Operator Ideals and the Principle of Local Reflexivity

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0.1. Introduction

Our aim is, to give necessary and sufficient conditions which allow us to transform the local reflexivity principle of Lindenstrauss and Rosenthal [Li–Rt] from the canonical operator norm $\|\cdot\|$ to p-Banach ideal norms $\|\cdot\|_{\mathscr{A}}$, where $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ is a given p-Banach ideal $(0 < p \leqq 1)$.

We will recognize two important facts:

- By a natural generalization of the *weak $\mathscr{A}$-local reflexivity principle* (introduced in [Oe1] and [Oe2]), we can omit the assumed maximality of the p-Banach ideal $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ in theorem 2.9. of [Oe2]. Moreover we are allowed to consider all $0 < p \leqq 1$ and not only the case $p = 1$.
- There are interesting relations between the above mentioned generalization of weak local reflexivity and structural properties of the ideal $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ such as *accessibility* (introduced in [D] and [D–F]). Hence, tensor norms are involved (cf. [Oe2]).

0.2. Notation and terminology

We shall use the common notations of Banach-space-theory; in particular B_E denotes the closed unit ball of a normed space E (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), E' the dual space of E and $\mathscr{L}(E, F)$ is the class of all (continuous) operators between the normed spaces E and F. Given $T \in \mathscr{L}(E, F)$, the dual operator of T is denoted by T'. *NORM*, *BAN* and *FIN* denotes the class of all normed spaces, Banach spaces and finite dimensional spaces respectively. *FIN*(E) is the class of all finite dimensional subspaces of a normed space E and *COFIN*(E) is the class of all finite codimensional subspaces of E. Concerning operator ideals we follow Pietsch's book ([P]). If $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ and $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ are both normed operator ideals, we sometimes use the abbreviation $\mathscr{A} = \mathscr{B}$ to indicate the equality $(\mathscr{A}, \|\cdot\|_{\mathscr{A}}) = (\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ and we write $\mathscr{A}^d$ instead of $\mathscr{A}^{dual}$. If $T: E \to F$ is an operator, we indicate that it is a metric injection ($\|Tx\| =$

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$= \|x\|$) by writing

$$T : E \overset{1}{\hookrightarrow} F$$

and that it is a metric surjection (F has the quotient norm of E via T) by

$$T : E \overset{1}{\twoheadrightarrow} F .$$

If there exists an isometric isomorphism between the spaces E and F, we write $E \cong F$. For $G \in FIN(E)$, $J_G^E : G \overset{1}{\hookrightarrow} E$ denotes the canonical metric injection and for $G \in COFIN(E)$, G closed, $Q_G^E : E \overset{1}{\twoheadrightarrow} E/G$ denotes the canonical metric surjection. We assume the reader to be familiar with the basics of the general theory of tensor norms as they are presented in [Gr], [D] and [D–F]. Another important tool to describe local properties of ideal components is given by the *trace* on a normed space E which is the linearization of the duality bracket

$$E' \times E \to \mathbb{K}$$
$$(a, x) \mapsto \langle x, a \rangle ,$$

whence

$$tr : E' \otimes E \to \mathbb{K}$$
$$\sum_{i=1}^{n} a_i \otimes x_i \mapsto \sum_{i=1}^{n} \langle x_i, a_i \rangle .$$

We recall that a Banach space E has the *metric approximation property* (short: m.a.p.) if for all compact sets $K \subseteq E$ and for all $\varepsilon > 0$ there is a finite dimensional operator $L \in \mathscr{F}(E, E)$ with $\|L\| \leqq 1$ such that $\|Lx - x\| \leqq \varepsilon$ for all $x \in K$.
Finally we remember the important

Principle of local reflexivity: Let M and F be Banach spaces, M finite dimensional and $T \in \mathscr{L}(M, F'')$. Then for every $\varepsilon > 0$ and $N \in FIN(F')$ there is an $R \in \mathscr{L}(M, F)$ such that

(i) $\|R\| \leqq (1 + \varepsilon) \|T\|$
(ii) $\langle Rx, b \rangle = \langle b, Tx \rangle \; \forall (x, b) \in M \times N$
(iii) $j_F Rx = Tx \; \forall x \in M$ with $Tx \in j_F(F)$.

1. The weak ($\mathscr{A}$)-local reflexivity principle

In the following, $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ always denotes a p-Banach ideal with $0 < p \leqq 1$ fixed. Recall, that the *adjoint ideal* $(\mathscr{A}^*, \|\cdot\|_{\mathscr{A}^*})$ is given by all operators $T \in \mathscr{L}(E, F)$ ($E, F \in BAN$) for which there exists a number $\varrho \geqq 0$ such that for all $E_0, F_0 \in BAN$ and for all $A \in \mathscr{F}(F, F_0)$, $S \in \mathscr{A}(F_0, E_0)$, $B \in \mathscr{F}(E_0, E)$

$$|tr(TBSA)| \leqq \varrho \cdot \|B\| \cdot \|S\|_{\mathscr{A}} \cdot \|A\| .$$

By definition, $\|T\|_{\mathscr{A}^*} := inf(\varrho)$ where the infimum is formed by all such $\varrho \geqq 0$ ([P]). According to [G-L-R] the *conjugate ideal* $(\mathscr{A}^{\Delta}, \|\cdot\|_{\mathscr{A}^{\Delta}})$ is given by all operators

$T \in \mathscr{L}(E, F)$ $(E, F \in BAN)$ for which there exists a number $\varrho \geqq 0$ such that for all $L \in \mathscr{F}(F, E)$

$$|tr(TL)| \leqq \varrho \cdot \|L\|_{\mathscr{A}} .$$

By definition, $\|T\|_{\mathscr{A}^\Delta} := inf(\varrho)$ where the infimum is formed by all such $\varrho \geqq 0$.

1.1. Definition: Let $\varepsilon > 0$, F be a Banach space, $M \in FIN$ and $N \in FIN(F')$. We are talking about the *weak* $(\mathscr{A})$*-local reflexivity principle* (short: (w)-$(\mathscr{A})$-$l.r.p.$) if for every $T \in \mathscr{L}(M, F'')$ there is an $S \in \mathscr{L}(M, F)$ such that

$$\langle b, Tx\rangle = \langle Sx, b\rangle \ \forall (x, b) \in M \times N$$

and

$$\|S\|_{\mathscr{A}} \leqq (1 + \varepsilon) \, \|T\|_{\mathscr{A}^{**}} .$$

Obviously the (w)-$(\mathscr{A})$-$l.r.p.$ always implies the (w) $\mathscr{A}$-$l.r.p.$, and if $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ is a maximal Banach ideal $(p = 1)$, then the (w) $\mathscr{A}$-$l.r.p.$ implies the (w)-$(\mathscr{A})$-$l.r.p.$ To prove our main theorem 1.5., we need the following

1.1. Lemma: Let $L \in \mathscr{F}(E, F)$, $A \in \mathscr{L}(N, E'')$ and $\varepsilon > 0$, where E, F are arbitrary Banach spaces and $\dim N < \infty$. Let $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ be a p-Banach ideal $(0 < p \leqq 1)$ such that the (w)-$(\mathscr{A})$-$l.r.p.$ holds. Then there is an operator $B \in \mathscr{L}(N, E)$ such that $\|\mathscr{B}\|_{\mathscr{A}} \leqq (1 + \varepsilon) \, \|A\|_{\mathscr{A}^{**}}$ and $L''A = L''j_E B = j_F LB$.

Proof: Since the range of L' is a finite dimensional subspace of E', there is an operator $B \in \mathscr{L}(N, E)$ such that $\langle L'b, Ay\rangle = \langle By, L'b\rangle$ for all $b \in F'$, $y \in N$ and $\|B\|_{\mathscr{A}} \leqq (1 + \varepsilon) \, \|A\|_{\mathscr{A}^{**}}$. Hence, for all $b \in F'$, $y \in N$ we have $\langle b, L''Ay\rangle =$ $= \langle LBy, b\rangle = \langle b, j_F LB y\rangle$. □

Easy to prove, but nevertheless of importance is the following

1.1. Lemma: Let $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ be a p-Banach ideal $(0 < p \leqq 1)$. Let $M \in FIN$ and $F \in BAN$. Then

$$\mathscr{A}^\Delta(F, M) \cong \mathscr{A}(M, F)'$$

where the isometric isomorphism is given by canonical trace duality. □

Remember, that $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ is called *left-accessible* if for all $(E, N) \in BAN \times FIN$, $T \in \mathscr{L}(E, N)$ and $\varepsilon > 0$ there are $L \in COFIN(E)$. $S \in \mathscr{L}(E/L, N)$ such that $T = SQ_L^E$ and $\|S\|_{\mathscr{A}} \leqq (1 + \varepsilon) \, \|T\|_{\mathscr{A}}$ ([D], [D–F]). By using tensor norm techniques (!) the following non-trivial result can be shown:

1.4. Proposition: Let $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ be a Banach ideal and $E, F \in BAN$ such that E' or F has the m.a.p. Then

$$\mathscr{B}^{\min}(E, F) \overset{1}{\hookrightarrow} (\mathscr{B}^{*\Delta})^{dd} (E, F) .$$

In particular $((\mathscr{A}^\Delta)^{dd}, \|\cdot\|_{(\mathscr{A}^\Delta)^{dd}})$ is left-accessible for each maximal Banach ideal $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$.

Prof: cf. [Oe1] and [Oe2]. □

Now we have all prepared to prove our main

1.5. Theorem: Let $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ be a p-Banach ideal $(0 < p \leqq 1)$. TFAE:

(1) $(\mathscr{A}^{\Delta}, \|\cdot\|_{\mathscr{A}^{\Delta}})$ is left-accessible

(2) $\mathscr{A}^{**}(M, F'') \cong \mathscr{A}(M, F)'' \ \forall (F, M) \in BAN \times FIN$

(3) The (w)-$(\mathscr{A})$-$l.r.p.$ holds.

Proof: $(1) \Rightarrow (2)$: Let $(\mathscr{A}^{\Delta}, \|\cdot\|_{\mathscr{A}^{\Delta}})$ be left-accessible and $(F, M) \in BAN \times FIN$. By [D−F, 25.2] it follows that $\mathscr{F} \circ \mathscr{A}^{\Delta} = (\mathscr{A}^*)^{\min}$ and so 1.3. implies that $\mathscr{A}(M, F)' \cong$ $\cong (\mathscr{A}^*)^{\min}(F, M)$. Hence dualization yields ([D−F, 22.6.]) $\mathscr{A}(M, F)'' \cong (A^{**})$. . (M, F'').

$(2) \Rightarrow (3)$: This implication follows by using Helly's lemma ([P]) and the canonical trace duality 1.3; namely, observe that by assumption (2)

$$\mathscr{A}(M, F)'' \xrightarrow{\cong} \mathscr{A}^{**}(M, F'')$$
$$\xi \longmapsto T_{\xi}$$

is an isometric isomorphism, where $\langle b, T_{\xi}x\rangle := \langle \operatorname{tr}((b \otimes x)\cdot), \xi\rangle$ $(x \in M, b \in F'')$. Let $\varepsilon > 0$, $N \in FIN(F')$ and $T \in \mathscr{L}(M, F'')$. Let $\{x_1, \ldots, x_n\}$ be a basis of M and $\{b_1, \ldots, b_m\}$ be a basis of $N \subseteq F'$. Let $L_{ij} := b_i \otimes x_j$. By 1.3., the linear span of $\{\operatorname{tr}(L_{ij}\cdot) : 1 \leqq i \leqq m, 1 \leqq j \leqq n\}$ is a finite dimensional subspace of $\mathscr{A}(M, F)'$. By assumption, there is a $\xi_0 \in \mathscr{A}(M, F)''$, such that $\langle b_i, Tx_j\rangle = \langle \operatorname{tr}(L_{ij}\cdot), \xi_0\rangle$ $\forall i \in$ $\in \{1, \ldots, m\}$ $\forall j \in \{1, \ldots, n\}$. By Helly, there exists an $S \in \mathscr{A}(M, F)$ with $\|S\|_{\mathscr{A}} \leqq$ $\leqq (1 + \varepsilon) \|\xi_0\| = (1 + \varepsilon) \|T\|_{\mathscr{A}^{**}}$ and with

$$\langle \operatorname{tr}(L_{ij}\cdot), \xi_0\rangle = \langle S, \operatorname{tr}(L_{ij}\cdot)\rangle = \operatorname{tr}(L_{ij}S) = \langle Sx_j, b_i\rangle$$

for all i and j. Hence − by linearity of T − the claim follows.

$(3) \Rightarrow (1)$: Let $\mathscr{B} := \mathscr{A}^{**}$. Since $((\mathscr{B}^{\Delta})^{dd}, \|\cdot\|_{(\mathscr{B}^{\Delta})^{dd}})$ is left-accessible (by 1.4.), it suffices to show that for all $(E, N) \in BAN \times FIN$ and for all $L \in \mathscr{L}(E, N)$ we have

$$(*) \qquad \|L''\|_{\mathscr{B}^{\Delta}} = \|L\|_{\mathscr{A}^{\Delta}}.$$

Obviously, $\|L\|_{\mathscr{A}^{\Delta}} \leqq \|L\|_{\mathscr{B}^{\Delta}} \leqq \|L''\|_{\mathscr{B}^{\Delta}}$. To prove the other inequality we use lemma 1.2.. Let $A \in \mathscr{F}(N'', E'')$ be given. By assumption we can choose an operator $B \in$ $\in \mathscr{L}(N'', E)$ as in lemma 1.2.. It follows that

$$|\operatorname{tr}(L''A)| = |\operatorname{tr}(j_F LB)| \leqq \|L\|_{\mathscr{A}^{\Delta}} \|B\|_{\mathscr{A}} \leqq (1 + \varepsilon) \|L\|_{\mathscr{A}^{\Delta}} \|A\|_{\mathscr{A}^{**}}.$$

Hence $\|L''\|_{\mathscr{B}^{\Delta}} \leqq \|L\|_{\mathscr{A}^{\Delta}}$ and $(*)$ is proven. □

Until now we do not know, if there exists a *maximal* Banach ideal $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ such that the (w)-$(\mathscr{A})$-$l.r.p.$ does not hold. In this case $(\mathscr{A}^{\Delta}, \|\cdot\|_{\mathscr{A}^{\Delta}})$ is not left-accessible, and especially $(\mathscr{A}^*, \|\cdot\|_{\mathscr{A}^*})$ would be another candidate for a non (left-) accessible maximal Banach ideal (since $(\mathscr{A}^{\Delta}, \|\cdot\|_{\mathscr{A}^{\Delta}}) \subseteq (\mathscr{A}^*, \|\cdot\|_{\mathscr{A}^*})$). Indeed, the hard problem of constructing such a candidate was open for a long time and had been recently solved by Pisier on the Oberwolfach meeting in September 1991, using a factorization over his own Pisier space P (cf. [D−F, 31.6.]). Therefore it seems also very non-trivial to construct a maximal Banach ideal $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$, for which the (w)-$(\mathscr{A})$-*l.r.p.* does not hold.

1.5. Remark: There exists a *minimal* Banach ideal $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ such that the (w)-$(\mathscr{A})$-*l.r.p.* does not hold.

Proof: Let $\mathscr{A} := \mathscr{A}_0^{\min}$, where $\mathscr{A}_0^*$ is Pisier's counterexample of a non left-accessible, maximal Banach ideal. Since in general $(\mathscr{B}^{\min}, \|\cdot\|_{\mathscr{B}^{\min}}) \subseteq ((\mathscr{B}^*)^{\Delta}, \|\cdot\|_{(\mathscr{B}^*)^{\Delta}})$, it follows for arbitrary $L \in \mathscr{F}(E, F)$ $(E, F \in BAN)$ that

$$\|L\|_{\mathscr{B}^*} \leqq \|L\|_{(\mathscr{B}^{\min})^{\Delta}} \leqq \|L\|_{(\mathscr{B}^*)^{\Delta\Delta}} \leqq \|L\|_{\mathscr{B}^*}.$$

Therefore the left-accessibility of $\mathscr{A}^{\Delta}$ would imply the left-accessibility of $\mathscr{A}_0^*$, which is a contradiction. □

1. The local reflexivity principle for operator ideals

Let $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ be a p-Banach ideal $(0 < p \leqq 1)$ such that the (w)-$(\mathscr{A})$-*l.r.p.* holds. Then it is possible to transfer the principle of local reflexivity in the following sense:

2.1. Theorem: Let $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ be a p-Banach ideal $(0 < p \leqq 1)$ such that the (w)-$(\mathscr{A})$-l.r.p. holds. Let $\varepsilon > 0$, $M \in FIN$, $F \in BAN$, $N \in FIN(F')$ and $S \in \mathscr{L}(M, F'')$. Then there exists an operator $R \in \mathscr{L}(M, F)$ such that

(i) $\|R\|_{\mathscr{A}} \leqq (1 + \varepsilon) \|S\|_{\mathscr{A}^{**}}$

(ii) $\langle Rx, b\rangle = \langle b, Sx\rangle \quad \forall (x, b) \in M \times N$

(iii) $j_F Rx = Sx \quad \forall x \in M \quad$ with $\quad Sx \in j_F(F)$.

Proof: Let $M_0 := \{x \in M : Sx \in j_F(F)\}$ and $J : M_0 \overset{1}{\hookrightarrow} M$ the canonical embedding. Let $S_0 : M_0 \to F$, $x \mapsto j_F^{-1}(Sx)$. Let $N \subseteq L \subseteq F'$ with $\dim L < \infty$ and $\varepsilon > 0$. By assumption there exists an $R_L \in \mathscr{F}(M, F)$ such that $\|R_L\|_{\mathscr{A}} \leqq (1 + \varepsilon) \|S\|_{\mathscr{A}^{**}}$ and $\langle R_L x, b\rangle = \langle b, Sx\rangle$ for all $(x, b) \in M \times L$. Hence

$$(*) \qquad \langle R_L Jx, b\rangle = \langle b, j_F S_0 x\rangle = \langle S_0 x, b\rangle \; \forall (x, b) \in M_0 \times L.$$

Let $\Phi := \{L \in FIN(F') : N \subseteq L\}$. By canonical set inclusion, Φ is a partially ordered set. Let $A = \sum_{i=1}^{n} b_i \otimes x_i \in \mathscr{L}(F, M_0)$ be arbitrary given $(b_1, \dots, b_n \in F'$ and $x_1, \dots, x_n \in M_0)$. Choose $L_0 \in \Phi$ such that $b_1, \dots, b_n \in L_0$. Hence, by $(*)$ we obtain for all $L \in \Phi$ with $L \supseteq L_0$:

$$\operatorname{tr}(R_L JA) = \sum_{i=1}^{n} \langle R_L Jx_i, b_i\rangle = \sum_{i=1}^{n} \langle S_0 x_i, b_i\rangle = \operatorname{tr}(S_0 A).$$

By the canonical trace duality 1.3., it follows that S_0 is the $\sigma(\mathscr{A}(M_0, F), \mathscr{A}(M_0, F)')$-limit of the net $(R_L J)_{L \in \Phi}$. Now we consider the set C, consisting of all operators UJ where $U \in \mathscr{L}(M, F)$, $\|U\|_{\mathscr{A}} \leqq (1 + \varepsilon) \|S\|_{\mathscr{A}^{**}}$ and $\langle Ux, b\rangle = \langle b, Sx\rangle$ for all $(x, b) \in M \times N$. Since $R_L J \in C$ for all $L \in \Phi$, S_0 is an element of the $\sigma(\mathscr{A}(M_0, F),$

$\mathscr{A}(M_0, F)')$-closure of the convex set C, hence S_0 is an element of the $\|\cdot\|_{\mathscr{A}}$-closure of C. Therefore to each $\delta > 0$ there exists an $U_0 J \in C$ such that $\|S_0 - U_0 J\|_{\mathscr{A}} < \delta$. Let $Q : M \twoheadrightarrow M_0$ an arbitrary projection. Then $\|Q\| \leqq 1$ and evidently the statements (ii) and (iii) are valid for the operator $R := (S_0 - U_0 J)\, Q + U_0 \in \mathscr{L}(M, F)$. Since

$$\|R\|_{\mathscr{A}} \leqq \|S_0 - U_0 J\|_{\mathscr{A}} + \|U_0\|_{\mathscr{A}} < \delta + (1 + \varepsilon)\, \|S\|_{\mathscr{A}^{**}},$$

statement (i) follows, and the theorem is proven. □

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