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Unconditionally Converging Holomorphic Mappings between Banach Spaces

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It ıs proved that every holomorphic mapping between complex Banach spaces takes unconditionally convergent series into unconditionally convergent series, and (locally) weakly unconditionally Cauchy series into weakly unconditionally Cauchy series. The class of unconditionally converging holomorphic mappings is ntroduced, as those mappings taking (locally) weakly unconditionally Cauchy series into unconditiona ly convergent series. It is shown that a holomorphic mapping is unconditionally converging if and only if all its derivatives at the origin are unconditionally converging polynomials. A characterızation is given of the spaces E such that the space $\mathscr{H}_b(E)$ of holomorphic functions of bounded type on E is reflexıve. Other properties of unconditionally converging holomorphic mappings are investigated. The analogous properties for polynomials are surveyed.

1. Introduction

It is well known that, in general, the polynomials between Banach spaces do not preserve the weak convergence. An easy example is the k-homogeneous polynomial $(k \geq 2)$ P $\ell_2 \rightarrow \ell_1$ given by $P((x_n)) = (x_n^k)_n$, where each coordinate is raisen to the power k. Indeed, P takes a weakly null sequence into the unit vector basis of ℓ_1. It might therefore seem unexpected that a polynomial preserves the weakly unconditionally Cauchy series, as well as the unconditionally convergent ones [8] (the definitions are recalled below). Moreover, this is true even for holomorphic mappings, in a sense to be made more precise below.

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We say that a polynomial $P : E \to F$ is *unconditionally converging* if for every weakly unconditionally Cauchy series $\sum x_i$ in E, the series $\sum P x_i$ is unconditionally convergent. Unconditionally converging holomorphic mappings are defined (locally) in an analogous way. There are pairs of Banach spaces E, F such that every (linear bounded) operator from E into F is unconditionally converging. It may be proved that exactly for those spaces, every polynomial from E into F is unconditionally converging too. The same is true for holomorphic mappings.

A polynomial $P : E \to F$ is *completely continuous* if for every sequence $(x_n) \subset E$ weakly convergent to some x, we have that $\|Px_n - Px\|$ converges to zero.

The polynomial $P : \ell_2 \to \ell_1$ given above shows that there are spaces E, F so that every operator from E into F is completely continuous, but not every polynomial from E into F is so. However, if E has the Dunford-Pettis property and every operator from E into F is weakly compact, then every polynomial from E into F is completely continuous. Therefore, if F contains no copy of ℓ_∞, then every polynomial from ℓ_∞ into F is completely continuous. It is easy to see that not every polynomial from ℓ_∞ into F is weakly compact, whenever F is nonreflexive.
In this paper, we survey the above mentioned properties of polynomials, that may be seen in [8, 9], and prove that they can be extended to the holomorphic case.

Throughout, E and F will denote complex Banach spaces (however, all the results given here for polynomials are also true in the real case), and $\mathbf{N}$ the natural numbers. A formal series $\sum x_i$ in E is *weakly unconditionally Cauchy* (w.u.C.) if for every ϕ in the dual space E^* we have $\sum |\phi(x_i(x_i)| < \infty$; equivalently, if

$$\sup_n \sup_{|\varepsilon_i| \leq 1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| < \infty .$$

The series is *unconditionally convergent* (u.c.) if any subseries is norm convergent; equivalently, if

$$\sup_{|\varepsilon_i| \leq 1} \left\| \sum_{i=n}^{n} \varepsilon_i x_i \right\| \to 0 \qquad \text{for} \quad n \to \infty .$$

We denote by $\mathscr{L}(E, F)$ the space of all (linear bounded) operators from E into F. In $\mathscr{L}(E, F)$, we shall be considering the following subspaces:
$\mathscr{W}\mathscr{C}o(E, F)$: the subspace of all weakly compact operators;
$\mathscr{C}\mathscr{C}(E, F)$: the completely continuous operators;
$\mathscr{U}\mathscr{C}(E, F)$: the unconditionally converging operators.

Recall that E has the *Dunford-Pettis property* (DPP for short) if for every F, we have $\mathscr{W}\mathscr{C}o(E, F) \subseteq \mathscr{C}\mathscr{C}(E, F)$. Classical examples of spaces with DPP are $L^1(\mu)$ and $C(K)$.

The notation $\mathscr{P}(^k E, F)$ stands for the space of all k-homogeneous (continuous) polynomials from E into F, and $\mathscr{P}_{cc}(^k E, F)$ for the subspace of completely continuous polynomials. When the space F is omitted, it is understood to be the scalar field $\mathbf{K}$, e.g., $\mathscr{P}(^k E) = \mathscr{P}(^k E, \mathbf{K})$.

The space of k-linear (continuous) mappings from E^k into F is denoted by $\mathscr{L}({}^kE, F)$. To each $P \in \mathscr{P}({}^kE, F)$ we can associate a unique symmetric $\hat{P} \in \mathscr{L}({}^kE, F)$ so that $P(x) = \hat{P}(x, \ldots, x)$ for all $x \in E$. For the general theory of polynomials on Banach spaces, we refer to [10].

2. Results on polynomials

The preservation of series by polynomials relies on a lemma, whose proof is included for completeness. For it, we need the generalized Rademacher functions, denoted by $s_n(t)$, $n \in \mathbf{N}$, which were defined in [4] as follows:

Fix $2 \leqq k \in \mathbf{N}$, and let $\alpha_1 = 1, \alpha_2, \ldots, \alpha_k$ denote the kth roots of unity.

Let $s_1 : [0, 1] \to \mathbf{C}$ be the step function taking the value α_j on $((j-1)/k, j/k)$ for $j = 1, \ldots, k$.

Then, assuming that s_{n-1} has been defined, define s_n as follows. Fix any of the k^{n-1} subintervals I of $[0, 1]$ used in the definition of s_{n-1}. Divide I into k equal intervals $I_1, \ldots, I_k$, and set $s_n(t) = \alpha_j$ if $t \in I_j$.

These functions are orthogonal [4, Lemma 1.2] in the sense that, for any choice of integers $i_1, \ldots, i_k$; $k \geqq 2$, we have

$$\int_0^1 s_{i_1}(t) \ldots s_{i_k}(t)\,\mathrm{d}t = \begin{cases} 1, & \text{if } i_1 = \ldots = i_k\,; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1. *Given a polynomial $P \in \mathscr{P}({}^kE, F)$, we have that, for every $x_1, \ldots, x_n \in E$,*

$$\sup_{|\varepsilon_j| \leqq 1} \left\| \sum_{j=1}^n \varepsilon_j P x_j \right\| \leqq \sup_{|v_j| \leqq 1} \left\| P\left(\sum_{j=1}^n v_j x_j \right) \right\|.$$

Proof. Observe that both suprema are attained for some $|\varepsilon_j| = |v_j| = 1$.

For any $x_1, \ldots, x_n \in E$ and any complex numbers ε_j with $|\varepsilon_j| = 1$, we can find $\psi \in F^*$, $\|\psi\| = 1$, such that

$$\left\| \sum_{j=1}^n \varepsilon_j P x_j \right\| = \psi\left(\sum_{j=1}^n \varepsilon_j P x_j \right).$$

Then, taking complex numbers δ_j such that $\delta_j^k = \varepsilon_j$, we obtain

$$\begin{aligned} \left\| \sum_{j=1}^n \varepsilon_j P x_j \right\| &= \psi\left(\sum_{j=1}^n P(\delta_j x_j) \right) \\ &= \int_0^1 \left(\sum_{j_1, \ldots, j_k = 1}^n s_{j_1}(t) \ldots s_{j_k}(t)\, \psi \circ \hat{P}(\delta_{j_1} x_{j_1}, \ldots, \delta_{j_k} x_{j_k}) \right) \mathrm{d}t \\ &= \int_0^1 \psi \circ \hat{P}\left(\sum_{j_1=1}^n \delta_{j_1} s_{j_1}(t)\, x_{j_1}, \ldots, \sum_{j_k=1}^n \delta_{j_k} s_{j_k}(t)\, x_{j_k} \right) \mathrm{d}t \end{aligned}$$

$$= \int_0^1 \psi \circ P\left(\sum_{j_1=1}^{n} \delta_j s_j(t)\, x_j\right) \mathrm{d}t$$

$$\leqq \sup_{|v_j|=1} \left\| P\left(\sum_{j=1}^{n} v_j x_j\right)\right\|,$$

and the proof is finished. □

We remark that if the spaces under consideration are real, then the right hand side of the inequality has to be multiplied by $(2k)^k/k!$. The proof, using complexifications of the spaces, is standard.

Now, thanks to the equivalent definitions of w.u.C. and u.c. series given in Section 1, applying Lemma, we easily obtain

Theorem 2. *Given a polynomial $P \in \mathscr{P}({}^kE, F)$, if $\sum x_i$ is a w.u.C. series (resp. an u.c. series) in E, then $\sum Px_i$ is w.u.C. (resp. u.c.) in F.*

Therefore, as said in the Introduction, it is natural to introduce the class of unconditionally converging polynomials. A polynomial is *unconditionally converging* if it takes w.u.C. series into u.c. series. The space of all k-homogeneous unconditionally converging polynomials from E into F is denoted by $\mathscr{P}_{uc}({}^kE, F)$.

The prototype of w.u.C. series which is not u.c. is the unit vector basis of c_0 Using Lemma 1 and the Bessaga-Pełczyński selection theorem, it is not difficult to prove the following result which will be needed in Section 3.

Lemma 3. *Suppose $P \in \mathscr{P}({}^kE, F)$ is not unconditionally converging. Then there is an injective isomorphism $i: c_0 \to E$ such that $P \circ i$ takes the unit vector basis of c_0 into a sequence equivalent to the unit vector basis of c_0. In particular, $P \circ i \in \mathscr{P}({}^kc_0, F)$ is not unconditionally converging.*

It is well known that the Dunford-Pettis property may be restated in terms of polynomials: a space E has the DPP if and only if for every $k \in \mathbf{N}$ and F, we have $\mathscr{P}_{wco}({}^kE, F) \subseteqq \mathscr{P}_{cc}({}^kE, F)$ [13]. However, the situation is very different as regards property (V). A space E has *property* (V) [12] if for every space F, we have $\mathscr{UC}(E, F) \subsetneqq \mathscr{WCo}(E, F)$. The $C(K)$ spaces and the reflexive spaces have property (V). In the polynomial case, the following result may be proved.

Theorem 4. *Given an integer $k > 1$ and a space E, we have that $\mathscr{P}_{uc}({}^kE, F) \subsetneqq \mathscr{P}_{wco}({}^kE, F)$ for every F if and only if the space $\mathscr{P}({}^kE)$ is reflexive.*

Only a few spaces satisfy that $\mathscr{P}({}^kE)$ is reflexive. Indeed, if E has a quotient isomorphic to ℓ_p $(1 \leqq p < \infty)$, then $\mathscr{P}({}^kE)$ contains a copy of ℓ_∞ [8, 6]. If $E = T^*$, Tsirelson's original space, then $\mathscr{P}({}^kE)$ is reflexive for every $k \in \mathbf{N}$ [1].

It is well known that if F contains no copy of ℓ_∞, then we have the equalities

$$\mathscr{L}(\ell_\infty, F) = \mathscr{WCo}(\ell_\infty, F) = \mathscr{CC}(\ell_\infty, F) = \mathscr{UC}(\ell_\infty, F)\,. \tag{1}$$

Contrarily to the linear case $(k = 1)$, whenever F is nonreflexive, for every integer

$k \geqq 2$, there is a polynomial $P \in \mathscr{P}_{cc}({}^k\ell_\infty, F)$ which is not weakly compact. It can be obtained as the composition of the following three mappings

$$\ell_\infty \xrightarrow{U} \ell_2 \xrightarrow{Q} \ell_1 \xrightarrow{T} F$$

where U is a completely continuous linear surjection, Q is the polynomial given by $Q((x_n)_n) = (x_n^k)_n$, and T is a quotient onto a separable nonreflexive subspace of F. Therefore, the equality (1) is not true in the polynomial case as far as the weakly compact polynomials are concerned.

It remains to ask if the equality holds for the unconditionally converging and the completely continuous polynomials. The following results give the answer:

Theorem 5. *Whenever $\mathscr{L}(E, F) = \mathscr{UC}(E, F)$, we also have $\mathscr{P}({}^kE, F) = - \mathscr{P}_{uc}({}^kE, F)$ for all $k \in \mathbf{N}$.*

We briefly mention the ideas of the proof. Firstly, the problem may be reduced to the case when E contains no complemented copy of c_0. Now, it may be proved by induction on k that, given $P \in \mathscr{P}({}^kE, F)$ and a w.u.C. series $\sum x_i$ in E, the mapping

$$Tx := (\hat{P}(x_m, \dots, x_m, x))_m$$

defines an operator $T: E \to c_0(F)$. From the fact that E contains no complemented copy of c_0, we can get that T is unconditionally converging, and then the result follows easily.

Theorem 6. *Suppose E has the* DPP, *and $\mathscr{L}(E, F) = \mathscr{WC}o(E, F)$. Given $k \in \mathbf{N}$ and $A \in \mathscr{L}({}^kE, F)$, let $(x_1^n), \dots, (x_k^n) \subset E$ be weak Cauchy sequences. Then the sequence $(A(x_1^n, \dots, x_k^n))_n$ is norm convergent.*

The proof may be reduced to the case when one of the sequences is weakly null. Assume for instance that (x_1^n) converges weakly to 0. Then the point is to prove that

$$Tz := (A(x_1^n, \dots, x_{k-1}^n, z))_n$$

defines a weakly compact operator $T: E \to c_0(F)$. Since E has the DPP, T is completely continuous, and this yields the result.

Corollary 7. *Suppose E has the* DPP *and $\mathscr{L}(E, F) = \mathscr{WC}o(E, F)$. Then we have $\mathscr{P}({}^kE, F) = \mathscr{P}_{cc}({}^kE, F)$ for all $k \in \mathbf{N}$.*

The last Theorem and his Corollary hold; for instance, in the following cases:

(a) $E = C(K)$ with K stonean (e.g. $E = \ell_\infty$), and $F \not\supset \ell_\infty$.

(b) $E = C(K)$ and $F \not\supset c_0$.

(c) E^* has the Schur property, and $F^* \not\supset \ell_1$.

(d) E^* has the Schur property, and F is weakly sequentially complete.

Finally, we give a result on extensions of polynomials to the bidual space. The following Proposition is needed:

Proposition 8. [5] *The dual space E^* has the* DPP *if and only if for every F and $T \in \mathscr{WC}o(E, F)$, the second adjoint $T^{**}: E^{**} \to F^{**}$ is completely continuous.*

From this, we can obtain.

Theorem 9. *Suppose* E^* *has the* DPP, *and* $\mathscr{L}(E, F) = \mathscr{WCo}(E, F)$. *Then each polynomial* $P \in \mathscr{P}({}^kE, F)$ *has an extension* $\tilde{P} \in \mathscr{P}_{cc}({}^kE^{**}, F)$, *with* $\|\tilde{P}\| = \|P\|$.

3. Holomorphic mapping

In this Section, we show that a holomorphic mapping $f: E \to F$ with $f(0) = 0$ takes w.u.C. series into w.u.C. series, provided that the unconditional partial sums of the series are all contained in a closed ball on which the convergence of the Taylor expansion of f at the origin is uniform. Moreover, we show that f takes u.c. series into u.c. series.

We introduce the class of unconditionally converging holomorphic mappings and prove that a mapping f which vanishes at the origin is unconditionally converging if and only if all the derivatives of f at the origin are unconditionally converging polynomials. As an application we obtain a new characterization of the spaces E such that the space $\mathscr{H}_b(E)$ of holomorphic functions of bounded type on E is reflexive. Finally, we extend Theorems 5 and 9, and Corollary 7 holomorphic mappings.

Given a holomorphic mapping $f: E \to F$ between complex Banach spaces, consider the Taylor expansion at the origin

$$f(x) = \sum_{k=0}^{\infty} P_k(x)$$

where $P_k \in \mathscr{P}({}^kE, F)$, and the convergence is uniform for x in a neighbourhood of 0. Its radius of convergence at 0 may be calculated by the Cauchy-Hadamard formula

$$\varrho(f) = \left(\limsup_{k\to\infty} \|P_k\|^{1/k}\right)^{-1}.$$

The space of all holomorphic mappings from E to F is denoted by $\mathscr{H}(E, F)$. We refer to [10, 11] for the general theory of holomorphic mappings on Banach spaces.

Given a mapping $f \in \mathscr{H}(E, F)$, we shall say that f *locally takes w.u.C. series into w.u.C. (u.c.) series* if for every w.u.C. series $\sum_{i=1}^{\infty} x_i$ in E with

$$\sup_n \sup_{|\varepsilon_i| \leq 1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| < \varrho(f),$$

we have that $\sum_{i=1}^{\infty} f(x_i)$ is a w.u.C. (u.c.) series.

We denote $\mathscr{H}_b(E, F)$ the space of holomorphic mappings of bounded type (i.e., bounded on bounded sets) from E into F, and by $\mathscr{H}_{wco}(E, F)$ the space of weakly compact holomorphic mapping (see [14]), defined as follows: $f \in \mathscr{H}(E, F)$ is weakly compact if every $x \in E$ has a neighbourhood V_x such that $f(V_x)$ is relatively weakly compact. We have $\mathscr{H}_b(E, F) = \{f \in \mathscr{H}(E, F) : \varrho(f) = \infty\}$ [10, Theorem 7.13].

Theorem 10. *Let $f \in \mathscr{H}(E, F)$ such that $f(0) = 0$. Then f locally takes w.u.C. series into w.u.C. series, and it takes u.c. series into u.c. series.*

Proof. Let $f = \sum_{k=1}^{\infty} P_k$ be the Taylor expansion of f at the origin, with $P_k \in \mathscr{P}({}^kE, F)$. If $\sum_{i=1}^{\infty} x_i$ is a w.u.C. series and we denote

$$M := \sup_n \sup_{|\varepsilon_i| \leqq 1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| < \varrho(f),$$

from the Cauchy-Hadamard formula we get $\limsup_k \left(M^k \|P_k\|\right)^{1/k} < 1$. Therefore, $\sum_{k=1}^{\infty} \|P_k\| \, M^k < \infty$.

Now, for any finite subset $\Delta \subset \mathbf{N}$,

$$\left\| \sum_{i \in \Delta} f(x_i) \right\| = \left\| \sum_{i \in \Delta} \left(\sum_{k=1}^{\infty} P_k \right) x_i \right\| \leqq \sum_{k=1}^{\infty} \left\| \sum_{i \in \Delta} P_k(x_i) \right\|;$$

by Lemma 1

$$\leqq \sum_{k=1}^{\infty} \sup_{|\varepsilon_i| \leqq 1} \left\| P_k \left(\sum_{i \in \Delta} \varepsilon_i x_i \right) \right\| \leqq \sum_{k=1}^{\infty} \|P_k\| \sup_{|\varepsilon_i| \leqq 1} \left\| \sum_{i \in \Delta} \varepsilon_i x_i \right\|^k \leqq \sum_{k=1}^{\infty} \|P_k\| \, M^k .$$

Hence $\sum_{i=1}^{\infty} f(x_i)$ is a w.u.C. series.

The proof for the u.c. case is analogous: If $\sum_{i=1}^{\infty} x_i$ is a u.c. series, then for a finite subset $\Delta \subset \mathbf{N}$ with $\min \Delta$ large enough, we have

$$\left\| \sum_{i \in \Delta} f(x_i) \right\| \leqq \sum_{k=1}^{\infty} \|P_k\| \sup_{|\varepsilon_i| \leqq 1} \left\| \sum_{i \in \Delta} \varepsilon_i x_i \right\|^k,$$

which converges to 0 when $\min \Delta$ goes to infinity since

$$\lim_{\min \Delta \to \infty} \sup_{|\varepsilon_i| \leqq 1} \left\| \sum_{i \in \Delta} \varepsilon_i x_i \right\| = 0 .$$

Hence $\sum_{i=1}^{\infty} f(x_i)$ is a u.c. series.

Corollary 11. *Any $f \in \mathscr{H}_b(E, F)$ such that $f(0) = 0$ takes w.u.C. (u.c.) series into w.u.C. (u.c.) series.*

Now it is natural to consider the class of unconditionally converging holomorphic mappings.

Definition 12. A mapping $f \in \mathscr{H}(E, F)$ is said to be *unconditionally converging* if $f(0) = 0$ and f locally takes w.u.C. series into u.c. series. We shall denote by $\mathscr{H}_{uc}(E, F)$ the subspace of all unconditionally converging holomorphic mappings from E into F.

Next we show that the unconditionally converging holomorphic mappings can be characterized in terms of the summands of their Taylor expansion at the origin.

Theorem 13. *Let* $f = \sum_{k=1}^{\infty} P_k \in \mathscr{H}(E, F)$. *Then* $f \in \mathscr{H}_{uc}(E, F)$ *if and only if* $P_k \in \mathscr{P}_{uc}$ *for every* $k \in \mathbf{N}$.

Proof. Assume $P_k \in \mathscr{P}_{uc}(^kE, F)$ for every $k \in \mathbf{N}$. Given a w.u.C. series $\sum_{i=1}^{\infty} x_i$ in E with

$$M := \sup_n \sup_{|\varepsilon_i| \leqq 1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| < \varrho(f),$$

we have $\sum_{k=1}^{\infty} \|P_k\| M^k < \infty$. Then, for every $\varepsilon > 0$ there exists k_ε such that $\sum_{k=k_\varepsilon+1}^{\infty} \|P_k\| M^k < \varepsilon/2$. On the other hand, since $\sum_{k=1}^{k_\varepsilon} P_k$ takes w.u.C. series into u.c. series, we can select n_ε such that, for $n_\varepsilon < \min \Delta$ we have $\left\| \sum_{i \in \Delta} \left(\sum_{k=1}^{k_\varepsilon} P_k \right) x_i \right\| < \varepsilon/2$. Then,

$$\begin{aligned} \left\| \sum_{i \in \Delta} f(x_i) \right\| &\leqq \left\| \sum_{i \in \Delta} \left(\sum_{k=1}^{k_\varepsilon} P_k \right) x_i \right\| + \left\| \sum_{i \in \Delta} \left(\sum_{k=k_\varepsilon+1}^{\infty} P_k \right) x_i \right\| \\ &< \frac{\varepsilon}{2} + \sum_{k=k_\varepsilon+1}^{\infty} \left\| \sum_{i \in \Delta} P_k x_i \right\| \\ &\leqq \frac{\varepsilon}{2} + \sum_{k=k_\varepsilon+1}^{\infty} \sup_{|\varepsilon_i| \leqq 1} \left\| P_k \left(\sum_{i \in \Delta} \varepsilon_i x_i \right) \right\| \\ &\leqq \frac{\varepsilon}{2} + \sum_{k=k_\varepsilon+1}^{\infty} \|P_k\| M^k \\ &< \varepsilon ; \end{aligned}$$

hence $\sum_{i=1}^{\infty} f(x_i)$ is u.c.

The proof of the converse is similar to that of [3, Proposition 1.8]. Let $f \in \mathscr{H}_{uc}(E, F)$; fix $k \in \mathbf{N}$ and $\varepsilon > 0$, and take a w.u.C. series $\sum_{i=1}^{\infty} x_i$ in E. We can assume (after multiplying by a suitable constant) that

$$\sup_n \sup_{|\varepsilon_i|\leqq 1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| < \varrho(f).$$

By Lemma 3, it is enough to show that exists $n_\varepsilon \in \mathbf{N}$ such that $\|P_k x_n\| \leqq \varepsilon$ for $n > n_\varepsilon$.

Claim: There exists $n_\varepsilon \in \mathbf{N}$ such that for $y \in \operatorname{aco}\{x_n : n \geqq n_\varepsilon\}$, the absolutely convex hull of $\{x_n : n \geqq n_\varepsilon\}$, we have $\|f(y)\| \leqq \varepsilon$.

Otherwise, for every $m \in \mathbf{N}$ we can select $y_m \in \operatorname{aco}\{x_n : n \geqq m\}$ with $\|f(y_m)\| > \varepsilon$, and passing to a subsequence we can assume $y_m \in \operatorname{aco}\{x_n : p_i \leqq n \leqq q_i\}$, with $q_i < p_{i+1}$. Then $\sum_{i=1}^{\infty} y_{m_i}$ is a w.u.C. series with

$$\sup_n \sup_{|\varepsilon_i|\leqq 1} \left\| \sum_{i=1}^{n} \varepsilon_i y_{m_i} \right\| < \varrho(f);$$

hence $\|f(y_{m_i})\| \to 0$, a contradiction which proves the claim.

Now, if there exists $n > n_\varepsilon$ such that $\|P_k x_n\| > \varepsilon$, then we can select $\varphi \in F^*$ such that $|\varphi(P_k x_n)| > 1$ and $|\varphi(z)| \leqq 1$ for every $z \in \varepsilon B_F$. Defining $g(\lambda) := \varphi(f(\lambda x_n))$, we obtain a function $g \in \mathscr{H}(\mathbf{C})$ for which we have

$$1 < |\varphi(P_k x_n)| = \left| \frac{g^{(k)}(0)}{k!} \right|.$$

However, the Cauchy inequalities yield

$$\left| \frac{g^{(k)}(0)}{k!} \right| \leqq \sup_{|\lambda|=1} |g(\lambda)| = \sup_{|\lambda|=1} |\varphi(f(\lambda x_n))| \leqq 1,$$

since $f(\lambda x_n) \in \varepsilon B_F$, and this contradiction finishes the proof. □

A similar result is true for $\mathscr{H}_{wco}(E, F)$ (see [14, Theorem 3.3]) and the space $\mathscr{H}_{cc}(E, F)$ of all $f \in \mathscr{H}(E, F)$ which *locally take weak Cauchy sequences into convergent sequences*, i.e., every $y \in E$ has a neighbourhood U_y so that whenever $(x_i) \subset U_y$ is a weak Cauchy sequence, then $f(x_i)$ is norm convergent [13]. Some properties of $\mathscr{H}_{cc}(E, F)$ may be seen in [7, §4].

Now we give a characterization of the reflexivity of $\mathscr{H}_b(E)$.

Theorem 14. *For a space E, the following assertions are equivalent:*

(a) For every F, $\mathscr{H}_{uc}(E, F) \subseteqq \mathscr{H}_{wco}(E, F)$.

(b) For every F and $k \in \mathbf{N}$, $\mathscr{P}_{uc}({}^kE, F) \subseteqq \mathscr{P}_{wco}({}^kE, F)$.

(c) For every $k \in \mathbf{N}$, $\mathscr{P}({}^kE)$ *is reflexive.*

(d) $\mathscr{H}_b(E)$ *is reflexive.*

Proof. $(a) \Leftrightarrow (b)$ follows from Theorem 13 for $\mathscr{H}_{uc}(E, F)$, and the analogous result for $\mathscr{H}_{wco}(E, F)$ [14, Theorem 3.2].

$(b) \Leftrightarrow (c)$ is Theorem 4.

For $(c) \Leftrightarrow (d)$, see [2, Corollary 3].

From Theorems 5 and 13, we get.

Theorem 15. *Suppose $\mathscr{L}(E, F) = \mathscr{U}\mathscr{C}(E, F)$. Then every $f \in \mathscr{H}(E, F)$ which vanishes at the origin belongs to $\mathscr{H}_{uc}(E, F)$.*

Theorem 6 yields.

Theorem 16. *Suppose E has the* DPP *and $\mathscr{L}(E, F) = \mathscr{W}\mathscr{C}o(E, F)$. Then, $\mathscr{H}(E, F) = \mathscr{H}_{cc}(E, F)$.*

Finally, we have.

Theorem 17. *Suppose E^* has the* DPP, *and $\mathscr{L}(E, F) = \mathscr{W}\mathscr{C}o(E, F)$. Then each $f \in \mathscr{H}(E, F)$ has an extension to a mapping $\tilde{f} \in \mathscr{H}_{cc}(E^{**}, F)$, and $\varrho(\tilde{f}) = \varrho(f)$.*

Proof. Given the Taylor expansion at the origin $f = \sum P_k$, we obtain the extensions $\dot{P}_k$ of P_k, by Theorem 9, and define

$$\tilde{f}(z) := \sum_{k=0}^{\infty} \tilde{P}_k(z) \qquad (z \in E^{**}).$$

The Cauchy-Hadamard formula yields $\varrho(\tilde{f}) = \varrho(f)$. Since $\tilde{P}_k \in \mathscr{P}_{cc}({}^kE^{**}, F)$, we have that $\tilde{f} \in \mathscr{H}_{cc}(E^{**}, F)$.

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