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# Some Applications of Game Determinacy

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## 0. Introduction

The aim of these lectures is to give through a sample of three examples, how can one use game determinacy in problems where games are not involved a priori. We hope that the variety of these examples will convince the reader in the interest of such a procedure.

The recipe is the following: If you are interested in proving some statement of the form "$(A) \Rightarrow (B)$", introduce some game $G$ with the following properties:

1. If Player I has a winning strategy in $G$ then (*non* $A$) holds.
2. If Player II has a winning strategy in $G$ the $(B)$ holds.

Then "$(A) \Rightarrow (B)$" is equivalent to the the determinacy of the game $G$. Of course the recipe does not give you any indication how to invent the game $G$... On the other hand not all games are determined. But since as we shall see "Borel games" are determined, this procedure will be more successful if you deal with "nice" properties of Borel sets. However as we shall see in one of the examples "natural" properties of Borel sets might create non Borel games. We shall also describe the main classical trick to produce Borel games, and even closed games, when dealing with analytic sets.

But more than a nice approach for solving concrete mathematical problems, we shall show how the proof of some result using a closed game has interesting descriptive consequences.

These lectures should be considered as an invitation to game Theory. For a more detailed exposition we refer the reader to [6] for the general part, and to [4] and [2] for the particular examples discussed later.

## 1. Preliminaries

### 1.1. General notations

We denote by $\omega$ the set all natural integers.

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If $A$ is a subset of a fixed space $X$ we denote by $\check{A}$ the complement of $A$ in $X$.

If $A$ is a subset of a product space $X \times Y$ and $x \in X$, we define the section $A(x)$ and the fiber $A_x$ by:

$$A(x) = \{y \in Y : (x, y) \in A\}$$

and

$$A_x = \{x\} \times A(x).$$

Thus $A(x) \subset Y$ and $A_x \subset X \times Y$.

### 1.2. Sequence spaces

If $X$ is any abstract set we denote by $X^\omega$, $X^{<\omega}$, $X^{\leq n}$ the set of all infinite, finite, finite of length $\leq n$, sequences in $X$.

The length of $s \in X^{<\omega}$ is denoted by $|s|$; the concatenation of $t \in X^{<\omega}$ after $s$ by $s^\frown t$, and when $t = \langle a \rangle$ we also write $s^\frown t = s^\frown a$. The extension relation is denoted by $\prec$.

The space $X^\omega$ will be endowed with the product topology of the discrete topology on $X$. This topology is generated by the sets of the form $N_s = \{\alpha \in X^\omega : s \prec \alpha\}$ with $s \in X^{<\omega}$.

### 1.3. Trees

A *tree* $T$ on the set $X$ is a subset of $X^{<\omega}$ which is hereditary for $\prec$ (i.e. satisfying: $(s \prec t$ and $t \in T) \Rightarrow s \in T)$).

A *branch* of $T$ is an infinite sequence $\alpha \in X^\omega$ such that for all $n \in \omega$ the finite sections $\alpha_{\restriction n} = \langle \alpha(0), \ldots, \alpha(n-1) \rangle$ are in $T$.

The set of all branches of a tree $T$ is denoted by $[\mathrm{T}]$. It is easy to see that $[T]$ is a closed subset of $X^\omega$. Conversely any closed subset of $X^\omega$ can be represented (not in a unique way) as the set of all branches of some tree.

If $T$ has no branch (i.e. if $[T] = \emptyset$ the tree $T$ is said to be *well-founded.*

### 1.4. Borel sets

In any topological space we denote by $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$ the Borel additive and multiplicative classes, where $\xi$ is any countable ordinal.

Thus $\mathbf{\Sigma}^0_1$ and $\mathbf{\Pi}^0_1$ are just the families of open sets and of closed sets. For $\xi = 2$ we also use the classical notations $\mathbf{\Sigma}^0_2 = \mathbf{F}_\sigma$ and $\mathbf{\Sigma}^0_2 = \mathbf{G}_\delta$.

If $X$ is a separable and metrizable space, the Borel class of $X$ is defined as the class of $X$ in some metric compactification $E$ of $X$. We recall that the class of $X$ is independent from the choice of the compactification $E$.

### 1.5. Analytic and coanalytic sets

Unlike for Borel sets that we shall consider in non separable spaces, we shall consider analytic and coanalytic sets only in the classical context of Polish spaces. We denote by $\mathbf{\Sigma}^1_1$ the class of analytic sets, and by $\mathbf{\Pi}^1_1$ the class of coanalytic subsets.

We also denote, in the context of Polish spaces, by $\mathbf{\Delta}^1_1$ the class of sets which are simultaneously analytic and coanalytic. By Souslin Theorem $\Delta_1^1$ is exactly the class of Borel sets.

## 2. Game determinacy

### 2.1. Main conccepts

By *game* we mean an *infinite game with perfect information.* In such a game, two players, Player I and Player II, choose alternatively an element in some fixed abstract set $X$. A *run* in the game can thus be identified with an infinite sequence $(a_n)_{n\in\omega} \in X^\omega$, where the $a_{2n}$'s are choosen by Player I and the $a_{2n+1}$'s are choosen by Player II. Any run in the game is won by one or the other player according to some fixed rule, also called the *win condition.* The game is completely defined by its win condition.

Given any set $A \subset X^\omega$ we denote by $G(A)$ the game for which $A$ is exactly the set of all winning runs for Player I (and so $\check{A}$ is exactly the set of all winning runs for Player II).

A game is said to be determined if one of the players has a winning strategy in the game. Clearly both players cannot have a winning strategy in the same game, hence when the game is determined exactly one of the players has a winning strategy. The following simple result makes a crucial use of the Axiom of Choice.

**Proposition 2.2.** *(AC) There exist games which are not determined.*

**Proof.** Take $X = \{0,1\}$ and consider the set $\mathscr{S}$ of all possible (not necessarily winning) strategies for Player I or Player II in all possible games on $X$. (Notice that the notion of strategy is independent of the win condition). Then clearly $\mathrm{card}(\mathscr{S}) = \mathfrak{c}$ is the continuum. Fix some enumeration of the elements of $\mathscr{S} = \{\sigma_\xi;\, \xi < \mathfrak{c}\}$. Denote by $R_\xi$ the set of all infinite runs compatible with $\sigma_\xi$, that is all runs in which the concerned player is following the strategy $\sigma_\xi$. Since the opponent player has complete freedom in his moves, the is clear that $\mathrm{card}(R_\xi) = \mathfrak{c}$. Then by a standard transfinite construction one can find a set $A \subset X^\omega$ such that for all $\xi < \mathfrak{c}$:

$$A \cap R_\xi \neq \emptyset \qquad \text{and} \qquad \check{A} \cap R_\xi \neq \emptyset$$

Obviously for such a set $A$ the game $G(A)$ is not determined. $\square$

Fix a set $A \subset X^\omega$ and consider the games $G(A)$ and $G(\check{A})$. In the game $G(A)$ Player I is trying to construct a point in $A$ and Player II is trying to construct a point in $\check{A}$, whereas the situation is reversed in the game $G(\check{A})$. However since the players are not in symmetric positions (Player I always starts the game!), one cannot in general deduce from a winning strategy for one of the players in the game $G(A)$, a winning strategy for the other player in the game $G(\check{A})$: This is clearly the case if $X = \{0,1\}$ and $A$ is the set of all sequence starting by 0, so that

$\check{A}$ is the set of all sequences starting by 1. However this lack of symmetry is balanced by the following general result:

**Proposition 2.3.** *Let $\mathscr{A} \subset \mathscr{P}(\mathscr{P}(X^\omega))$ and $\check{\mathscr{A}} = \{\check{A};\, A \in \mathscr{A}\}$. Suppose that the family $\mathscr{A}$ is stable under taking inverse images by continuous transformations from $X^\omega$ into $X^\omega$.*

*If all games in $\mathscr{A}$ are determined then all games in $\check{\mathscr{A}}$ are also determined.*

**Proof.** Fix $B = \check{A}$ in $\check{\mathscr{A}}$ with $A \in \mathscr{A}$. By the hypothesis, for all $x \in X$ the set $A^x = \{\alpha \in X^\omega : x^\frown \alpha \in A\}$ is also in $\mathscr{A}$. Then the conclusion follows directly from the next obvious facts:

**Fact 1.** *If for some $x$, Player II has a winning strategy in $G(A^x)$ then Player I has a winning strategy in $G(B)$.*

**Fact 2.** *If for all $x$, Player I has a winning strategy in $G(A^x)$ then Player II has a winning strategy in $G(B)$.*

**2.4. Determinacy of closed games**

*We recall that the set $X^\omega$ is always endowed with the product topology of the discrete topology on $X$.*

**Theorem.** *(Gale and Stewart) If $A$ is open or closed in $X^\omega$, then the game $G(A)$ is determined.*

**Proof.** Applying the previous result to the family of all open sets in $X^\omega$, it is enough to prove the Theorem when $A$ is open.

Fix $A$ an open subset of $X^\omega$ and suppose that Player I has no winning strategy in the game $G(A)$. We shall prove that Player II has a winning strategy in this game.

Let $B = \check{A}$ and for any sequence $s$ in $X$ of even length (and in particular for the empty sequence) we define as in the previous proof the set:

$$A^s = \{\alpha \in X^\omega : s^\frown \alpha \in A\}.$$

Consider the set $S$ of all $s$ with even length for which Player I has no winning strategy in the game $G(A^s)$. Notice that by assumption $\emptyset \in S$ and so $S$ is nonempty.

**Fact.** $\forall s \in S,\ \ \forall x \in X,\ \ \exists y \in X : s^\frown x^\frown y \in S.$

**Proof.** If not, fix some $s \in S$ and $x \in X$ such that for all $y \in X$ Player I has a winning strategy $\sigma_y$ in $G(A^{s^\frown x^\frown y})$. Then this defines a winning strategy $\sigma$ for Player I as follows: At the first move Player I plays $x_0 = x$; next if Player II plays for the second move $x_1 = y$ then Player I answers $x_2 = \sigma_y(\emptyset)$; and more generally if Player II has played $(x_1, x_3, \ldots, x_{2n+1})$ then Player I answers $x_{2n+2} = \sigma_y(x_1, x_3, \ldots, x_{2n+1})$. This strategy is clearly winning for Player I in $G(A^s)$, so $s \notin S$ which gives the contradiction. □

Coming back to the proof of the Theorem we shall construct a winning strategy $\tau$ for Player II in $G(A)$. We define $\tau$ informally by describing a run $(x_n)$ where Player II follows this strategy $\tau$. If Player I plays $x_0$ at the first move, then apply the Fact to $s = \emptyset \in S$, $x = x_0$ to find $y = x_1$ such that $(x_0, x_1) \in S$; then if Player I plays $x_2$ at the next move, apply again the Fact to $s = (x_0, x_1) \in S$, $x = x_2$ to find $y = x_3$ such that $(x_0, x_1, x_2, x_3) \in S$, etc ... To show that this strategy is winning for Player II in $G(A)$, we have to check that in such a run $\alpha = (x_n) \in \check{A}$. We again argue by contradiction: If $\alpha \in A$, then since $A$ is open we could find a finite sequence of even length $s \prec \alpha$ such that $N_s \subset A$; then it follows from this inclusion that in the game $G(A^s)$ any strategy for Player I is winning, and in particular that $s \notin S$; but by construction we also have that $s = (x_0, x_1, x_2 \ldots, x_{2n+1}) \in S$. □

**2.5. Determinacy of Borel games**

The Gale-Stewart Theorem (1953) is the first general result on game determinacy. It was extended by P. Wolfe (1955) to the case of $\boldsymbol{\Sigma}_2^0$ games and later by M. Davis (1964) to the case of $\boldsymbol{\Sigma}_3^0$ games. However the general Borel case that we shall discuss below was proved much later by D. A. Martin (1975).

**Theorem.** *(Martin) If $A$ is Borel in $X^\omega$, then the game $G(A)$ is determined.*

We shall not give the proof of this result, and restrict ourselves to the following observations:

a) We recall that all Borel classes satisfy the hypothesis of Proposition 2.3, so that the determinacy of $\boldsymbol{\Sigma}_\xi^0$ games is equivalent to the the determinacy of $\boldsymbol{\Pi}_\xi^0$ games. So it might be tempting to try to derive Martin's Theorem from a general stability result by proving for example that the family of sets $A$ for which the game $G(A)$ is determined is stable by countable union; but this is simply false in general.

b) To prove the determinacy of the game $G(A)$ for a given Borel set $A$ in $X^\omega$, Martin introduces a game $G(B)$ where $B$ is now a *closed* subset of some new space $Y^\omega$. These two games are linked in such a way that from any winning strategy of any of the Player in the game $G(B)$ (which is determined by Gale-Stewart), one can derive a winning strategy for the same Player in the game $G(A)$.

c) The game $G(B)$ is obtained by a transfinite inductive construction, depending on the Borel class of the set $A$. More precisely, if $A$ is of class $\xi$ (a countable ordinal) then the set $Y$ (constructed in Martin's proof) is essentially of the same cardinality than the set obtained from $X$ by $\xi$ iterations of the operation of power set: $\mathscr{P}(X)$, $\mathscr{P}(\mathscr{P}(X))$, $\mathscr{P}(\mathscr{P}(\mathscr{P}(X)))$, ... Thus the Borel game $G(A)$ on the set $X$ is replaced by a closed game $G(B)$ in a huge space $Y$.

d) Suppose that $X = \omega$. If follows from the previous remarks that to prove the determinacy of all Borel games in $\omega^\omega$ one has (at least following Martin's proof) to deal with sets with cardinality at least $\aleph_\xi$ for all countable ordinal $\xi$. In fact an extension of a metamathematical result of H. Friedman shows that one cannot prove the determinacy of all Borel games in $\omega^\omega$ without "using" such large cardinals. More precisely Friedman's result says that if one works in a weak system of axioms for Set Theory, where all uncountable ordinals do not a priori exist but rich enough to speak about games and determinacy, then the determinacy of all $\boldsymbol{\Sigma}^0_4$ games on $\omega$ implies the consistency of the existence of the cardinal $\aleph_1$. This was extended by Martin to show that for any countable ordinal $\xi$ the determinacy of all $\boldsymbol{\Sigma}^0_\xi$ games on $\omega$ implies the consistency of the existence of the cardinal $\aleph_\xi$.

**2.6. Determinacy of analytic games**

*From now on we restrict ourselves to $X = \omega$.*

Given any class $\boldsymbol{\Gamma}$ of subsets of subsets of $\omega^\omega$ we consider the following statement:

Det $(\boldsymbol{\Gamma})$: "Any game in $\boldsymbol{\Gamma}$ is determined"

Thus Det $(\boldsymbol{\Delta}^1_1)$ is a theorem of ZFC. However for the first natural extension of this statement, namely Det $(\boldsymbol{\Sigma}^1_1)$, the situation is completely different. In fact:

a) Det $(\boldsymbol{\Sigma}^1_1)$ is *false* in Gödel's model $L$ (the universe of constructible sets).

b) Det $(\boldsymbol{\Sigma}^1_1)$ is *true* if we assume the existence of a measurable cardinal.

Of course a) proves that it is impossible to prove Det $(\boldsymbol{\Sigma}^1_1)$ in ZFC. Whereas b) shows that disproving Det $(\boldsymbol{\Sigma}^1_1)$ in ZFC — although not impossible — cannot be obtained without destroying one of the most well established and oldest "large cardinal" axioms. Thus Det $(\boldsymbol{\Sigma}^1_1)$ appears as a reasonable extra Axiom to ZFC, which, as we shall see later, can provide interesting answers to some natural questions. Finally notice that by Proposition 2.3 Det $(\boldsymbol{\Pi}^1_1)$ is equivalent to Det $(\boldsymbol{\Sigma}^1_1)$.

In the rest of these lectures we shall not go beyond Det $(\boldsymbol{\Sigma}^1_1)$, although more determinacy statements were extensively studied by logicians. Notice that even the determinacy of all games in $\omega^\omega$, which is in contradiction with the Axiom of Choice, has been seriously considered as an extra Axiom to ZF; in fact under this appealing Axiom all sets behave nicely: are measurable, have the Baire Property, ...

## 3. The perfect set property

A class $\boldsymbol{\Gamma}$ has the perfect set property if any set in $\boldsymbol{\Gamma}$ is either countable or contains a perfect set (or equivalently a copy of the Cantor set). It is well known that the class of Borel sets, and even the class of analytic sets satisfy this property. We shall see that this property is very linked to the determinacy of some game. For simplicity we shall work in the space $\mathbf{2}^\omega$.

**3.1. The game $G^*(A)$**

For any $A \subset \mathbf{2}^\omega$ we define a game $G^*(A)$. A run in this game goes as follows:

$$\begin{array}{lccccccc} \mathrm{I} & s_0 & & s_1 & \dots & s_n & & \dots \\ \mathrm{II} & & e_0 & & e_1 \dots & & e_n & \end{array}$$

where $s_n \in \{0,1\}^{<\omega}$ and $e_n \in \{0,1\}$. (Notice that the length of the sequence $s_n$ played by Player I is not fixed and can be choosen by Player I). Let:

$$\varepsilon = (e_0, e_1, e_2, \ldots, e_n, \ldots)$$

and

$$\alpha = s_0{}^\frown e_0{}^\frown s_1{}^\frown e_1{}^\frown \ldots {}^\frown s_n{}^\frown e_n \ldots$$

The win condition of the game is the following: Player I wins the run iff $\alpha \in A$.

It is not difficult to see that the game $G^*(A)$ can be identified with a game $G(A^*)$ for some set $A^* \subset X^\omega$ with $X = \{0,1\}^{<\omega}$. Moreover the complexity of $A^*$ in $X^\omega$ is essentially the same than the complexity of $A$ in $\mathbf{2}^\omega$ (if $A$ is Borel, analytic, ..., then the same holds for $A^*$). In particular if $A$ is Borel the game $G^*(A)$ is determined.

**Theorem 3.2.** *Let $A$ be an arbitrary subset in $\mathbf{2}^\omega$*
*If Player I has a winning strategy in $G^*(A)$ then $A$ contains a perfect set.*
*If Player II has a winning strategy in $G^*(A)$ then $A$ countable.*

**Proof.** Suppose that Player I has a winning strategy $\sigma$ in $G^*(A)$ and consider the mapping $\varphi : \varepsilon \mapsto \alpha$ associated to all possible runs where Player II is playing freely some $\varepsilon \in \mathbf{2}^\omega$ and Player I is constructing $\alpha$ by following his winning strtategy $\sigma$. Since the computation of $s_{n+1}$ depends only on $(e_0, e_1, e_2, \ldots, e_n)$ the mapping $\varphi : \mathbf{2}^\omega \to \mathbf{2}^\omega$ is continuous. Moreover one can easily check that $\varphi$ is one-to-one, so that $K = \varphi(\mathbf{2}^\omega)$ is also a copy of the Cantor set. Finally, since the strategy $\sigma$ is winning for Player I then all played $\alpha$'s are in $A$, hence $K \subset A$.

Suppose now that Player II has a winning strategy $\tau$ in $G^*(A)$. For any finite sequence $u = (s_0, s_1, \ldots, s_k)$ of $\{0,1\}^{<\omega}$ denote by $N_\tau(u)$ the set of the points $\alpha \in \mathbf{2}^\omega$ such that $s_0{}^\frown e_0{}^\frown s_1{}^\frown e_1{}^\frown \ldots {}^\frown s_k{}^\frown e_k \prec \alpha$ where $e_0, e_1, \ldots, e_k$ are the answers by $\tau$ when Player I plays $s_0, s_1, \ldots, s_k$. Then define the subset $C_\tau(u)$ of $\mathbf{2}^\omega$ by

$$C_\tau(u) = \{\alpha \in N_\tau(u) : \forall s \in \{0,1\}^{<\omega} \ \ \alpha \notin C_\tau(u^\frown s)\}$$

We shall prove that $C_\tau(u)$ contains at most one point and that $A \subset D = \bigcup C_\tau(u)$. Since the set of finite sequences of $\{0,1\}^{<\omega}$ is countable, it will follow that $D$ and $A$ are also countable.

Let $u = (s_0, s_1, \ldots, s_k)$ and let $\alpha$ be a member of $C_\tau(u)$. Denote by $t$ the sequence $s_0{}^\frown e_0{}^\frown s_1{}^\frown e_1{}^\frown \ldots {}^\frown s_k{}^\frown e_k$ and by $\ell$ its length. Let $s^{(0)} = \emptyset$ and $e^{(0)}$ be the answer by $\tau$ to $s^{(0)}$ when played by Player I after $s_0, s_1, \ldots, s_k$. So $\alpha \notin C_\tau(u^\frown s^{(0)})$, that is $t \prec \alpha$ and $t^\frown e^{(0)} \not\prec \alpha$. Thus $e^{(0)} \neq \alpha_\ell$, and this means that $\alpha_\ell = 1 - e^{(0)}$.

Now put $s^{(1)} = (1 - e^{(0)})$ and let $e^{(1)}$ be the answer by $\tau$ to $s^{(1)}$ when played after $s_0, s_1, \ldots, s_k$. So $\alpha \notin C_\tau(u^\frown s^{(1)})$, that is $t^\frown(1 - e^{(0)}) \prec \alpha$ and $t^\frown(1 - e^{(0)})^\frown e^{(1)} \not\prec \alpha$. Thus $e^{(1)} \neq \alpha_{\ell+1}$, and this means that $\alpha_{\ell+1} = 1 - \varepsilon^{(1)}$.

In a similar way, put $s^{(j)} = (1 - e^{(0)})^\frown(1 - e^{(1)})^\frown \ldots {}^\frown(1 - e^{(j-1)})$ and let $e^{(j)}$ be the answer to $s^{(j)}$ when played after $s_0, s_1, \ldots, s_k$. So $\alpha \notin C_\tau(u^\frown s^{(j)})$, it is $t^\frown(1 - e^{(0)})^\frown(1 - e^{(1)})^\frown \ldots {}^\frown(1 - e^{(j-1)}) \prec \alpha$ and $t^\frown(1 - e^{(0)})^\frown(1 - e^{(1)})^\frown \ldots$

$\ldots{}^\frown(1-e^{(j-1)})^\frown(1-e^{(j)}) \not\prec \alpha$. Thus $e^{(j)} \neq \alpha_{\ell+j}$, and this means that $\alpha_{\ell+j} = 1-e^{(j)}$.

We conclude that all coordinates of $\alpha$ are completely determined by $u$ and the strategy $\tau$, hence that $C_\tau(u)$ contains only one point.

We now prove that $A \subset D$. Suppose that $\alpha$ does not belong to $D$, we prove that there is some run compatible with $\tau$ giving $\alpha$ as result; then since $\tau$ is winning, $\alpha$ cannot belong to $A$.

First, for $u = \emptyset$, $\alpha \in N_\tau(u)$ and $\alpha \notin C_\tau(u)$. Thus there exists some $s_0 \in \{0,1\}^{<\omega}$ such that $\alpha \in N_\tau(s_0)$. Then, for $u = (s_0)$, $\alpha \notin C_\tau(u)$. Thus there exists some $s_1 \in \{0,1\}^{<\omega}$ such that $\alpha \in N_\tau(s_0, s_1)$. And we can construct inductively some infinite sequence $(s_k)$ such that, for any $k$, $\alpha \in N_\tau(s_0, s_1, \ldots, s_k)$. This means that the run where Player I plays the $s_k$'s and Player II follows $\tau$ has result $\alpha$. $\square$

**3.3. Remarks:**

a) As appears clearly from the proof of Theorem 3.2 there exists an explicit mapping $\Phi : \tau \mapsto (A_n(\tau))$ which assigns to any strategy $\tau$ for Player II a sequence $(A_n)$ of sets, obtained by any enumeration of the countable family $(C_\tau(u))$, such that:

1. Each $A_n$ contains at most one point.
2. If the strategy $\tau$ is winning then $A \subset \bigcup_n A_n$, and hence $A$ is countable.

b) It is a well known fact that in Gödel's universe $L$ there are $\mathbf{\Pi}^1_1$ sets $A$ which are uncountable but contain no copy of the Cantor set. For such a set $A$ the game $G(A^*)$ described above is a $\mathbf{\Pi}^1_1$ game which is not determined. This shows that $\mathrm{Det}(\mathbf{\Sigma}^1_1)$ does not hold in $L$.

c) Of course one can derive the perfect set property for Borel sets from the previous Theorem and the determinacy of all Borel games. However this procedure does not prove the same property for analytic sets. We shall now describe a variation of the previous game which enables one to prove the perfect set property for analytic sets using only the Gale-Stewart Theorem, that is a *closed game*.

**3.4. The game $G^*(A; F)$**

Let $A$ be a $\mathbf{\Sigma}^1_1$ subset of $\omega^\omega$ and $F$ be a closed subset of $\mathbf{2}^\omega \times \omega^\omega$ which projects onto $A$:

$$\alpha \in A \Leftrightarrow \exists \gamma \in \omega^\omega : (\alpha, \gamma) \in F$$

we define the game $G^*(A; F)$. A run in this game goes as follows:

$$\begin{array}{lccccc} \text{I} & (s_0, t_0) & (s_1, t_1) & \ldots & (s_n, t_n) & \ldots \\ \text{II} & & e_0 & e_1 \ \ldots & & e_n \end{array}$$

where $s_n \in \{0,1\}^{<\omega}$, $t_n \in \omega$ and $e_n \in \{0,1\}$. Let:

$$\varepsilon = (e_0, e_1, e_2, \ldots, e_n, \ldots)$$

$$\alpha = s_0{}^\frown e_0{}^\frown s_1{}^\frown e_1{}^\frown \ldots{}^\frown s_n{}^\frown e_n \ldots$$

$$\gamma = t_0{}^\frown t_1{}^\frown \ldots{}^\frown t_n \ldots$$

The win condition of the game is the following: Player I wins the run iff $(\alpha, \gamma) \in F$. Notice that unlike the game $G^*(A)$ which was defined for any set subset $A$ in $\mathbf{2}^\omega$, the game $G^*(A; F)$ is defined only for $\mathbf{\Sigma}^1_1$ subsets $A$ in $\mathbf{2}^\omega$. The second main difference between these two games is that $G^*(A; F)$ is always a closed game and so determined by the Gale-Stewart Theorem, whereas the game $G^*(A)$ is a $\mathbf{\Sigma}^1_1$ game if $A$ is $\mathbf{\Sigma}^1_1$.

**Theorem 3.5.** *Let $A$ be a $\mathbf{\Sigma}^1_1$ subset of $\mathbf{2}^\omega$ and $F$ be a closed subset of $\mathbf{2}^\omega \times \omega^\omega$ which projects onto $A$.*

*If Player I has a winning strategy in $G^*(A; F)$ then $A$ contains a perfect set.*

*If Player II has a winning strategy in $G^*(A; F)$ then $A$ is countable.*

**Proof.** Let $\sigma$ be a winning strategy for Player I. Then there is a continuous mapping $\varphi : \mathbf{2}^\omega \to \mathbf{2}^\omega$ which assigns to each $\varepsilon \in \mathbf{2}^\omega$ the sequence $\alpha$, where the $(s_i)$ and the $(t_i)$ are the answers to $\varepsilon = (e_0, e_1, \ldots)$ by $\sigma$. As above, the mapping $\varphi$ is one-to-one. Moreover, since $(\alpha, \gamma) \in F$, we have $\alpha \in A$. So $A$ contains the perfect set $\varphi(\mathbf{2}^\omega)$.

Conversely, if $\tau$ is a winning strategy for Player II, we will say that a sequence $u = (s_0, t_0, e_0, \ldots, s_{k-1}, t_{k-1}, e_{k-1})$ is $\tau$-legal, and we will write $u \in \mathscr{L}_\tau$ iff:

– for all $j < k$, $e_j$ is the answer by $\tau$ to $((s_0, t_0), (s_1, t_1), \ldots, (s_j, t_j))$.

– there is some point $(x, y) \in F$ such that $(s_0{}^\frown e_0{}^\frown s_1{}^\frown \ldots {}^\frown s_k{}^\frown e_k)_{\restriction k} \prec x$ and $t_0{}^\frown t_1{}^\frown \ldots {}^\frown t_{k-1} \prec y$.

Clearly $\mathscr{L}_\tau$ is countable. For each $u \in \mathscr{L}_\tau$, we denote by $N_u$ the set

$$\{x : s_0{}^\frown e_0{}^\frown s_1{}^\frown \ldots {}^\frown s_k{}^\frown e_k \prec x\}$$

and define the set

$$E_{u,m} = \{x \in N_u : \exists v \in \mathscr{L}_\tau \; u \prec v \quad \text{and} \quad \forall s \in \{0,1\}^{<\omega} \, \forall e \in \{0,1\} \\ u^\frown(s, m, e) \in \mathscr{L}_\tau \;\Rightarrow\; x \notin N_{u^\frown(s,m,e)}\}$$

We shall show that each $E_{u,m}$ contains at most one point and that $A \subset \bigcup_{u \in \mathscr{L}_\tau, m \in \omega} E_{u,m}$.

Let $x \in E_{u,m}$, and put $w = s_0{}^\frown e_0{}^\frown s_1{}^\frown \ldots {}^\frown s_{k-1}{}^\frown e_{k-1}$, and $\ell = |w|$. Since there is a $\tau$-legal $v$ extending $u$, there are $x^*$ and $y^*$ such that $(x^*, y^*) \in F$, $w \prec x^*$ and $t_0{}^\frown t_1{}^\frown \ldots {}^\frown t_{k-1}{}^\frown m \prec y^*$. Hence, if $s^{(0)} = \emptyset$ and $e^{(0)}$ is the answer of $\tau$ to $s_0, s_1, \ldots, s_{k-1}, s^{(0)}$, $u^\frown(s^{(0)}, m, e^{(0)}) \in \mathscr{L}_\tau$, so $w^\frown e^{(0)} \not\prec x$, it is $x(\ell) = 1 - e^{(0)}$. Like in the proof of Theorem 3.2, we define inductively sequences $s^{(j)}$ in $\{0,1\}^{<\omega}$ and numbers $e^{(j)}$ such that $e^{(j)}$ is the answer of $\tau$ to $(s_0, s_1, \ldots, s_{k-1}, s^{(j)})$, and $s^{(j+1)} = s^{(j)\frown}(1 - e^{(j)})$. Then one can check that $u^\frown(s^{(j)}, m, e^{(j)}) \in \mathscr{L}_\tau$ and deduce that $x(\ell + j)$ should be equal to $1 - e^{(j)}$. This shows that $x$ is completely determined by $u, m$ and $\tau$, hence that $E_{u,m}$ contains at most one point.

To prove that $A \subset \bigcup_{u \in \mathscr{L}_\tau, m \in \omega} E_{u,m}$, suppose by contradiction that there exists some $\alpha \in A \setminus \bigcup_{u \in \mathscr{L}_\tau, m \in \omega} E_{u,m}$, and fix $\gamma = (t_k)$ such that $(\alpha, \gamma) \in F$. Then we can define inductively sequences $s_k$ such if $e_k$ is the answer by $\tau$ to $((s_0, t_0), (s_1, t_1), \ldots, (s_k, t_k))$, then $u_k = (s_0, t_0, e_0, \ldots, s_{k-1}, t_{k-1}, e_{k-1})$ is $\tau$-legal and satisfies $\alpha \in N_{u_k}$ and

$\alpha \notin E_{u_k, t_k}$. This means that the run where Player I plays the $s_k$'s and the $t_k$'s Player II follows $\tau$ is won by Player I, a contradiction since $\tau$ is winning.

## 4. $\Sigma^0_2$-separation

In all the sequel the notation $X \approx Y$ will mean that the topological spaces $X$ and $Y$ are homeomorphic. We shall more particularly be interested in spaces which are homeomorphic to $\mathbb{Q}$ the space of all rational numbers. We recall that a space $X$ is homeomorphic to $\mathbb{Q}$ iff $X$ is countable and dense in itself (has no isolated points).

**4.1. Hurewicz Theorem**

The following classical result is due to Hurewicz (1928).

**Theorem.** *Let $A_0$ and $A_1$ be two disjoint subsets of a Polish space, and suppose that $A_0$ is $\Sigma^1_1$. Then:*

– *either there exists a $\Sigma^0_2$-set containing $A_0$ and disjoint from $A_1$,*

– *or there exists a compact set $K \subset A_0 \cup A_1$ with $K \approx 2^\omega$ and $K \cap A_1 \approx \mathbb{Q}$.*

Notice that in the second alternative we also have that $K \cap A_0 \approx \omega^\omega$.

We shall now show that Hurewicz Theorem is linked to the determinacy of some game.

**4.2. The game $H(A_0, A_1)$**

In the sequel we identify $\mathbb{Q}$ with the subset of the Cantor set $2^\omega$ constituted of all infinite sequences which are eventually null.

Given two disjoint subsets $A_0$ and $A_1$ in $\omega^\omega$ we define a game $H(A_0, A_1)$. A run in this game goes a follows:

$$\begin{array}{lcccccccc} \mathrm{I} & e_0 & & e_1 & \dots & e_n & \dots & \\ \mathrm{II} & & s_0 & & s_1 & \dots & s_n & \end{array}$$

where $s_n \in \omega$ and $e_n \in \{0,1\}$. Let:

$$\varepsilon = (e_0, e_1, e_2, \dots, e_n, \dots)$$

$$\alpha = s_0{}^\frown s_1{}^\frown \dots {}^\frown s_n \dots$$

The win condition of the game is the following: Player I wins the run iff

$$(\varepsilon \in \mathbb{Q} \quad \text{and} \quad \alpha \in A_0) \qquad \text{or} \qquad (\varepsilon \notin \mathbb{Q} \quad \text{and} \quad \alpha \in A_1)$$

**Theorem 4.3.** *Let $A_0$ and $A_1$ be two disjoint sets in $\omega^\omega$.*

*If Player I has a winning strategy in $H(A_0, A_1)$ then there exists a $\Sigma^0_2$-set containing $A_0$ and disjoint from $A_1$.*

*If Player II has a winning strategy in $H(A_0, A_1)$ then there exists a compact set $K \subset A_0 \cup A_1$, with $K \approx 2^\omega$ and $K \cap A_1 \approx \mathbb{Q}$.*

**Proof.** Suppose that Player I has a winning strategy $\sigma$ in $H(A_0, A_1)$ and consider the mapping $\varphi : \alpha \mapsto \varepsilon$ associated to all possible runs where Player II is playing freely some $\alpha \in \mathbf{2}^\omega$ and Player I is constructing $\varepsilon$ by following his winning strategy $\sigma$. Then $\varphi$ is continuous and so $S = \varphi^{-1}(\mathbb{Q})$ is a $\Sigma^0_2$ set in $\omega^\omega$. And since the strategy $\sigma$ is winning for Player I, then $S \supset A_0$ and $S \cap A_1 = \emptyset$.

Suppose that Player II has a winning strategy $\tau$ in $H(A_0, A_1)$ and consider the mapping $\psi : \varepsilon \mapsto \alpha$ associated to all possible runs where Player I is playing freely some $\varepsilon \in \mathbf{2}^\omega$ and Player II is constructing $\alpha$ by following his winning strategy $\tau$. Then $K = \psi(\mathbf{2}^\omega)$ is a compact set, and since the strategy $\tau$ is winning for Player II then:

$$K \cap A_0 = \psi(\mathbf{2}^\omega \setminus \mathbb{Q}) \qquad \text{and} \qquad K \cap A_1 = \psi(\mathbb{Q})\,.$$

So $K \cap A_1$ is countable and dense in itself, hence $K \cap A_1 \approx \mathbb{Q}$; it follows then that $\overline{K \cap A_1} = K \approx \mathbf{2}^\omega$. □

Notice that the complexity of the game $H(A_0, A_1)$ is essentially of the same level as the complexity of the sets $A_0$ and $A_1$. In particular since in the classical Hurewicz Theorem no assumption is made about the set $A_1$, one cannot derive this result from the previous Theorem. However as we shall see next, if we impose on both sets $A_0$ and $A_1$ to be $\Sigma^1_1$, then one can obtain interesting informations via games.

**4.4. The game** $H(A_0, A_1; F_0, F_1)$

Let $A_0$ and $A_1$ be two disjoint $\Sigma^1_1$ subsets of $\omega^\omega$, and let $F_0$ and $F_1$ be two closed subsets of $\omega^\omega \times \omega^\omega$ which project onto $A_0$ and $A_1$. A run in the game $H(A_0, A_1; F_0, F_1)$ goes as follows:

$$\begin{array}{llllll} \text{I} & (s_{-1}, t_{-1}) & (s_0, t_0) & (s_1, t_1) & \dots & (s_n, t_n) \\ \text{II} & & e_0 & e_1 \quad \dots & e_n & \dots \end{array}$$

where $s_n, t_n \in \omega$ and $e_n \in \{0,1\}$. Let:

$$\varepsilon = (e_0, e_1, e_2, \dots, e_n, \dots)$$

$$\alpha = (s_0, s_1, \dots, s_n, \dots)$$

$$\gamma = (t_0, t_1, \dots, t_n \dots)$$

Let us define trees $T_0$ and $T_1$ on $\{0,1\} \times \omega$ by letting, for $s \in \{0,1\}^{<\omega}$ and $u \in \omega^{<\omega}$

$$(s, u) \in T_i \Leftrightarrow |s| = |u| \quad \text{and} \quad \exists(\alpha^*, \gamma^*) \in F_i \quad \text{with} \quad s \prec \alpha^* \quad \text{and} \quad u \prec \gamma^*$$

In order to win, Player I has to obey to the following rules for every $k$:

- if $e_k = 0$ and if

$$\{j_0, j_1, \dots, j_p\} = \{j \le k : e_j = 0\}$$

then

$$(\langle s_0, s_1, \ldots, s_p\rangle, \langle t_{j_0}, t_{j_1}, \ldots, t_{j_p}\rangle) \in T_1$$

— if $e_k = 1$ and if $e_p = e_{p+1} = \ldots = e_k = 1$ with $p = 0$ or $e_{p-1} = 0$, then

$$(\langle s_0, s_1, \ldots, s_{k-p}\rangle, \langle t_p, t_{p+1}, \ldots, t_k\rangle) \in T_0$$

And Player II wins if for some $k$ these rules are not satisfied. It is clear that the game $H(A_0, A_1; F_0, F_1)$ is a closed game.

**Theorem 4.5.** *Let $A_0$ and $A_1$ be two disjoint analytic sets in $\omega^\omega$ and let $F_0$ and $F_1$ be two closed subsets of $\omega^\omega \times \omega^\omega$ which project onto $A_0$ and $A_1$.*

*If Player II has a winning strategy in $H(A_0, A_1; F_0, F_1)$ then there exists a $\mathbf{\Sigma}^0_2$-set containing $A_0$ and disjoint from $A_1$.*

*If Player I has a winning strategy in $H(A_0, A_1; F_0, F_1)$ then there exists a compact set $K \subset A_0 \cup A_1$, with $K \approx \mathbf{2}^\omega$ and $K \cap A_1 \approx \mathbb{Q}$.*

**Proof.** Suppose that Player I has a winning strategy $\sigma$ in $H(A_0, A_1; F_0, F_1)$ and consider the mapping $\psi : \varepsilon \mapsto \alpha$ associated to all possible runs where Player II is playing freely some $\varepsilon \in \mathbf{2}^\omega$ and Player I is constructing $\alpha$ by following his winning strategy $\sigma$, and ignoring the sequence $\gamma$, as well as $s_{-1}$. Then exactly as in the proof of Theorem 4.3 one can check that $K = \psi(\mathbf{2}^\omega)$ has all the desired properties: if $\varepsilon \in \mathbb{Q}$, Player II constructs a branch of the tree $T_1$, hence $\psi(\varepsilon) \in A_1$; and if $\varepsilon \notin \mathbb{Q}$, Player II constructs a branch of $T_0$, hence $\psi(\varepsilon) \in A_0$.

Now if we suppose that Player II has a winning strategy $\tau$ the arguments of Theorem 4.3 cannot be resumed, but we shall define a countable family of closed sets whose union separates $A_0$ from $A_1$.

Let $\alpha \in \mathbf{2}^\omega$. We say that a sequence $u = (t_0, t_1, \ldots, t_n)$ is $(\tau, \alpha)$-legal and write $u \in \mathscr{L}_{\tau,\alpha}$ if the run where Player II follows $\tau$ and I plays $((0, 0), (\alpha_0, t_0), (\alpha_1, t_1), \ldots, (\alpha_n, t_n))$ is legal up to $n$. Let us then denote by $\varrho(u)$ the answer $e_{n+1}$ by $\tau$ to $((0, 0), (s_0, t_0), (s_1, t_1), \ldots, (s_n, t_n))$, and

$$C_u = \{\alpha \in \omega^\omega : u \in \mathscr{L}_{\tau,\alpha} \text{ and } \varrho(u) = 1\}$$

$$C = \{\alpha : \exists u\ \alpha \in C_u \text{ and } \forall v\, \forall m\ \alpha \in C_v \text{ and } v^\frown m \in \mathscr{L}_{\tau,\alpha} \Rightarrow \exists w \succ v^\frown m\ \alpha \in C_w\}$$

It is clear that, for any $u$, the set $C_u$ is clopen. Thus $C$ is $\mathbf{\Pi}^0_2$. We want to prove that $\check{C}$ is a $\mathbf{\Sigma}^0_2$-set separating $A_0$ from $A_1$. We have to prove that $C$ separates $A_1$ from $A_0$, that is $C \cap A_0 = \emptyset$ and $\check{C} \cap A_1 = \emptyset$.

Suppose $\alpha \in A_1 \cap \check{C}$. We shall construct a run won by Player I against $\tau$, a contradiction. Fix some $\gamma$ such that $(\alpha, \gamma) \in F_1$. We distinguish two cases:

— If for every $u$ $\alpha \notin C_u$, then for every $k$, $\gamma_{\restriction k} \in \mathscr{L}_{\tau,\alpha}$ since $\tau$ never answers 1. Hence Player II wins against $\tau$ playing $(\alpha, \gamma)$.

— If there is $v$ such that $\alpha \in C_v$, then fix $m$ such that $v^\frown m$ is $(\tau, \alpha)$-legal but for any extension $w$ of $v^\frown m$ $\alpha$ does not belong to $C_w$. Then Player II plays $\alpha$ and $v^\frown m^\frown \gamma$ and this is legal since $\varrho(v) = 1$ and $\tau$ answers 0 each time after this move, so nothing more is needed with respect to $T_0$. Thus each position of Player I is legal and Player II looses.

Now suppose $\alpha \in A_0 \cap C$ and fix $\gamma$ such that $(\alpha, \gamma) \in F_0$. There is a sequence $u_0$ of minimal length $k_0$ such that $\alpha \in C_{u_0}$, that is $\varrho(u_0) = 1$; but, by minimality, this move is the first one where $\tau$ answers 1. So Player II can play $s_{k_0}$ and $m_0 = \gamma(0)$; then $(s_0, m_0) \in T_0$ and the move is legal.

Since $u_0 ^\frown m_0$ is $(\tau, \alpha)$-legal and $\alpha \in C$ there is a $u_1$ of minimal length $k_1 > k_0$ extending $u_0 ^\frown m_0$ such that $\varrho(u_1) = 1$, and, by minimality, it is the second time $\tau$ answers 1. So Player II is expected to play $s_{k_1}$ and $m_1$ such that $((s_0, s_1), (m_0, m_1) \in T_0$, and Player II can choose $m_1 = \gamma(1)$. Repeating this argument, Player I can play legally, and wins against $\tau$. This completes the proof. $\square$

**Remark 4.6.**

There is an explicit mapping $\Phi : \tau \to (E_n(\tau))$ which assigns to each strategy $\tau$ for Player II a countable family of closed sets in $\omega^\omega$ such that the union of the $E_n(\tau)$'s separates $A_0$ from $A_1$ whenever $\tau$ winning in the game $H(A_0, A_1; F_0, F_1)$.

## 5. The strategic basis Theorem for closed games

*We consider only games on $\omega$.*

### 5.1. Families of games

We shall now present a theorem of Martin which roughly speaking, asserts the following: In a closed game $G(A)$, if Player II (who is trying to go in the open set $\check{A}$) has a winning strategy, then he has a winning strategy depending in a Borel way on $A$.

To give a precise statement we shall replace the set $A$ by a set $A(p)$ depending on some parameter $p$. Or in other terms we fix some auxiliary space $P$ and a set $A \subset P \times \omega^\omega$, so that for all $p \in P$ the section $A(p)$ is a subset of $\omega^\omega$ and we can consider the the game $G_A(p) = G(A(p))$. Notice that if $A$ is closed then all the games $G(A(p))$ are also closed.

### 5.2. The space of strategies

For simplicity we shall consider only strategies for Player II. Let $S_1$ (respectively $S_2$) denote the set of all finite sequences in $\omega$ with odd (respectively even) length. A strategy for Player II can be viewed as a mapping $\tau : S_1 \to S_2$ of the form:

$$t \mapsto \tau(t) = t^\frown a$$

A sequence $(a_n)$ is compatible with $\tau$ if:

$$\tau(\langle a_0, a_1, \ldots, a_{2_n}\rangle) = (\langle a_0, a_1, \ldots, a_{2_n}, a_{2_{n+1}}\rangle)$$

So the set $\Sigma$ of all strategies (for Player II) can be viewed as a subset of the space $S_0^{S_1}$. We endow this space with the product topology of the discrete topology on $S_0$. Thus $S_0^{S_1} \approx \omega^\omega$, and it is easy to see that $\Sigma$ is a closed subset of this space. Hence $\Sigma$ is a Polish space.

**Theorem 5.3.** *Let $P$ be a Polish space and $A$ be a closed subset of $P \times \omega^\omega$. Then*

*a) The set $Q = \{p \in P$: such that Player II has a winning strategy in $G_A(p)\}$ is $\mathbf{\Pi}_1^1$.*

*b) There exist $\Phi_1$ and $\Phi_1$ two $\mathbf{\Sigma}_1^1$ subsets of $P \times \Sigma$ such that for all $p \in Q$, $\Phi_1(p) = \check{\Phi}_2(p) = \{\varphi(p)\}$ where $\varphi(p)$ is a winning strategy for Player II in $G_A(p)$.*

*c) If $Q$ is Borel then there exists a Borel mapping $\varphi : Q \to \Sigma$ such that for all $p \in Q$, $\varphi(p)$ is a winning strategy for Player II in $G_A(p)$.*

Of course *c)* is an immediate consequence of *b)*. In fact the most useful case will be for us the case where $Q = P$. The proof of *a)* can be derived from the proof of the Gale Stewart Theorem, where it follows from the arguments that in a closed game $G(B)$, the question whether Player II has a winning strategy can be reduced to the question whether some tree — depending continuously on $B$ — is well founded. The proof of *b)* requires some nontrivial descriptive arguments.

We now give several applications of the previous Theorem.

**Theorem 5.4.** *Let $P$ be a Polish space and $A$ be a $\mathbf{\Sigma}_1^1$ subset of $P \times \mathbf{2}^\omega$. Then*

*a) The set $Q = \{p \in P : A(p)$ is countable$\}$ is $\mathbf{\Pi}_1^1$.*

*b) If $A(p)$ is countable for all $p \in P$ (i.e. if $Q = P$), then $A$ is the union of a countable family of Borel graphs (in particular $A$ is Borel).*

To prove these results, fix a closed set $F$ in $P \times \omega^\omega \times \mathbf{2}^\omega$ which projects on $A$. Then consider the family of closed games $G^*(A(p), F(p))$ for $p \in P$. Then the Theorem is an immediate consequence of Theorem 3.5 and Theorem 5.3 (see Remark 3.3.a).

In a "similar" way, using Theorem 4.5 and Remark 4.6, one can prove the following:

**Theorem 5.5.** *Let $P$ be a Polish space, and let $A_0$ and $A_1$ be two disjoint analytic sets in $P \times \omega^\omega$. Then*

*a) The set $Q = \{p \in P : \exists C$ a $\mathbf{\Sigma}_2^0$-set with $C \supset A_0(p)$ and $C \cap A_1(p) = \emptyset\}$ is $\mathbf{\Pi}_1^1$.*

*b) If for all $p \in P$ there exists a $\mathbf{\Sigma}_2^0$-set containing $A_0(p)$ and disjoint from $A_1(p)$ (i.e. if $Q = P$), then there exists a Borel set $B$ containing $A_0$ and disjoint from $A_1$, and of the form $B = \bigcup_n B_n$ where each $B_n$ is a Borel set and all its sections $B_n(p)$ are closed.* □

The main argument in this proof is the following observation: If $\Phi$ is the mapping defined in Remark 4., then for all $n$ the set $\{(\alpha, \tau) \in \omega^\omega \times \Sigma : \alpha \in E_n(\tau)\}$ is closed in the space $\omega^\omega \times \Sigma$.

**Corollary 5.6.** *If $B$ is a Borel set in $P \times \omega^\omega$ with $\mathbf{\Sigma}_2^0$ sections then $B = \bigcup_n B_n$ where each $B_n$ is a Borel set with closed sections.*

Consider a set $X \subset 2^\omega \times 2^\omega$ and $Y = \pi(X) \subset 2^\omega$ its projection on the first factor. We shall denote by

$$\pi_X : X \to Y$$

the restriction of the projection mapping to $X$. We are interested in the comparison of the following two properties for $f = \pi_X$:

**(CC):** "$\forall L$ compact $\subset Y$, $\exists K$ compact $\subset X$ such that $f(K) = L$"

and

**(IP):** "$\exists X' \subset X$ such that $f(X') = Y$ and $\forall L$ compact $\subset Y$,

the set $X' \cap f^{-1}(L)$ is compact."

the mapping $f$ is said to be *compact covering* if it satisfies **(CC)** and *inductively perfect* if it satisfies **(IP)**. Notice that since $f$ is continuous, **(IP)** states exactly that the restriction of $f$ to some (necessarily closed) subset of $X$ is onto and perfect.

Clearly "**(IP)** $\Rightarrow$ **(CC)**"; the problem whether the converse holds was first raised, for some particular cases, by E. Michael. In fact the implication "**(CC)** $\Rightarrow$ **(IP)**" is now known to be true under several different supplementary hypothesis among which:

**($\mathbf{H_0}$):** "$X$ is Polish"

or

**($\mathbf{H_1}$):** "$Y$ is $\mathbf{K}_\sigma$"

The case ($\mathbf{H_0}$) was proved several years ago (1972) independently by J. P. R. Christensen and the second author, and ($\mathbf{H_1}$) more recently (1990) by A. V. Ostrovsky and in the particular (already non-trivial) case where $Y$ is countable by W. Just and H. Wicke. Notice that none of these two cases can be derived from the other; moreover the methods of their proofs are completely different.

On the other hand, one can construct counter-examples to "**(CC)** $\Rightarrow$ **(IP)**" using the Axiom of Choice; but in these examples the spaces $X$ and $Y$ do not have nice definability properties. It is then natural to ask whether the implication "**(CC)** $\Rightarrow$ **(IP)**" holds under the hypothesis:

**(H):** "$X$ and $Y$ are Borel."

or under the weaker hypothesis:

**(H′):** "$X$ is Borel."

In fact, assuming $(\mathrm{Det}\,(\Sigma_1^1))$, we shall prove the following.

**Theorem 6.1.** (Det $(\Sigma_1^1)$). *Suppose that $X$ is $\mathbf{\Pi}_1^1$, and that $Y$ is either $\mathbf{\Pi}_1^1$ or $\mathbf{\Sigma}_1^1$. Then $\pi_X : X \to Y$ is compact covering if and only if $\pi_X$ is inductively perfect.*

**Corollary 6.2.** (Det $(\Sigma_1^1)$). *Suppose that $X$ is Borel. Then $\pi_X : X \to Y$ is compact covering if and only if $\pi_X$ is inductively perfect.*

These results will be obtained by considering the following game.

**6.3. The game $G_c(X, Y)$**

For any $X \subset \mathbf{2}^\omega \times \mathbf{2}^\omega$ and $Y \subset \mathbf{2}^\omega$, we define a game $G_c(X, Y)$. A run in this game is always infinite and goes as follows:

$$\begin{array}{lcccccc} \text{I} & k_0 & k_1 & \dots\ k_{n-1} & & k_n\ \dots & \\ \text{II} & & S_1 & S_2\ \dots & & S_n & \dots \end{array}$$

where $k_n \in \{0,1\}$ and $\emptyset \neq S_n \subset \{0,1\}^n$ is such that $T_n = \bigcup_{0 \le p \le n} S_p$ is a tree (where by convention $S_0 = \{\emptyset\}$). A run in this game will be identified with the sequence $(k_n, S_n)_{n\in\omega}$, although $S_0 = \{\emptyset\}$ is not formally "played" by Player II. Let:

$$y = (k_n)_{n\in\omega} \quad \text{and} \quad T = \bigcup_{n\in\omega} T_n = \bigcup_{n\in\omega} S_n\,.$$

Player I wins the run if:

$$y \in Y \quad \text{and} \quad \exists z \in [T] \quad \text{such that} \quad x = (y, z) \notin X\,.$$

Notice that there is no relation a priori between $X$ and $Y$, but obviously if $Y \not\subset \pi(X)$ then Player I wins the game.

Theorem 6.1 is now an immediate consequence of the following result.

**Theorem 6.4.** *If Player I has a winning strategy in $G_c(X, Y)$, then $\pi_X$ is not compact covering.*

*If Player II has a winning strategy in $G_c(X, Y)$, then $\pi_X$ is inductively perfect.*

**Proof.** Suppose that Player I has a winning strategy, and let $L$ be the set of all possible $y = (k_n)_{n\in\omega} \in \mathbf{2}^\omega$ played by $\sigma$ (in all possible runs). Since Player II has at each step only finitely many possible choices $(\le 2^{2^n})$ then $L$ is compact, and since $\sigma$ is winning then necessarily $L \subset Y$.

Suppose that $L \subset \pi(K)$ for some compact set $K \subset X$. For any $u \in 2^{<\omega}$ let:

$$S(u) = \{v \in 2^{<\omega} : |v| = |u| \text{ and } K \cap (N_u \times N_v) \neq \emptyset\}\,.$$

Consider the run where Player I follows $\sigma$ and Player II answers $S_n = S(u_n)$ where $u_n = (k_0, \dots, k_n)$ is already played by Player I; then (with the notations of 6.1) we have:

$$y \in L \subset Y \quad \text{and} \quad \{y\} \times [T] \subset K \subset X$$

so Player II wins the run which is a contradiction. $\square$

Suppose now that Player II has a winning strategy $\tau$. For any $y \in \mathbf{2}^\omega$ let $T(y)$ denote the tree constructed by Player II in the run where Player I plays $y = (k_n)_{n\in\omega}$ and Player II follows $\tau$. Then

$$H = \{(y,z) \in \mathbf{2}^\omega \times \mathbf{2}^\omega : z \in [T(y)]\}$$

is clearly closed (compact) in $\mathbf{2}^\omega \times \mathbf{2}^\omega$ and since $\tau$ is winning, then for any $y \in Y$ we have:

$$H_y = \{y\} \times [T(y)] \subset X . \quad \square$$

**Remark 7.1.**

Assuming the existence of an uncountable $\mathbf{\Pi}^1_1$ set which contains no perfect subset, we construct in [2] a $\mathbf{\Pi}^1_1$ set $X \subset \mathbf{2}^\omega \times \mathbf{2}^\omega$ such that $Y = \pi(X)$ is also $\mathbf{\Pi}^1_1$ and $\pi_X$ is compact covering but not inductively perfect.

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