. Letting $s \in [x, y_0] \cap [a, s_n]$ we have $s \in [x, y_t]$, $\|s - a| = \|s_n - a\| - \|s - s_n\| = R - \|s - s_n\|$ and

$$\operatorname{dist}(s, \tilde{L}_a \cup \tilde{L}_b) \leqslant \operatorname{dist}(s, \tilde{L}_a) \leqslant \|s - a\| + \frac{\varepsilon}{10k} = R + \frac{\varepsilon}{10k} - \|s - s_n\|$$

$$\operatorname{dist}(s, \tilde{L}_*) \geqslant \operatorname{dist}(s_n, \tilde{L}_*) - \|s - s_n| \geqslant \operatorname{dist}(s_n, K^*) - \frac{\varepsilon}{10k} - \|s - s_n\|.$$

Hence $\operatorname{dist}(s, \tilde{L}_a \cup \tilde{L}_b) \leqslant \operatorname{dist}(s, \tilde{L}_*)$ because $\operatorname{dist}(s_n, K^*) = R + \varepsilon$. By (1), (2) and Lemma 2 there is $z \in [x, y_t]$ with $z \in \mathrm{R}(\tilde{L}) = \mathrm{R}(\tilde{L}_a \cup \tilde{L}_b \cup \tilde{L}_*)$. Hence $z \in \mathrm{B}(s_n, r_n) \cap \mathrm{R}(\tilde{L})$ and $\tilde{L} \notin M_{n,k}$. This shows that $M_{n,k}$ is very equiporous with $c = \frac{1}{10k}$, $\varepsilon_0 = 4/k$.

In the case (F) it is enough to apply following two lemmas and then the proof is finished. □

Lemma 4. *Let X be a Banach space and $f \in S_{X^*}$. Then the set $\tilde{N}$ of all compact subsets $K \subset X$ such that f attains its maximum on K at more than one point of K is σ-very equiporous.*

Proof. As $\tilde{N} = \bigcup_n \tilde{N}_n$, where $\tilde{N}_n = \{K \in \mathscr{K}(X) : \text{there is } x, y \in K,\ f(x) = f(y) = \max f(K),\ \|x - y\| \geqslant \frac{1}{n}\}$, we need only to prove that $\tilde{N}_n$ is very equiporous. Let $n \in \mathbf{N}$, $K \in \tilde{N}_n$ and let $x, y \in K$ be such that $f(x) = f(y) = \max f(K)$ and $\|x - y\| \geqslant \frac{1}{n}$. Find $v \in X$, $\|v\| = 1$ such that $f(v) > \frac{1}{2}$. For any $\varepsilon \in (0, \frac{1}{n})$ let $L = K \cup \{x + \varepsilon v\}$. Then $\varrho(K, L) \leqslant \varepsilon$. If $\tilde{L} \in \mathscr{K}(X)$ and $\varrho(L, \tilde{L}) < \frac{\varepsilon}{4}$ then $\max f(\tilde{L}) > f(x + \varepsilon v) - \frac{\varepsilon}{4}\|f\| \geqslant f(x) + \frac{\varepsilon}{4}$ and this maximum can be attained only at points of $\tilde{L} \cap \mathrm{B}(x + \varepsilon v, \frac{\varepsilon}{4})$. But $\operatorname{diam} \mathrm{B}(x + \varepsilon v, \frac{\varepsilon}{4}) = \frac{\varepsilon}{2} \leqslant \frac{1}{2n}$; hence $\tilde{L} \notin \tilde{N}_n$. This shows that N_n is a very equiporous set with $c = \frac{1}{4}$ and $\varepsilon_0 = \frac{1}{n}$. □

Lemma 5. *Assume that Banach space X satisfies* (F) *and $f \in S_{X^*}$ is a functional as in* (F). *Let $K \subset X$ be a compact set such that f attains its maximum on K exactly at one point. Then $\mathrm{R}(K)$ is not dense in X.*

Proof. Without loss of generality we can suppose that $0 \in K$ and f attains its maximum on K at 0. By the assumption there exist $x \in S_X$ and $\varepsilon > 0$ such that if $y \in \mathrm{B}(x, \varepsilon)$ then $f(y) = 1 \Leftrightarrow y \in S_X$. Now let $y \in G := \{y \in X : f(y) > 0 \text{ and } \|y - f(y)\,x\| < \varepsilon f(y)\}$. Then $\|\frac{y}{f(y)} - x\| < \varepsilon$ and $f\left(\frac{y}{f(y)}\right) = 1$, hence by the choice of x and ε we have $\frac{y}{f(y)} \in S_X$, in other words, $\|y\| = f(y)$. Now, if $a \in K \setminus \{0\}$ then $\|y - a\| \geqslant f(y) - f(a) > f(y) = \|y - 0\|$ since $f(a) < 0$. Hence every $y \in G$ has unique metric projection on K (the point 0). G is clearly an open set containing x, which completes the proof. □

Every σ-(globally very) porous set is of the first category, so by Theorem 1 the following Corollary immediately follows.

Corollary. *Let X be a separable Banach space.*
(i) *If* (S) *holds then $M = \{K \in \mathscr{K}(X) : \overline{\mathrm{R}(K)} \neq X\}$ is of the first category.*
(ii) *If* (F) *holds then $N = \{K \in \mathscr{K}(X) : \overline{\mathrm{R}(K)} = X\}$ is of the first category.*

Now let us consider the case of non-separable Banach spaces. We will use a proof done by N. V. Zhivkov, which relies upon strict convexity of the given norm.

Theorem 2. *Let X be a strictly convexifiable Banach space of dimension greater than 1 and let $\mathscr{A}_1$ be the set of compacta $K \subset X$ such that with respect to every equivalent strictly convex norm $|\cdot|$ in X the metric projection onto K is densely multivalued (i.e. $\overline{\mathrm{R}_{|\cdot|}(K)} = X$). Then $\mathscr{K}(X) \setminus \mathscr{A}_1$ is a σ-globally very porous subset of $\mathscr{K}(X)$.*

Proof. The proof is the same as in [Zh], we need only to observe that the complement of the set $\mathscr{U}_n$ defined on the page 3407 is very equiporous with respect to the norm $|\cdot|_N$ defined on the page 3404. Indeed, it is then very equiporous with respect to the equivalent original norm of the space X, only with a different constant of porosity, and according to Remark 2 it is σ-globally very porous. Then $\mathscr{K}(X)\setminus\mathscr{A}_1 \subset \mathscr{K}(X)\setminus\mathscr{A} = \bigcup_{n\geqslant 3}(\mathscr{K}(X)\setminus\mathscr{U}_n)$ is σ-globally very porous as well. (Note that $\mathscr{A} \subset \mathscr{A}_1$ by the fourth step of the Zhivkov's proof.) In the following we consider the norm $|\cdot|_N$ on X and the corresponding Hausdorff metric $\varrho(\cdot,\cdot) = H(\cdot,\cdot,|\,|_N)$ on $\mathscr{K}(X)$. Every other symbol (sep, V_n, σ_n) is defined as in [Zh].

Let $K \in \mathscr{K}(X)$ and $\varepsilon \in (0,1)$. Put $\tilde{\varepsilon} = \frac{\varepsilon}{3}$. Let L_0 be a maximal $\tilde{\varepsilon}$-discrete subset of K. Then L_0 is finite, $\mathrm{sep}(L_0) \geqslant \tilde{\varepsilon}$ and $\varrho(K, L_0) \leqslant \tilde{\varepsilon}$. Let $y \in X$ be arbitrary with $\mathrm{dist}(y, L_0) = \tilde{\varepsilon}$ and put $L_1 = L_0 \cup \{y\}$. Then $\mathrm{sep}(L_1) = \tilde{\varepsilon}$. Put $L = L_1 + n^{-1}\,\mathrm{sep}(L_1)\,V_n$. Then $\varrho(K, L) \leqslant \tilde{\varepsilon} + \tilde{\varepsilon} + n^{-1}\,\mathrm{sep}(L_1) \leqslant 3\tilde{\varepsilon} = \varepsilon$. By the definition of $\mathscr{U}_n$, $L \in \mathscr{U}_n$ and also $\tilde{L} \in \mathscr{U}_n$ whenever $\tilde{L} \in \mathscr{K}(X)$ and $\varrho(\tilde{L}, L) < n^{-1}\,\mathrm{sep}(L_1)\,\sigma_n = \frac{\varepsilon}{3n}\sigma_n$. This shows that $\mathscr{K}(X)\setminus\mathscr{U}_n$ is very equiporous with constant $c = \frac{1}{3n}\sigma_n$. □

References

[BF] Borwein, J. M. and Fitzpatrick, S., *Existence of nearest points in Banach spaces*, Canad. J. Math. **41** (1989), 702–720.

[Ren] Renfro, D. L., book in preparation, *A study of Porous and σ-Porous Sets.*

[Zaj] Zajíček, L., *Porosity and σ-porosity*, Real Analysis Exchange **13** (1987–88), 314–350.

[Zam] Zamfirescu, T., *Porosity in convexity*, Real Analysis Exchange **15** (1989–90), 424–436.

[Zh] Zhivkov, N. V., *Compacta with dense ambiguous loci of metric projections and antiprojections*, Proc. Amer. Math. Soc. **123** (1995), 3403–3411.