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On the Limits of Sequences of Darboux a.e. Quasi-Continuous Functions

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It is proved that every cliquish real function of real variable is the limit of a sequence of Darboux a.e. quasi-continuous functions, where a.e. denotes an O'Malley's topology.

Let E denote the set of all reals and let T be a topology in E. A function $f: E \to E$ is said to be T-quasi-continuous (T-cliquish) at a point $x \in E$ if for every $\varepsilon > 0$ and for every T-open neighbourhood U of x there is a nonempty set $V \in T$ such that $V \subset U$ and $|f(t) - f(x)| < \varepsilon$ for every $t \in V$ ($|f(t) - f(u)| < \varepsilon$ for all $t, u \in V$) [6].

Let $A \subset E$ be a measurable set (in the Lebesgue sense). The lower (upper) density $d_l(A, x)$ ($d_u(A, x)$) of the set A at a point $x \in E$ is defined as $\liminf_{r \to 0} \mu(A \cap (x - r, x + r))/2r$ ($\limsup_{r \to 0} \mu(A \cap (x - r, x + r))/2r$), where μ denotes the Lebesgue measure in E. The family of all measurable sets $A \subset E$ such that if $x \in A$ then $d_l(A, x) = 1$ is a topology called the density topology T_d [2, 7]. The family $T_{a.e.} = \{A \in T_d; \mu(A - intA) = 0\}$ ($intA$ denotes the euclidean interior of A) is also a topology [7]. Moreover, denote by T_e the euclidean topology in E. In [3] it is proved that every T_e-cliquish function $f: E \to E$ is the limit of a sequence of Darboux T_e-quasi-continuous functions. In this article I show that every T_e-cliquish function $f: E \to E$ is the limit of a sequence of Darboux $T_{a.e.}$-quasi-continuous functions. This new result is stronger, since the family of all $T_{a.e.}$-quasi-continuous functions forms a newhere dense closed subset in the space of all T_e-quasi-continuous functions with the metric $\varrho(f, g) = \min(1, \sup_{x \in E} |f(x) - g(x)|)$.

If T is a topology in E then let $Q(T)$ ($P(T)$) denote the family of all T-quasi-continuous (T-cliquish) functions $f: E \to E$. Since $intA \neq \emptyset$ for every nonempty set $A \in T_{a.e.}$, we may observe that $P(T_e) = P(T_{a.e.})$ and $Q(T_{a.e.}) \subset Q(T_e)$.

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Remark 1. *$Q(T_{a.e.})$ is a nowhere dense closed subset of the space $Q(T_e)$ with the metric* $\varrho(f, g) = \min(1, \sup_{x \in E} |f(x) - g(x)|)$.

Proof. Since the convergence in ϱ is the uniform convergence and the family $Q(T_{a.e.})$ is uniformly closed [6], we obtain that $Q(T_{a.e.})$ is a closed subset of $Q(T_e)$. Fix $\varepsilon > 0$ $(\varepsilon < 1)$ and $f \in Q(T_{a.e.})$. There are a point $t \in E$ and intervals $I_n = [a_n, b_n]$ such that f is continuous at t and at every a_n, b_n, $n = 1, 2, \ldots$, $t = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, t is not in $[a_n, b_n]$ for $n = 1, 2, \ldots$ and $d_u(\bigcup I_n, t) = 0$. Then the function $g(x) = f(x) + \varepsilon/2$ for $x \in I_n$, $n = 1, 2, \ldots$, and for $x = t$, and $g(x) = f(x)$, otherwise in E, belongs to $Q(T_e) - Q(T_{a.e.})$ and $\varrho(f, g) = \varepsilon/2 < \varepsilon$. This completes the proof.

Let D denote the family of all Darboux functions.

Remark 2. *$DQ(T_{a.e.})$ is a nowhere dense closed subset of the space $DQ(T_e)$ with the metric ϱ.*

Proof. By Remark 1, the set $DQ(T_{a.e.})$ is closed in $DQ(T_e)$. Fix $f \in DQ(T_{a.e.})$ and $\varepsilon > 0$, $\varepsilon < 1$. There are a continuity point x of f and a sequence of closed intervals $I(n) = [a(n), b(n)]$, $n = 1, 2, \ldots$, such that $a(1) < b(1) < a(3) < b(3) \ldots < a(2n-1) < b(2n-1) < \ldots \to x$, $b(2) > a(2) > b(4) > a(4) > \ldots > b(2n) > a(2n) > \ldots \to x$, $osc_{I(n)} f < \varepsilon/8$, and $d_u(\bigcup I(n), x) = 0$. Let $g((a(n) + b(n)))/2) = f((a(n) + b(n))/2) + \varepsilon/2$, $n = 1, 2, \ldots$, let $g(t) = f(t)$ if $t \neq x$ is not in $(a(n), b(n))$, $n = 1, 2, \ldots$, let $g(x) = f(x) + \varepsilon/2$, and let g be linear in the intervals $[a(n), (a(n) + b(n))/2]$ and $[(a(n) + b(n))/2, b(n)]$, $n = 1, 2, \ldots$ Then $g \in DQ(T_e) - DQ(T_{a.e.})$ and $\varrho(f, g) < \varepsilon$. This completes the proof.

Lemma 1. *Let $A \subset E$ be a nowhere dense T_e-closed set and let $U \supset A$ be an T_e-open set. Let $g: E \to [-r, r]$, where $r \geq 0$, be a function. Then there is a Darboux function $f \in Q(T_{a.e.})$ such that $f(E) = [-r, r]$, $f(x) = g(x)$ for every $x \in A$, $f(x) = 0$ for every x which is not in U, f is continuous at each x which is not in A, and for every nondegenerate interval I such that $I \cap A \neq \emptyset$ we have $f(I - A) = [-r, r]$.*

Proof. For each $n = 1, 2, \ldots$ let Φ_n be the family of all intervals $[(k-1)/2^n, k/2^n]$, where $k = 0, 1, -1, 2, -2, \ldots$ Let $\Phi = \Phi_1 \cup \Phi_2 \cup \ldots$ We may assume, without a loss of generality, that A is compact.

Step 1st. For each $x \in A$ there is an interval $I(1, x) \in \Phi$ containing x in its interior $intI(1, x)$ and contained in U or there are two intervals $J(1, x)$, $K(1, x) \in \Phi$ such that x is the right endpoint of $J(1, x)$ and the left endpoint of $K(1, x)$ and $I(1, x) = J(1, x) \cup K(1, x) \subset U$. There are points $x(1, 1), \ldots, x(1, k(1)) \in A$ such that $A \subset intI(1, x(1, 1)) \cup \ldots \cup intI(1, x(1, k(1)))$. In every open interval $intI(1, x(1, i))$, $i = 1, \ldots, k(1)$, there are closed intervals

$$I(1, i, j), J(1, i, j) \subset intI(1, x(1, i)) - A - \bigcup_{r < i, j \leq j(1, r)} J_{1,r,j}\,, \; j = 1, \ldots, j(1, i)$$

such that:

- $I(1, i, j) \subset intJ(1, i, j)$, $j \leq j(1, i)$;
- $J(1, i, j(1)) \cap J(1, i, j(2)) = \emptyset$ for $j(1) \neq j(2), j(1), j(2) \leq j(1, i)$;
- $$\frac{\mu((I(1, x(1, i)) \cap (A \cup \bigcup_{r<i, j \leq j(1,r)} I(1, r, j))) \cup \bigcup_{j \leq j(1,i)} I(1, i, j))}{\mu(I(1, x(1, i))} > 1 - 8^{-1}.$$

Step nth. For each $x \in A$ there is an interval $I(n, x) \in \Phi$ containing x in its interior and contained in $V = U - \bigcup_{r<n, i \leq k(r), j \leq j(r,i)} J(r, i, j)$ or there are two intervals $J(n, x)$, $K(n, x) \in \Phi$ such that x is the right endpoint of $J(n, x)$ and the left endpoint of $K(n, x)$ and $I(n, x) = J(n, x) \cup K(n, x) \subset V$. There are points $x(n, 1), \ldots, x(n, k(n)) \in A$ such that $A \subset intI(n, x(n, 1)) \cup \ldots \cup intI(n, x(n, k(n)))$. In every open interval $intI(n, x(n, i))$, $i \leq k(n)$, we find closed intervals

$$I(n, i, j), J(n, i, j) \subset intI(n, x(n, i)) - A - \bigcup_{r<i, j \leq j(n,r)} J(n, r, j),$$

$(j = 1, \ldots, j(n, i))$, such that:

(1) $I(n, i, j) \subset intJ(n, i, j)$, $j \leq j(n, i)$;

(2) $J(n, i, j(1)) \cap J(n, i, j(2)) = \emptyset$ for $j(1) \neq j(2), j(1), j(2) \leq j(n, i)$;

(3) $$\frac{\mu((I(n, x(n, i)) \cap (A \cup \bigcup_{r<i, j \leq j(n,r)} I(n, r, j))) \cup \bigcup_{j \leq j(n,i)} I(n, i, j))}{\mu(I(n, x(n, i))} > 1 - 8^{-n}.$$

Moreover, in every component (a, b) of the set $U - A$ with $a \in A$ and $b \in A$ we find two sequences of closed intervals $L(1, n, a, b) = [a(1, n, a, b), b(1, n, a, b)]$,

$$L(2, n, a, b) = [a(2, n, a, b), b(2, n, a, b)] \subset (a, b) - \bigcup_{n=1,2,\ldots, i \leq k(n), j \leq j(n,i)} J(n, i, j)$$

such that:

- $$\frac{a+b}{2} > b(1, 1, a, b) > a(1, 1, a, b) > \ldots > b(1, n, a, b) > a(1, n, a, b) > \ldots \to a,$$

and

- $$\frac{a+b}{2} < a(2, 1, a, b) < b(2, 1, a, b) < \ldots < a(2, n, a, b) < b(2, n, a, b) < \ldots \to b,$$

If a component (a, b) of the set $U - A$ is such that a or b is not in A then we find only one corresponding sequence. For each component (a, b) of the set $U - A$ there is a continuous function $g_{(a,b)}: (a, b) \to [-r, r]$ such that $g_{(a,b)}(L(i, n, a, b)) = [-r, r]$, $i = 1, 2$ and $n = 1, 2, \ldots$ and $g_{(a,b)}(x) = 0$ if x is not in any $L(i, n, a, b)$, $i = 1, 2$ and $n = 1, 2, \ldots$ Let $w(1), w(2), \ldots$ be an enumeration of all rationals of the interval $[-r, r]$ and let $(u(1), u(2), \ldots) = (w(1), w(1), w(2), \ldots, w(n), w(1), \ldots, w(n+1), \ldots)$. For $n \geqq 1$, $i \leqq k(n)$ and $j \leqq j(n, i)$, by (1), (2), there are continuous functions

$$f_{n,i,j}: J(n, i, j) \to [\min(0, u(n)), \max(0, u(n))]$$

such that $f_{n,i,j}(x) = u(n)$ for $x \in I(n, i, j)$ and $f_{n,i,j}(x) = 0$ on the boundary of $J(n, i, j)$. Let $f(x) = g(x)$ for $x \in A$, $f(x) = f_{n,i,j}(x)$ for $x \in J(n, i, j)$, $n \geqq 1$, $i \leqq k(n)$, $j \leqq j(n, i)$, $f(x) = g_{(a,b)}(x)$ if (a, b) is a component of the set $U - A$ and $x \in L(i, n, a, b)$, $i = 1, 2$, $n \geqq 1$, and $f(x) = 0$ otherwise on E. Obviously, f is continuous at each point $x \in E - A$. Fix $x \in A$, a set $W \in T_{a.e.}$ containing x and $\varepsilon > 0$. Let $w(n)$ be such that $|f(x) - w(n)| < \varepsilon$. From the construction of f, by (3), it follows that $d_u(A \cup f^{-1}(w(n)), x) = 1$. If $d_u(A, x) > 0$ then $intW \cap A \neq \emptyset$. From the construction of f it follows that $intW \cap intf^{-1}(w(n)) \neq \emptyset$. If $d_u(A, x) = 0$ then $d_u(f^{-1}(w(n)), x) = 1$ and consequently, $intW \cap intf^{-1}(w(n)) \neq \emptyset$. So, $f \in Q(T_{a.e.})$. Since f is continuous at each point $x \in E - A$ and for every nondegenerate interval I such that $A \cap I \neq \emptyset$ we have $f(I - A) = [-r, r]$, the function f has the Darboux property. Evidently, $f(x) = g(x)$ for each $x \in A$ and $f(x) = 0$ for each $x \in E - U$. This completes the proof.

Lemma 2. *Let $A \subset E$ be a nowhere dense T_e-closed set and let $U \supset A$ be an T_e-open set. Let $g: E \to E$ be a function. Then there is a Darboux function $f \in Q(T_{a.e.})$ such that $f(x) = g(x)$ for each $x \in A$, $f(x) = 0$ for each $x \in E - U$, f is continuous at each point $x \in E - A$, and for each nondegenerate interval I such that $I \cap A \neq \emptyset$ we have $f(I - A) = E$.*

Proof. The proof is analogous as the proof of Lemma 1. It suffices only to take as $(w(n))$ a sequence of all rationals and to assume that $g_{a,b)}(L(i, n, a, b)) \supset [-n, n]$.

Theorem 1. *Let $f \in P(T_e)$ be a function. There is a sequence of functions $f_n \in DQ(T_{a.e.})$, $n = 1, 2, \ldots$, which pointwise converges to f.*

Proof. We may suppose that the set of discontinuity points of f is nonempty. Since the set of all continuity points of f is dense, there is a Baire 1 function $g: E \to E$ such that the set $\{x \in E; f(x) \neq g(x)\}$ is of the first category [5], p. 341. Let $h = f - g$. Then $h \in P(T_e)$ and $h(x) = 0$ at each point x at which it is continuous. Let $A_n = cl(\{x \in E; |h(x)| \geqq 1/n\})$, $n = 1, 2, \ldots$, and cl denotes the closure operation in the topology T_e. Every set A_n, $n \geqq 1$, is T_e-closed and nowhere dense. Consequently, every set $A_{n+1} - A_n$, $n \geqq 1$, is the union of pairwise disjoint closed sets $B_{n,k}$ [8]. Let $F(2), F(3), \ldots$ be the sequence of all nonempty sets $B_{n,k}$ such that $F(n) \neq F(m)$ for $n \neq m$, $n, m = 2, 3, \ldots$ and let $F(1) = A_1$. For each $n > 1$ let $r(n) = 1/k$, where $k \geqq 1$ is such that $F(n) \subset A_{k+1} - A_k$. Since the sets $F(k)$,

$k \geqq 1$, are pairwise disjoint, for every $n \geqq 1$ there are pairwise disjoint T_e-open sets $U(n, 1), \ldots, U(n, n)$ such that $F(i) \subset U(n, i)$ for $i \leqq n$ and such that $\sup\{dist(x, F(i)) = \inf_{t \in F(i)} |x - t|; x \in U(n, i)\} < 1/(n + i)$. By Lemmata 1, and 2, there are Darboux functions $f_{n,1}: E \to E$ and $f_{n,i}: E \to [-r(i), r(i)]$, $i = 2, \ldots, n$, belonging to $Q(T_{a.e.})$ and such that for each $i \leqq n$ the reduced functions $f_{n,i}/F(i)$ are the same, $f_{n,i}(x) = 0$ for $x \in E - U(n, i)$, $f_{n,i}$ is continuous on $E - F(i)$, for each nondegenerate interval I such that $I \cap F(1) \neq \emptyset$, $f_{n,1}(I) = E$, and for each nondegenerate interval I such that $I \cap F(i) \neq \emptyset$, $i = 2, \ldots, n$, $f_{n,i}(I) = [-r(i), r(i)]$. Let $h_n(x) = f_{n,i}(x)$ if $x \in U(n, i)$, $i \leqq n$, and let $h_n(x) = 0$ otherwise. Since $h_n = f_{n,1} + \ldots + f_{n,n}$ and all functions $f_{n,i} \in Q(T_{a.e.})$, $i \leqq n$, are continuous at $x \in E - U(n, i)$, we have $h_n \in Q(T_{a.e.})$ [4]. Evidently, h_n has the Darboux property. If $x \in F(k)$ for some $k \geqq 1$ then $h_n(x) = h(x)$ for $n > k$ and $\lim_{n \to \infty} h_n(x) = h(x)$. In the contrary case, if x is not in any $F(k)$, $k \geqq 1$, then $h(x) = 0$ and x is not in any A_k, $k \geqq 1$. Fix $\varepsilon > 0$. Let $m > 1$ be such that $1/m < \varepsilon$. Since x is not in A_m and A_m is T_e-closed, there is a positive number δ such that $[x - \delta, x + \delta] \cap A_m = \emptyset$. Let $k > m$ be such that $1/k < \delta$. Then, if $n > k$ and $F(n) \subset A_m$ then x is not in $U(i, n)$ for $i \geqq n$. Consequently, $|h_n(x)| < 1/m < \varepsilon$ for $n > k$ and $\lim_{n \to \infty} h_n(x) = h(x) = 0$. So, the sequence (h_n) pointwise converges to h. Since g is of Baire class 1, there is a sequence $(g_n)_n$ of continuous functions $g_n: E \to E$ which pointwise converges to g. Every function $f_n = g_n + h_n$, $n \geqq 1$, belongs to $Q(T_{a.e.})$ [4] and $\lim_{n \to \infty} f_n = \lim_{n \to \infty} g_n + \lim_{n \to \infty} h_n = g + h = f$. Fix $n \geqq 1$ and observe that f_n is continuous at each point $x \in E - \bigcup_{i \leqq n} F(i)$ and at each point $x \in \bigcup_{i \leqq n} F(i)$ the sets of all right-hand sided (left-hand sided) limit points of the function f_n and of the reduced function $f_n/(E - \bigcup_{i \leqq n} F(i))$ are the same. This means that every point $x \in \bigcup_{i \leqq n} F(i)$ is a Darboux point of f_n [1], and consequently f_n has the Darboux property. This finishes the proof.

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