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Josef Kolomý

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Resolvents and Selections of Accretive Mappings in Banach Spaces

J. KOLOMÝ*)

Czechoslovakia

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Introduction

The theory of monotone and accretive mappings, intensively studied in the last period, has fruitful applications in the theory of nonlinear partial, ordinary differential and integral equations.

This note deals with properties of asymptotic behavior of resolvents of maximal accretive multivalued mappings in smooth Banach spaces having X^* Fréchet smooth. Furthermore, we extend the result of Barbu [1] concerning the almost main selections for maximal accretive mappings in Fréchet smooth Banach spaces and, moreover, we derive the result on approximations of resolvents of accretive mappings, which is connected with the assertion of Brézis and Pazy [2]. Let us remark that other results concerning the asymptotic properties of resolvents of accretive mappings have been obtained for instance by Gobbo [8], Reich [12–14], Takahashi and Ueda [16]. For the basic properties of accretive mappings, we refer to Barbu [1], Browder [3], Ciorănescu [4] and Kato [10].

Definitions and notation

Let X be a real normed linear space, X^* its dual, $\langle , \rangle$ the pairing between X and X^*, $S_1(O)$ the unit sphere in X. By R, R_+, we denote the set of all real and non-negative numbers, respectively. We shall use the notions of Giles [7] for rotund (i.e. strictly convex) and uniformly rotund spaces, convex functions, Gâteaux and Fréchet differentials and derivatives. Recall that X is said to be: (i) smooth (Fréchet smooth), if the norm of X is Gâteaux (Fréchet) differentiable on $S_1(O)$; (ii) uniformly (uniformly Fréchet) smooth, if the norm of X is uniformly Gâteaux (uniformly Fréchet) differentiable on $S_1(O)$; (iii) an (H)-space, if for each $(u_n) \subset X$, $u_n \to u$

*) Matematický Ústav University Karlovy, Sokolovská 83, 186 00 Praha 8 - Karlín, Czechoslovakia

weakly, $u \in X$, $\|u_n\| \to \|u\|$, we have that $u_n \to u$ in the norm of X. Let $A: X \to 2^Y$ be a multivalued mapping (Y denotes a normed linear space, 2^Y the system of all subsets of Y), $D(A) = \{u \in X: A(u) \neq \emptyset\}$ its domain, $G(A) = \{(u, v) \in X \times Y: u \in D(A), v \in A(u)\}$ its graph in the space $X \times Y$. A duality mapping $J: X \to 2^{X^*}$ is defined by $J(u) = \{u^* \in X^*, \langle u^*, u\rangle = \|u\|^2, \|u^*\| = \|u\|\}$ for each $u \in X$. Recall that $J(u)$ is a nonempty convex weakly* compact subset of X^* for each $u \in X$ and that X is smooth (Fréchet smooth) if and only if J is singlevalued (continuous) on X (see [6]). A mapping $A: X \to 2^X$ is said to be: (i) accretive if $I + \lambda A$, where I is an identity mapping in X, is expansive for each $\lambda > 0$; i.e. if for each $u, v \in D(A)$ and each $x \in A(u)$, $y \in A(v)$, there is $\|(u - v) + \lambda(x - y)\| \geqq \|u - v\|$ for each $\lambda > 0$ (equivalently if for each $u, v \in D(A)$ and each $x \in A(u)$, $y \in A(v)$ there exists an element $x^* \in J(u - v)$ such that $\langle x - y, x^*\rangle \geqq 0$); (ii) maximal accretive, if A is accretive and if $(u, x) \in X \times X$ is a given element such that for each $v \in D(A)$ and $y \in A(v)$ there exists a point $x^* \in J(u - v)$ such that $\langle x - y, x^*\rangle \geqq 0$, then $u \in D(A)$ and $x \in A(u)$; (iii) m-accretive, if A is accretive and the range $R(I + \lambda A)$ of $I + \lambda A$ is equal to X for some $\lambda > 0$. Let I denote an identity mapping in X, $A: X \to 2^X$ an accretive mapping with $D(A) \subseteqq X$. Then the so-called resolvent $J_\lambda = (I + \lambda A)^{-1}$ of A exists for each $\lambda > 0$ and is singlevalued with the domain $D(J_\lambda) = R(I + \lambda A)$ and the range $R(J_\lambda) = D(A)$. The Yoshida approximations A_λ of A are defined by $A_\lambda = \lambda^{-1}(I - J_\lambda)$ for each $\lambda > 0$. For a subset $G \subset X$ we define (see [5])

$$|G| = \begin{cases} \inf\{\|v\|: v \in G\} & \text{if } G \neq \emptyset, \\ +\infty & \text{if } G = \emptyset. \end{cases}$$

Let $A: X \to 2^X$ be a mapping. We set $A^0u = \{v \in A(u): \|v\| = |A(u)|\}$ for each $u \in D(A)$ and $D(A^0) = \{u \in D(A): A^0u \neq \emptyset\}$. In general, A^0 is a multivalued mapping from $D(A^0)$ into 2^X. If $D(A^0) = D(A)$ and A^0u is a singleton for each $u \in D(A)$, then A^0 is a selection of A with the property that for each fixed $u \in D(A)$ we have that $\|A^0u\| \leqq \|x\|$ for each $x \in A(u)$. In this case, A^0 is called a canonical restriction of A (see [1], [5]). Let X be a normed linear space, $A: X \to 2^X$ a maximal accretive mapping with $D(A) \subseteqq X$. Recall that: (i) if X is smooth, then $A(u)$ is convex and closed for each $u \in D(A)$; (ii) if X is Fréchet smooth, then $G(A)$ is closed in the space $(X, \|\cdot\|) \times (X, \sigma(X, X^*))$, where $\sigma(X, X^*)$ denotes the weak topology on X.

Results

The basic properties of the operators J_λ and A_λ are contained in the following lemma, where the assertions e), f) are expressed in a more general form (compare [1], [4]).

Lemma 1. *Let X be a normed linear space, $A: X \to 2^X$ an accretive mapping with $D(A) \subseteqq X$. Then:*

(a) $\|J_\lambda x - J_\lambda y\| \leqq \|x - y\|$ *for each* $x, y \in D(A_\lambda)$;

(b) A_λ *is accretive on* $D(A_\lambda)$ *and* $\|A_\lambda x - A_\lambda y\| \leqq 2\lambda^{-1}\|x - y\|$ *for each* $x, y \in \in D(A_\lambda)$;

(c) $A_\lambda x \in AJ_\lambda x$ *and* $|AJ_\lambda x| \leqq \|A_\lambda x\|$ *for each* $x \in D(A_\lambda)$. *If* $x \in D(A) \cap D(A_\lambda)$, *then* $\|A_\lambda x\| \leqq |Ax|$;

(d) *if* $x \in D(A) \cap (\bigcap\{D(A_\lambda): \lambda > 0\})$, *then* $\lim_{\lambda \to 0_+} J_\lambda x = x$. *If* $x \in \bigcap\{D(A_\lambda): \lambda > 0\} \setminus \setminus \overline{D(A)}$, *then* $\|A_\lambda x\| \to +\infty$ *as* $\lambda \to +\infty$;

(e) *if X is smooth and* $x \in D(A) \cap (\bigcap\{D(A_\lambda): \lambda > 0\})$, *then the function* $\lambda \to \|A_\lambda x\|$ *is nonincreasing on* $[0, +\infty)$;

(f) *if X is reflexive and smooth and the graph* $G(A)$ *of A is closed in* $(X, \|\cdot\|) \times \times (X, \sigma(X, X^*))$, *then* $\lim_{\lambda \to 0_+} \|A_\lambda x\| = |Ax|$ *for each* $x \in D(A) \cap (\bigcap\{D(A_\lambda): \lambda > 0\})$.

In particular, the assertion f) is valid if X is a reflexive Fréchet smooth Banach space and A is maximal accretive.

Lemma 2. *Let X be a reflexive smooth and rotund Banach space,* $A: X \to 2^X$ *a maximal accretive mapping with* $D(A) \subseteqq X$. *Then there exists a unique canonical restriction* A^0 *of A.*

Proof. Since X is smooth and A is maximal accretive, $A(u)$ is a convex closed set for each $u \in D(A)$. Fix $u \in D(A)$ and set $d = \inf\{\|x\|: x \in A(u)\}$. Then there exists $(x_n) \subset A(u)$ such that $\|x_n\| \to d$. Then there exists a subsequence of (x_n), say (x_n), such that $x_n \to x_0$ weakly in X. Then $\|x_0\| \leqq \liminf_{n\to\infty} \|x_n\| = \lim \|x_n\| = d$. Since $A(u)$ is weakly closed, $x_0 \in A(u)$ and $\|x_0\| = d$. But the rotundity of X and the convexity of $A(u)$ imply that x_0 is the unique point of $A(u)$ with the minimum norm. Setting $x_0 = A^0 u$ (where $u \in D(A)$), we see that A^0 is the unique canonical restriction of A.

Theorem 1. *Let X be a smooth Banach space such that* X^* *is Fréchet smooth,* $A: X \to 2^X$ *a maximal accretive mapping with* $D(A) \subseteqq X$. *If* $\overline{R(A)}$ *is convex and* $\bigcap\{D(A_\lambda): \lambda > 0\} \supset D(A)$, *then the strong limit* $\lim_{\lambda\to+\infty} (1/\lambda) J_\lambda(u)$ *exists for each* $u \in D(A)$ *and* $\lim_{\lambda\to+\infty} (1/\lambda) J_\lambda(u) = -a^0$ *for each* $u \in D(A)$, *where* a^0 *is the unique element of* $\overline{R(A)}$ *with the minimum norm.*

Proof. Since X^* is Fréchet smooth, X is reflexive rotund and (H)-Banach space. By Lemma 2, A has the unique canonical restriction A^0. By Lemma 1, $A_\lambda = \lambda^{-1}(I - J_\lambda)$ is lipschitzian with constant $2\lambda^{-1}$ from $D(A_\lambda)$ into X for each $\lambda > 0$ and the function $(0, \infty) \ni \lambda \to \|A_\lambda(u)\|$ is nonincreasing for each fixed $u \in \bigcap\{D(A_\lambda): \lambda > 0\}$. Moreover, $\|A_\lambda(u)\| \leqq \|A^0(u)\|$ for each $u \in D(A)$. Hence there exists $\lim_{\lambda\to+\infty} \|A_\lambda u\| = p(u)$ for each $u \in D(A)$. We have that $\|A_\lambda(v)\| \leqq \|A_\lambda(v) - A_\lambda(u)\| + \|A_\lambda(u)\| \leqq \leqq 2\lambda^{-1}\|u - v\| + \|A_\lambda(u)\|$. Therefore $p(v) = \lim_{\lambda\to+\infty} \|A_\lambda(v)\| \leqq \lim_{\lambda\to+\infty} \|A_\lambda(u)\| = p(u)$

for each $u, v \in D(A)$. This implies that $p(u) = c = \text{const}$ for each $u \in D(A)$. Since X is reflexive and rotund and $\overline{R(A)}$ is convex, there exists the unique point a^0 in $\overline{R(A)}$ with the minimum norm. If $u \in D(A)$ is an arbitrary point, then $J_\lambda(u) \in D(A)$ and $A_\lambda(u) \in A\, J_\lambda(u) \subset R(A)$ for each $\lambda > 0$. Hence $\|a^0\| \leqq A\|_\lambda(u)\|$ for each $u \in D(A)$ and $\|a^0\| \leqq c$. On the other hand $\|A_\lambda(u)\| \leqq \|A^0 u\| = \min\{\|v\|: v \in A(u)\}$ for each $u \in D(A)$. Hence $c \leqq \|A^0(u)\|$ for each $u \in D(A)$. By definition $R(A) = \bigcup\{A(u): u \in D(A)\}$ and therefore $c \leqq \|v\|$ for each $v \in R(A)$. If $v_0 \in \overline{R(A)}$, there exists a sequence $(v_n) \subset R(A)$ such that $v_n \to v_0$ and therefore $\|v_n\| \geqq c$ for each n. Then $\|v_0\| \geqq c$ and $c \leqq \|z\|$ for each $z \in \overline{R(A)}$. In particular, $c \leqq \|a^0\|$ and hence $c = \|a^0\|$. Let (λ_n) be a sequence of positive numbers such that $\lambda_n \to +\infty$, $u \in D(A)$ and set $x_n = A_{\lambda_n} u$. Then $\lim_{n \to \infty} \|x_n\| = \|a^0\|$ and there exists a subsequence of (x_n), say (x_{n_k}), and a point $v_0 \in X$ such that $x_{n_k} \to v_0$ weakly in X. Then $\|v_0\| \leqq \liminf_{k \to \infty} \|x_{n_k}\| = \lim_{n \to \infty} \|x_n\| = \|a^0\|$. Moreover, $x_{n_k} \in A\, J_{\lambda_{nk}}(u) = A(u_{n_k}) \in R(A)$, where $u_{n_k} = J_{\lambda_{nk}}(u) \in D(A)$. Therefore $v_0 \in \overline{R(A)}$ and $\|a^0\| \leqq \|v_0\|$ and hence $\|a^0\| = \|v_0\|$. Since X is rotund and $\overline{R(A)}$ convex, $a^0 = v_0$. Hence $x_{n_k} \to a^0$ weakly in X, which together with the reflexivity of X implies that the whole sequence (x_n) converges weakly to a^0. As $\|x_n\| \to \|a^0\|$ and X is an (H)-space, we conclude that $x_n \to a^0$ in the norm of X. Hence

$$x_n = A_{\lambda_n} u = \lambda_n^{-1} u - \lambda_n^{-1} J_{\lambda_n}(u) \to a^0 \quad \text{for each} \quad u \in D(A),$$

which gives our assertion.

If X is a real Hilbert space, $A: X \to 2^X$ a maximal monotone operator with $D(A) \subseteqq X$, then $\overline{R(A)}$ is convex and $D(A_\lambda) = R(I + \lambda A) = X$ for each $\lambda > 0$ and the conclusion of Thm. 1 is valid (compare Morosanu [11]). The asymptotic properties of resolvents of accretive operators were intensively studied for instance by Gobbo [8], Reich [12–13], Takahashi and Ueda [16]. By a quite different proof method, S. Reich [14] proved another generalization of the Morosanu result. His result is as follows. *Let X be a uniformly Gâteaux smooth Banach space such that X^* is Fréchet smooth, $A: X \to 2^X$ an accretive mapping such that $R(I + \lambda A) \supset \overline{D(A)}$ for each $\lambda > 0$ and that $\overline{D(A)}$ is nonexpanasive retract of X. Then for each x in $\overline{D(A)}$ the strong $\lim_{\lambda \to +\infty} \lambda^{-1} J_\lambda x = -v$, where v is the element of the least norm in $\overline{R(A)}$.* Recall that if X is a reflexive Banach space, then X^* is Fréchet smooth if and only if X is a rotund (H)-space. Let $X = L_\Phi(G)$ be an Orlicz space provided by the Orlicz norm, where $G \subset R^n$, mes $G < +\infty$ and R^n denotes an n-dimensional Euclidean space. If N-function Φ is strictly convex on $[0, \infty]$ and both Φ and its dual function Φ^* satisfy the Δ_2-condition for large arguments, then X satisfies the assumptions of Thm. 1 (see [15]). A similar conclusion is valid if X is provided by the Luxemburg norm (compare Hudzik [9]).

The following lemma and theorem give an extension of the Barbu result [1] where we do not assume that X, X^* are both uniformly rotund and A is m-accretive. Moreover, our proofs are rather different.

Lemma 3. *Let X be a Banach space such that X and X^* are Fréchet smooth $A: X \to 2^X$ a maximal accretive mapping with $D(A) \subseteqq X$. Then $\lim_{\lambda \to 0+} A_\lambda u = A^0 u$ in the norm of X for each fixed $u \in D(A) \cap (\bigcap\{D(A_\lambda): \lambda > 0\})$.*

Proof. By Lemma 2 there exists the unique canonical restriction A^0 of A. Fix $u \in D(A) \cap (\bigcap\{D(A_\lambda): \lambda > 0\})$, then for $\mu > 0$ we have $\|A_\mu(u)\| \leqq |A(u)| = \|A^0(u)\|$. Let $\lambda_n > 0$, $\lambda_n \downarrow 0$ as $n \to \infty$. Since $(A_{\lambda_n}(u))$ is a bounded sequence, without loss of generality, one can assume that $A_{\lambda_n}(u) \to x$ weakly in X for some $x \in X$. We have that $(J_{\lambda_n}(u), A_{\lambda_n}(u)) \in G(A)$ and assume that (v, y) is an arbitrary point in $G(A)$. Then $\langle y - A_{\lambda_n}(u), J(v - J_{\lambda_n}(u))\rangle \geqq 0$ for each n. Since $J_{\lambda_n} u \to u$ in the norm of X as $n \to \infty$ (Lemma 2), we get that $\langle y - x, J(v - u)\rangle \geqq 0$ for each $(v, y) \in G(A)$. As A is maximal accretive, we conclude that $x \in A(u)$ and $\|x\| \geqq \|A^0(u)\|$. On the other hand $\|x\| \leqq \liminf_{n\to\infty} \|A_{\lambda_n}(u)\| \leqq \|A^0(u)\|$ by Lemma 2. Therefore $\|x\| = \|A^0(u)\|$. Since X is rotund and $A(u)$ is convex, $x = A^0(u)$. Therefore the whole sequence $(A_{\lambda_n}(u))$ converges weakly to $A^0(u)$ and $\limsup_{n\to\infty} \|A_{\lambda_n}(u)\| \leqq \|A^0(u)\|$. Hence

$$\|A^0(u)\| \leqq \liminf_{n\to\infty} \|A_{\lambda_n}(u)\| \leqq \limsup_{n\to\infty} \|A_{\lambda_n}(u)\| \leqq \|A^0(u)\| .$$

Therefore $\|A_{\lambda_n}(u)\| \to \|A^0(u)\|$ as $n \to \infty$. Since X is an (H)-space, $A_{\lambda_n}(u) \to A^0(u)$ in the norm of X, which proves our lemma.

Lemma 4 ([5]). *Let X be a smooth normed linear space, K a nonempty closed convex subset in X. Then $x_0 \in K$ is the point of minimal norm in K if and only if $\|x_0\|^2 \leqq \langle v, J(x_0)\rangle$ for each $v \in K$.*

Let X be a real normed space, $A: X \to 2^X$ an accretive mapping with $D(A) \subseteqq X$. *Recall* ([2]) *that a singlevalued mapping $A': D(A) \to X$ is said to be a main selection of A if the following two conditions are satisfied: (i) A' is a selection of A; (ii) if $(u_0, x_0) \in \overline{D(A)} \times X$ is such that for each $u \in D(A)$ there exists $x^* \in J(u_0 - u)$ such that $\langle x_0 - A'(u), x^*\rangle \geqq 0$, then $(u_0, x_0) \in G(A)$.*

Theorem 2. *Let X be a Banach space such that X and X^* are Fréchet smooth, $A: X \to 2^X$ a maximal accretive mapping with $D(A) \subseteqq X$ and that $\bigcap\{D(A_\lambda): \lambda > 0\} \supset D(A)$. Let $\tilde{A}$ be a selection of A such that there exists a nondecreasing function $\omega: [0, \infty) \to R_+$ with the property that $\|\tilde{A}(u)\| \leqq \omega(\|A^0(u)\|)$ for each $u \in D(A)$. If $(u_0, x_0) \in D(A) \times X$ is such that $\langle x_0 - \tilde{A}(u), J(u_0 - u)\rangle \geqq 0$ for each $u \in D(A)$, then $x_0 \in A(u_0)$, i.e. $\tilde{A}$ is "almost the main selection" of A.*

Proof. By Lemma 2, A has the unique canonical restriction A^0. If $u \in D(A)$, then $u \in \bigcap\{D(A_\lambda): \lambda > 0\}$ and $A_\lambda u \to A^0 u$ in the norm of X, when $\lambda \to 0_+$ in view of Lemma 3. Since X is Fréchet smooth, the duality mapping $J: X \to X^*$ is continuous and hence $J(A_\lambda u) \to J(A^0 u)$ as $\lambda \to 0_+$. Without loss of generaltiy one can assume that $x_0 = 0$. (If $x_0 \neq 0$, we consider a mapping $A_1 = A - x_0$ instead of A which has the same properties as A.) Then $-\langle \tilde{A}(u), J(u_0 - u)\rangle \geqq 0$ for each $u \in D(A)$. Since $J_\lambda(u_0) \in D(A)$, we have that $\langle \tilde{A}(J_\lambda(u_0)), J(A_\lambda(u_0))\rangle \leqq 0$. As $A_\lambda u_0 \in AJ_\lambda u_0$ (Lemma 1) and $A^0 J_\lambda u_0$ is a point of the minimal norm of the set $A(J_\lambda(u_0))$, then $\|\tilde{A}(J_\lambda(u_0))\| \leqq \omega(\|A^0(J_\lambda(u_0))\|) \leqq \omega(\|A_\lambda(u_0)\|) \leqq \omega(\|A^0(u_0)\|)$ for each $\lambda > 0$. Let (λ_n) be a sequence of positive real numbers such that $\lambda_n \to 0$. Since $\tilde{A}\, J_{\lambda_n}(u_0)$ is a bounded sequence in X, without loss of generality one can assume that $\tilde{A}\, J_{\lambda_n}(u_0) \to \to y$ weakly in X. Put $u_n = J_{\lambda_n} u_0$, then $u_n \in D(A)$ and $u_n \to u_0$ in view of Lemma 1, because $u_0 \in D(A)$. Since the graph $G(A)$ of A is closed in $(X, \|\cdot\|) \times (X, \sigma(X, X^*))$ and $\tilde{A}(u_n) \in A(u_n)$ and $\tilde{A}(u_n) \to y$ weakly in X, we get $y \in A(u_0)$. Passing to the limit in the inequality $\langle \tilde{A}(u_n), J(A_{\lambda_n}(u_0))\rangle \leqq 0$, we get that $\langle y, J(A^0(u_0))\rangle \leqq 0$. Since $A(u_0)$ is a convex closed set and $A^0(u_0)$ is the unique point of $A(u_0)$ with the minimum norm, we have, in view of Lemma 4, that $\langle v, J(A^0(u_0))\rangle \geqq \|A^0(u_0)\|^2$ for each $v \in A(u_0)$. Now the last inequality and $y \in A(u_0)$ imply that $A^0(u_0) = 0$. As $A^0(u_0) \in \in A(u_0)$, then $0 \in A(u_0)$, which proves the theorem.

Theorem 3. *Let X be a reflexive (H)-Banach space such that the graph $G(J)$ of the duality mapping $J: X \to 2^{X^*}$ is sequentially closed in $(X, \sigma(X, X^*)) \times \times (X^*, \sigma(X^*, X))$. Let $A: X \to 2^X$, $A_\alpha: X \to 2^X$ be accretive mappings with $D(A) \subseteqq \subseteqq X$, $D(A_\alpha) \subseteqq X$ for each $\alpha > 0$, A' a main selection of A and let $D \subset X$. Assume that A_α satisfies the following two conditions: (i) $J_\lambda^\alpha = (I + \lambda A_\alpha)^{-1}: D \to \overline{D(A)}$ for each $\lambda > 0$ and $\alpha > 0$; (ii) for each $u \in D(A)$ there exists $y_\alpha \in A_\alpha u$ such that $y_\alpha \to \to A'(u)$ as $\alpha \downarrow 0$.*

Then $J_\lambda^\alpha x \to J_\lambda x$, $A_\lambda^\alpha x \to Ax$ for each fixed $x \in D$ and fixed $\lambda > 0$ as $\alpha \downarrow 0$, where $A_\lambda^\alpha = \lambda^{-1}(I - J_\lambda^\alpha)$.

Proof. For fixed $x \in D$ and $\lambda > 0$ we set $u_\alpha = (I + \lambda A_\alpha)^{-1}\, x$. Then $u_\alpha \in \overline{D(A)}$. Let $u \in D(A)$ be an arbitrary (but fixed) element, $y_\alpha \in A_\alpha u$ be such that $y_\alpha \to A'(u)$, when $\alpha \downarrow 0$. Since A_α are accretive and $(x - u_\alpha)\, \lambda^{-1} \in A_\alpha u_\alpha$, $y_\alpha \in A_\alpha u$, we have that for each $\alpha > 0$ there exists $x_\alpha^* \in J(u_\alpha - u)$ such that $\langle \lambda^{-1}(x - u_\alpha) - y_\alpha, x_\alpha^*\rangle \geqq 0$. Then $\langle \lambda y_\alpha - x + (u_\alpha - u) + u, x_\alpha^*\rangle \leqq 0$ for all $\alpha > 0$. Hence $\|u_\alpha - u\|^2 \leqq \langle x - u - \lambda y_\alpha, x_\alpha^*\rangle \leqq \|x - u - \lambda y_\alpha\| \,.\, . \|u_\alpha - u\|$. Thus $\|u_\alpha - u\| \leqq \|x - u - \lambda y_\alpha\|$ and $\|x_\alpha^*\| = \|u_\alpha - u\|$. Suppose that (α_n) is a sequence such that $\alpha_n > 0$ and $\alpha_n \to 0$ as $n \to \infty$. The sequences (u_{α_n}), $(x_{\alpha_n}^*)$ are bounded in X and X^*, respectively. Without loss of generality, one can assume that $u_{\alpha_n} \to u_0$ weakly in X and $x_{\alpha_n}^* \to x_0^*$ weakly in X^*. We show that $u_{\alpha_n} \to u_0$ in the norm of X. Since $\|u_0\| \leqq \liminf_{n\to\infty} \|u_{\alpha_n}\|$, it is sufficient to prove that $\limsup_{n\to\infty} \|u_{\alpha_n}\| \leqq \leqq \|u_0\|$. First of all we have that $\|u_{\alpha_n}\| \leqq \|u\| + \langle x - u - \lambda y_{\alpha_n}, x_{\alpha_n}^*\rangle^{1/2}$. Hence $\limsup_{n\to\infty} \|u_{\alpha_n}\| \leqq \|u\| + \limsup_{n\to\infty} \langle x - u - \lambda y_{\alpha_n}, x_{\alpha_n}^*\rangle^{1/2} = \|u\| + \lim_{n\to\infty} \langle x - u - \lambda y_{\alpha_n}, x_{\alpha_n}^*\rangle^{1/2} = \|u\| + \langle x - u - \lambda A'(u), x_0^*\rangle^{1/2}$. As $u_{\alpha_n} - u \to u_0 - u$ weakly in X, $x_{\alpha_n}^* \in J(u_{\alpha_n} - u)$ and $x_\alpha^* \to x_0^*$ weakly in X^* and $G(J)$ is sequentially closed in

$(X, \sigma(X, X^*)) \times (X^*, \sigma(X^*, X))$, we conclude that $x_0^* \in J(u_0 - u)$. Let us set in the last inequality $u = u_0$. Since $x_0^* \in J(0)$, $x_0^* = 0$ and $\limsup_{n\to\infty} \|u_{\alpha_n}\| \leqq \|u_0\|$. Therefore $\|u_{\alpha_n}\| \to \|u_0\|$ as $n \to \infty$. As $u_{\alpha_n} \to u_0$ weakly in X and X is an (H)-space, $u_{\alpha_n} \to u_0$ in the norm of X. Passing to the limit in the inequality $\langle \lambda^{-1}(x - u_{\alpha_n}) - y_{\alpha_n}, x_{\alpha_n}^* \rangle \geqq$ $\geqq 0$, we get that $\langle \lambda^{-1}(x - u_0) - A'(u), x_0^* \rangle \geqq 0$ for each $u \in D(A)$. Now $u_{\alpha_n} \in \overline{D(A)}$ $(n = 1, 2, \ldots)$ and $u_{\alpha_n} \to u_0$ imply that $u_0 \in \overline{D(A)}$. According to our hypothesis A' is a main selection of A. Hence $u_0 \in D(A)$ and $\lambda^{-1}(x - u_0) \in A(u_0)$, i.e. $u_0 =$ $= (I + \lambda A)^{-1}\, x = J_\lambda x$. Since the limit point u_0 is uniquely determined, the whole sequence (u_{α_n}) converges to u_0 and $u_\alpha \to u_0$ when $\alpha \downarrow 0$. Furthermore, $A_\lambda^\alpha x =$ $= \lambda^{-1}(x - J_\lambda^\alpha x) \to \lambda^{-1}(x - J_\lambda x) = A_\lambda x$ in the norm of X for each fixed $x \in D$ and $\lambda > 0$, when $\alpha \downarrow 0$, which proves the theorem.

If X is a Banach space such that X and X^* are uniformly rotund and $A: X \to 2^X$ is m-accretive, then $\overline{D(A)}$ is convex (see [1, chapt. V] and [4, chapt. II]). More generally, we get the following

Proposition 1. *Let X be a Banach space such that X^* is Fréchet smooth, $A: X \to 2^X$ an accretive mapping with $D(A) \subseteqq X$. If $\bigcap\{D(A_\lambda): \lambda > 0\} \supset \operatorname{conv} D(A)$, then $\overline{D(A)}$ is convex.*

The proof of this assertion relies on the following statement: Let X be a Banach space such that X^* is Fréchet smooth, $x, y \in X$, $x \neq y$ and $(z_n) \subset X$ are such that there exist the limits $\lim_{n\to\infty} \|x - z_n\|$ and $\lim_{n\to\infty} \|y - z_n\|$. If $\lim_{n\to\infty} \|x - z_n\| +$ $+ \lim_{n\to\infty} \|y - z_n\| = \|x - y\|$, then there exists $\lim_{n\in\omega} z_n = z$ in the norm of X and $z =$ $= \lambda_0 x + (l - \lambda_0)\, y$ for some $\lambda_0 \in [0, 1]$ (compare [4, chapt. II]). Furthermore, the proof is based on almost the same arguments as in [4].

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