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PETTIS INTEGRATION

Kazimierz Musiał

1. INTRODUCTION. Recently, Geitz [4] has proved a Lebesgue Dominated Convergence type theorem for the Pettis integral defined on a finite perfect measure space. His proof is based on theorems due to Fremlin [2] and James [5]. We show here that Geitz's theorem holds for arbitrary finite measure spaces. Our proof imitates his one, however, instead of Fremlin's theorem we use the following well known theorem of Mazur: If X is a normed space and $\{x_n : n \in N\}$ is weakly convergent in X to $x \in X$, then there exist finite sets $a_1^m, \ldots, a_{k(m)}^m$, $m = 1,2,\ldots$ of non-negative numbers such that $\sum_{j=1}^{k(m)} a_j^m = 1$ and $\lim_m \sum_{j=1}^{k(m)} a_j^m x_{j+m} = x$ in the norm topology of X.

The second problem we consider here is the problem of the approximation of a Pettis integrable function by a sequence of simple functions. In [7] it has been proved that if (S,Σ,μ) is a finite measure space, X is a Banach space, and $f\colon S \to X$ is Pettis integrable then f is approximated (in the Pettis norm) by a sequence of simple functions $\{f_n : n \in N\}$ if and only if the indefinite Pettis integral of f has norm relatively compact range. In particular we have for such a function the following scalar approximation: $x^*f_n \to x^*f$ in measure μ, for every functional $x^* \in X^*$.
In this paper we present necessary and sufficient conditions for a Pettis integrable function to be approximable by simple functions in the above scalar sense.

2. TERMINOLOGY. Throughout X stands for a Banach space (real or complex), $B(X)$ for its closed unit ball and X^* for the conjugate space. S denotes a non-empty set, Σ is a σ-algebra of subsets of S, and μ is a finite measure on Σ. $N(\mu)$ denotes the family of μ-*null sets*.

A function $f\colon S \to X$ is *weakly measurable* if the scalar function x^*f is measurable for each $x^* \in X^*$ (i.e. $(x^*f)^{-1}(\mathcal{B}_R) \subset \Sigma$).

The function f is *scalarly integrable* if $x^*f \in L_1(\mu)$ for each $x^* \in X^*$.

The function f is *Pettis integrable* on Σ (or on (S,Σ,μ)) if there exists a set function $\nu\colon \Sigma \to X$ such that

$$x^*\nu(E) = \int_E x^*fd\mu$$

for all $x^* \in X^*$ and $E \in \Sigma$. In that case we write

$$\nu(E) = \int_E fd\mu$$

and ν is called the *indefinite Pettis integral of* f on Σ (or on (S,Σ,μ)).

A function $f\colon S \to X$ is *weakly uniformly bounded* if there is a constant M such that $|x^*f| \le M\|x^*\|$ μ-a.e. (the exceptional set may vary with x^*).

A family H of scalar integrable functions is *uniformly integrable* if

$$\lim_{\mu(E)\to 0} \int_E |h|d\mu = 0$$

uniformly for $h \in H$.

(S,Σ,μ) is said to be *separable* if it is separable in the Frechet-Nikodym metric $(\rho(E,F) = \mu(E \Delta F))$.

If Σ_0 is a sub-σ-algebra of Σ, then $E(h|\Sigma_0)$ denotes the *conditional expectation* of h with respect to Σ_0.

If $F \subset P(S)$ then $\sigma(F)$ is the *σ-algebra generated* by F. B_R denotes the σ-algebra of Borel subsets of the real line R.

3. LIMIT THEOREMS. The theorem we are going to present now is a Pettis analogue of Vitali's convergence theorem. Conditions (a) and (b) of this theorem guarantee that for each $x^* \in X^*$ and $E \in \Sigma$ the sequence $\{\int_E x^*f_n d\mu\colon n \in N\}$ is convergent to $\int_E x^*fd\mu$, and that the set $\{x^*f\colon x^* \in B(X^*)\}$ is weakly compact in $L_1(\mu)$. The conditions (a) and (b) may be replaced by any others guaranteeing the above weak compactness and the convergence of the appropriate scalar integrals.

THEOREM 1. (VITALI CONVERGENCE THEOREM FOR PETTIS INTEGRAL). *Let $f\colon S \to X$ be a function. If there exists a sequence $\{f_n\colon n \in N\}$ of X-valued Pettis integrable functions on S such that:*

(*a*) *The set $\{x^*f_n\colon x^* \in B(X^*),\ n \in N\}$ is uniformly integrable,*

(*b*) $\lim_n x^*f_n = x^*f$ *in measure, for each $x^* \in X^*$,*

then f is Pettis integrable and $\lim_n \int_E f_n d\mu = \int_E fd\mu$ weakly in X, for each $E \in \Sigma$.

PROOF. Assume at the beginning that X is a real Banach space. Fix $E \in \Sigma$, and let C be the weak closure of the set

$\{\int_E f_n d\mu: n \in N\}$. Since Vitali's convergence theorem guarantess that $\lim_n \int_E x^* f_n d\mu = \int_E x^* f d\mu$ for each $x^* \in X^*$, we see that C is bounded and $C \setminus \{\int_E f_n d\mu: n \in N\}$ consists of at most one point. In order to prove our assertion it is sufficient to show that C is weakly compact, since this yields the existence of the weak limit of $\{\int_E f_n d\mu: n \in N\}$ in X. Clearly the limit can only be equal to $\int_E f d\mu$, and so we shall be able to conclude that f is Pettis integrable on E, and hence an the whole of Σ.

Suppose therefore that C is not weakly compact. Then, according to a theorem of James ([5], Th.1) there exist a bounded sequence $\{x_n^*: n \in N\}$, a sequence $\{x_n: n \in N\} \subset C$, and $\theta > 0$, such that $x_k^*(x_n) = 0$ for $k > n$ and $x_k^*(x_n) > \theta$ for $k \leq n$.

Consequently, we can find a subsequence $\{g_m: m \in N\}$ of $\{f_n: n \in N\}$ and a subsequence $\{y_m^*: m \in N\}$ of $\{x_n^*: n \in N\}$, such that

(i) $\quad \int_E y_k^* g_m d\mu = 0$ for $k > m$,

(ii) $\quad \int_E y_k^* g_m d\mu > \theta$ for $k \leq m$,

(iii) $\quad \lim_m \int_E x^* g_m d\mu = \int_E x^* f d\mu$, for all $x^* \in X^*$.

Consider now the set $\{y_m^* f: m \in N\}$. It easily follows from (a) that this set is uniformly integrable and bounded in $L_1(\mu)$. Hence, it is relatively weakly compact. This yields the existence of a function $h \in L_1(\mu)$ and a subsequence $\{z_j^*: j \in N\}$ of $\{y_m^*: m \in N\}$ such that $\lim_j z_j^* f = h$ weakly in $L_1(\mu)$. Applying (iii) for all z_j^* we get an inequality $\int_E z_j f d\mu \geq \theta$ and hence $\int_E h d\mu \geq \theta$.

Now we shall appeal to the theorem of Mazur. Let $a_1^m, \ldots, a_{k(m)}^m$, $m \in N$, be non-negative numbers, such that $\sum_j a_j^m = 1$ and $\lim_m (\sum_j a_j^m z_{j+m}^* f) = h$ in $L_1(\mu)$. Without loss of generality, we may assume, that the above convergence holds μ-a.e.. Clearly, if z_o^* is a weak* cluster point of the sequence $\{\sum_j a_j^m z_{j+m}^*: m \in N\}$, then $h = z_o^* f$ μ-a.e. In particular, we have

(iv) $\quad \int_E z_o^* f d\mu \geq \theta$.

On the other hand, since each g_n is Pettis integrable, the functional $x^* \to \int_E x^* g_n d\mu$ is weak* continuous. Hence, if $\{w_{n\alpha}^*\}$ is a subnet of $\{\sum_j a_j^m z_{j+m}^*: m > n\}$ that converges weak* to z_o^*, then, applying (i), we get

$$0 = \lim_\alpha \int_E w_{n\alpha}^* g_n d\mu = \lim_\alpha w_{n\alpha}^* \int_E g_n d\mu = z_o^* \int_E g_n d\mu = \int_E z_o^* g_n d\mu.$$

Since this holds for each $n \in N$, we see from (iii), that $\int_E z_o^* f d\mu = 0$. But this contradicts the inequality (iv).

It follows that C is weakly compact and so the real part of the theorem is proved.

Assume now that X is a complex Banach space, and denote by X_R^* the real conjugate of X. According to the real valued version already proved there exists a set function $\nu: \Sigma \to X$ such that for each $z^* \in X_R^*$ and $E \in \Sigma$ the equality

$$z^*\nu(E) = \int_E z^* f d\mu$$

holds. Consider $x^* \in X^*$, then, there is a (unique) $z^* \in X_R^*$ such that

$$x^*(x) = z^*(x) - iz^*(ix)$$

for all $x \in X$. Then, consider an operator $T: X \to X$ given by $T(x) = ix$. As T is R-linear and continuous we have

$$i\nu(E) = T\nu(E) = \int_E Tfd\mu = \int_E ifd\mu,$$

where the integrals are taken with respect to X_R^*.

It follows that

$$\begin{aligned} x^*\nu(E) &= z^*\nu(E) - iz^*[i\nu(E)] = \\ &= \int_E z^*fd\mu - i\int_E z^*(if)d\mu = \\ &= \int_E [z^*f - iz^*(if)]d\mu = \int_E x^*fd\mu \end{aligned}$$

Thus, the theorem is completely proved.

As a direct consequence of Theorem 1 we get the following generalization of the classical Lebesgue Dominated Convergence Theorem:

THEOREM 2 (LEBESGUE DOMINATED CONVERGENCE THEOREM FOR PETTIS INTEGRAL). *Let $f: S \to X$ be a function satisfying the following two conditions:*

(α) *There exists a sequence of Pettis integrable functions $f_n: S \to X$, $n \in N$, such that $\lim_n x^*f_n = x^*f$ in measure, for each $x^* \in X^*$,*

(β) *There exists a Pettis integrable function $g: S \to X$ such that $|x^*f_n| \le |x^*g|$ μ-a.e, for each $x^* \in X^*$ and $n \in N$ (the exceptional set depends on x^*).*

Then f is Pettis integrable and $\lim_n \int_E f_n d\mu = \int_E fd\mu$ weakly for all $E \in \Sigma$.

PROOF. If $g: S \to X$ is Pettis integrable then the family $\{x^*g: x^* \in B(X^*)\}$ is uniformly integrable and bounded in $L_1(\mu)$ (this is an easy consequence of the countable additivity and the μ-continuity of the indefinite Pettis integral of g (cf.[1], Theorem II.3.5)

It follows that the assumptions of Theorem 1 are satisfied.

REMARK 1. Replacing the function g in Theorem 2 by a function $h \in L_1(\mu)$ satisfying for each x^* and $n \in N$ a condition $|x^*f_n| \le |h|$ μ-a.e., we get the same conclusion concerning f. But such a form of Theorem 2 is essentially weaker from the orginal one. Namely, it follows from Musiał ([6], Proposition 1) that if g is Pettis integrable, then one can only find a measurable partition $\{E_n: n \in N\}$ of S and, a sequence of functions $\phi_n \in L_1(\mu)$, $n \in N$, such that for each n and x^* the inequality $|x^*g\chi_{E_n}| \le |\phi_n|$ μ-a.e., holds. If the variation of the indefinite integral of g is infinite then the functions ϕ_n, $n \in N$, cannot be replaced by a single function $\phi \in L_1(\mu)$.

4. SEQUENTIAL APPROXIMATION BY SIMPLE FUNCTIONS. It has been proved by Musiał ([7], Remark 1 and Corollary 1) that if ν is the indefinite Pettis integral of $f: S \to X$, then $\nu(\Sigma)$ is a norm relatively compact set if and only if f can be approximated by simple functions in the sense of Pettis norm, i.e. if there is a sequence $f_n: S \to X$, $n \in N$, of simple functions, such that

$$\lim_n \sup\{\int_S |x^*f_n - x^*f|d\mu: x^* \in B(X^*)\} = 0$$

In this section, we show, that if one does not order the uniform convergence on $B(X^*)$, then one gets a condition which is equivalent to the separability of $\nu(\Sigma)$.

THEOREM 3. *Let* $f: S \to X$ *be a Pettis integrable function on* (S,Σ,μ) *and, let* $\nu: \Sigma \to X$ *be its indefinite integral. Then, the following conditions are equivalent:*

(*i*) $\{x^*f: x^* \in B(X^*)\}$ *is a separable subset of* $L_1(\mu)$,

(*ii*) *There exists a* σ*-algebra* $\Sigma_0 \subset \Sigma$ *such that* $(S,\Sigma_0,\mu|\Sigma_0)$ *is separable and* f *is weakly measurable with respect to* Σ_0,

(*iii*) *There exists a sequence* $\{f_n: n \in N\}$ *of* X*-valued simple functions, such that for each* $x^* \in X^*$ *one of the following conditions is satisfied:*

- (*a*) $\{x^*f_n: n \in N\}$ *is uniformly integrable and* μ*-a.e. convergent to* x^*f,
- (*b*) $\{x^*f_n: n \in N\}$ *is uniformly integrable and convergent in* μ*-measure to* x^*f,
- (*c*) $\{x^*f_n: n \in N\}$ *is convergent to* x^*f *in* $L_1(\mu)$,
- (*d*) $\{x^*f_n: n \in N\}$ *is convergent to* x^*f *weakly in* $L_1(\mu)$,

(*iv*) $\nu(\Sigma)$ *is a separable subset of* X.

PROOF. ($i \to ii$) Assume that the set $\{x^*f\colon x^* \in B(X^*)\}$ is separable. Then, there exists a sequence $\{x_n^*\colon n \in N\}$ in $B(X^*)$, such that $\{x_n^*f\colon n \in N\}$ is dense in $\{x^*f\colon \|x^*\| \le 1\}$. If $\Sigma_0 = \sigma[\cup_{n=1}^{\infty}(x_n^*f)^{-1}(B_R) \cup N(\mu)]$ then clearly $\mu|\Sigma_0$ is separable.

Take an arbitrary $x^* \in B(X^*)$. Then, by the assumption, there exists a sequence $\{x_{n_k}^*\colon k \in N\}$, such that $x_{n_k}^*f \to x^*f$ in $L_1(S,\Sigma,\mu)$. It follows that there is a subsequence of $\{x_{n_k}^*f\colon k \in N\}$ converging to x^*f, on a set $S|N$ with $\mu(N) = 0$. But $N(\mu) \subset \Sigma_0$, and so $N \in \Sigma_0$. It follows that x^*f is Σ_0-measurable.

($ii \to iiia$) Assume that f is weakly measurable with respect to a separable $(S,\Sigma_0,\mu|\Sigma_0)$ and, let $\tilde{\Sigma} = \sigma(\{E_n\colon n \in N\}) \subset \Sigma_0$ be a countably generated σ-algebra which is $\mu|\Sigma_0$-dense in Σ_0. Moreover, let π_n be the partition of S generated by the sets $E_1,\dots,E_n$.

Put for each n

$$f_n = \Sigma_{E\in\pi_n} \frac{\nu(E)}{\mu(E)}\chi_E \qquad (0/0 = 0)$$

It is well known that $\{f_n,\sigma(\pi_n)\}_{n=1}^{\infty}$ is an X-valued martingale and $x^*f_n \to E(x^*f|\tilde{\Sigma})$ in $L_1(S,\tilde{\Sigma},\mu|\tilde{\Sigma})$ (cf. [8], Ex. IV. 3.2) and $\mu|\tilde{\Sigma}$-a.e. (cf. [1], V. 2.8). Moreover, the conditional expectation operator is a contraction on $L_1(\mu|\tilde{\Sigma})$ and so we have $\int |x^*f_n|d\mu \le$ $\le \int |x^*f|d\mu$ for all $n \in N$. This yields the uniform integrability of $\{x^*f_n\colon n \in N\}$. As by the assumption $\tilde{\Sigma}$ is dense in Σ_0, we have $E(x^*f|\tilde{\Sigma}) = x^*f$ μ-a.e., and so $x^*f_n \to x^*f$ $\mu|\Sigma_0$-a.e.

This completes the proof.

The implications ($a \to b \to c \to d$) are obvious, and so it remains to prove that ($iiid$) yields (iv).

($iiid \to v$) The condition ($iiid$) means exactly that for each $E \in \Sigma$ the sequence $\{\int_E f_n d\mu\colon n \in N\}$ is weakly convergent to $\int_E f d\mu$. Hence $\nu(\Sigma)$ is contained in the weak closure of the set $\cup_n \nu_n(\Sigma)$, where ν_n is the indefinite Pettis integral of f_n. As each set $\nu_n(\Sigma)$ is finite dimensional, the union is weakly separable. But according to the well known result of Mazur, the weak and norm separability in Banach spaces coincide.

($iv \to i$) Suppose that $\{x^*f\colon \|x^*\| \le 1\}$ is not separable. We shall prove that $\nu(\Sigma)$ is non-separable. To do it take an arbitrary $x_1^* \in X^*$ with $\|x_1^*\| = 1$ and $h_1 \in L_\infty(\mu)$, such that $\langle h_1, x_1^*f\rangle = 1$ ($\langle x^*,x\rangle$ denotes the value of x^* on x). Then, assume that we have already constructed for an ordinal $\beta < \omega_1$ a family $\{(x_\alpha^*,h_\alpha)\colon \alpha < \beta\}$ with the following properties:

(α) $x_\alpha^* \in X^*$ and $\|x_\alpha^*\| = 1$,

(β) $h_\alpha \in L_\infty(\mu)$,

(γ) $x_\gamma^*f \notin \overline{\mathrm{lim}}\{x_\alpha^*f\colon \alpha < \gamma\}$ for each $\gamma < \beta$,

$$(\delta)\quad \langle h_\gamma, x_\alpha^* f\rangle = \begin{cases} 1 & \text{if } \alpha = \gamma < \beta \\ 0 & \text{if } \alpha < \gamma < \beta \end{cases}$$

Since $\{x^*f\colon \|x^*\| \le 1\}$ is non-separable, we can find $x_\beta^* \in X^*$, such that $\|x_\beta^*\| = 1$ and $x_\beta^* f \notin \overline{\mathrm{lin}}\{x_\alpha^* f\colon \alpha < \beta\}$. Then, applying the Hahn-Banach theorem we get $h_\beta \in L_\infty(\mu)$ such that $\langle h_\beta, x_\beta^* f\rangle = 1$ and $\langle h_\beta, x_\alpha^* f\rangle = 0$ for all $\alpha < \beta$.

Consequently, we get a net $\{(x_\alpha^*, h_\alpha)\colon \alpha < \omega_1\}$ satisfying (α)-(β) for all α, β, γ less then ω_1.

Consider now an operator $T\colon X^* \to L_1(\mu)$ given by $Tx^* = x^*f$. It is well known (and easy to see) that T is continuous.

It is easy to see that for $\alpha < \beta$ we have

$$\|T^*h_\beta - T^*h_\alpha\| \ge 1$$

and so the set $T^*L_\infty(\mu)$ is non-separable in X^{**}.

But $\mathrm{lin}\{\chi_E\colon E \in \Sigma\}$ is norm dense in $L_\infty(\mu)$ and so $\mathrm{lin}\,\nu(\Sigma)$ is norm dense in $T^*L_\infty(\mu)$. It follows that $\nu(\Sigma)$ is non-separable.

This completes the proof of the whole theorem.

REMARK 2. The uniform integrability of the sets $\{x^*f_n : n \in N\}$ appearing in conditions (*iiia*) and (*iiib*) may be replaced by the uniform integrability of the set $\{x^*f_n\colon n \in N,\ x^* \in B(X^*)\}$. This follows easily from the proof of (*ii* $\to$ *iiia*) if one applies the uniform integrability of the set $\{x^*f\colon \|x^*\| \le 1\}$.

REMARK 3. Theorem 3 holds for arbitrary normed spaces. The proof needs no change.

Combining Theorem 1 with Theorem 3 and Remark 2 we get the following characterization of Pettis integrability in the case of separable measure spaces:

THEOREM 4. *Let* (S, Σ, μ) *be a measure space and let* $f\colon S \to X$ *be a function. Then,* f *is Pettis integrable on* Σ *and weakly measurable with respect to a separable measure space* $(S, \Sigma_0, \mu|\Sigma_0)$ *if and only if there exists a sequence* $\{f_n\colon n \in N\}$ *of* X*-valued simple functions on* S *such that:*

(*a*) *The family* $\{x^*f_n\colon n \in N,\ x^* \in B(X^*)\}$ *is uniformly integrable,*

(*b*) *For each* $x^* \in X^*$ $\lim_n x^*f_n = x^*f$ μ*-a.e.*

In the particular case of bounded functions we get the following result:

THEOREM 5. *Let* (S,Σ,μ) *be a measure space and let* $f: S \to X$ *be a weakly uniformly bounded function. Then,* f *is Pettis integrable on* Σ *and weakly measurable with respect to a separable measure space* $(S,\Sigma_0,\mu|\Sigma_0)$ *if and only if there exists a bounded sequence* $\{f_n: n \in N\}$ *of X-valued simple functions such that* $\lim_n x^*f_n = x^*f$ μ*-a.e., for all* $x^* \in X^*$ *(the exceptional sets depend on* x^**).*

REMARK 4. As it has been proved by Stegall, the range of an indefinite Pettis integral of a function defined on a perfect measure space is norm relatively compact. It follows from Theorem 3 that such a function is weakly measurable with respect to a separable measure space. Hence, Theorems 6 and 7 of Geitz [4] are particular cases of Theorem 4 and 5 respectively.

Let come back to Theorem 3. It is a natural question whether the separability condition (*ii*) can be replaced by the following stronger one:

There exists a countably generated σ-algebra $\tilde{\Sigma} \subset \Sigma$ such that f is weakly measurable with respect to the $\mu|\tilde{\Sigma}$-completion of $\tilde{\Sigma}$.

Unfortunately, the answer in negative (at least if one assumes the validity of Martin's Axiom). We begin with an easy consequence of Theorem 34 of Talagrand [9].

PROPOSITION 1 (MA). *Assume that* (S,Σ,μ) *is such that* Σ *is contained in a* $\mu|\tilde{\Sigma}$*-completion of a countably generated* σ*-algebra* $\tilde{\Sigma}$. *If* $f: S \to X$ *is Pettis integrable then the indefinite Pettis integral of* f *has norm relatively compact range.*

PROOF. Without loss of generality we may assume f to be weakly uniformly bounded. Let $H = \{x^*f: x^* \in B(X^*)\}$. As H is compact in the topology of pointwise convergence we can apply Theorem 34 of [9]. Thus, if $\{x_n^*: n \in N\}$ is a sequence in $B(X^*)$, then there is a subsequence $\{x_{n_k}^*: k \in N\}$ such that $\{x_{n_k}^*f: k \in N\}$ is μ-a.e. convergent. It follows that an operator $T: X^* \to L_1(\mu)$ given by $Tx^* = x^*f$ is compact. Hence T^* is compact as well, and this yields the relative compactness of the Pettis integral.

Now we are ready to prove the existence of a *Pettis integrable function* f *such that its Pettis integral is separable but* f *is not weakly measurable with respect to any* Σ_0 *which would be contained in a* $\mu|\tilde{\Sigma}$*-completion of a countably generated* $\tilde{\Sigma}$.

EXAMPLE (MA). Let W be an infinite set. Identifying PW with the set of characteristic functions $\{0,1\}^W$ we can introduce on PW the product topology and the Haar measure μ. It has been proved by Talagrand ([9], Theorem 10) that there exists an extension $\bar{\mu}$ of μ to a σ-algebra Σ such that all non-measurable filters on W are of $\bar{\mu}$-measure one.

Let $\lambda = \bar{\mu} \times \bar{\mu}$ be the direct product measure on $PW \times PW$ and let $f\colon PW \times PW \to l_\infty(W)$ be given by

$$f(a,b) = \chi_a - \chi_b ,$$

where χ_c is the characteristic function of a set $c \subset W$.

It is proved in ([3], 2D) that f is Pettis integrable on $\sigma(\Sigma \times \Sigma)$, the range of its indefinite Pettis integral is always non relatively compact, and for uncountable W it is even non-separable. Thus *if W is uncountable, then f cannot be approximated by any sequence of simple functions, in the sense considered in this paper.*

Assume now that W is countable, and denote by $\mathcal{B}$ the σ-algebra of Borel subsets of $PW \times PW$. Clearly $\mathcal{B}$ is countably generated.

Suppose that there is a countably generated $\tilde{\Sigma}$ such that f is weakly measurable with respect to a $\Sigma_0 \subset \sigma(\Sigma \times \Sigma)$ being the completion of $\tilde{\Sigma}$ with respect to $\lambda|\tilde{\Sigma}$. Without loss of generality, we may assume that $\mathcal{B} \subset \Sigma_0$. But then, it follows from the construction of Σ that Σ_0 is λ-dense in $\sigma(\Sigma \times \Sigma)$. In particular the indefinite Pettis integral of f on Σ_0 (which is relatively compact by Proposition 1) coincides with the indefinite Pettis integral on $\sigma(\Sigma \times \Sigma)$ (which is non relatively compact).

Thus, we have got a contradiction, which proves that the σ-algebra with respect to which f is weakly measurable cannot be too small.

REMARK 5. Let us also observe that the function f used in the above example in the case of countable W gives an answer for a long outstanding *question concerning the existence of conditional expectations of Pettis integrable functions.* Indeed, if there existed the conditional expectation $E(f|\mathcal{B})$ of f with respect to $\mathcal{B}$, then the equality

$$\int_E E(f|\mathcal{B})d\lambda = \int_E f d\lambda$$

would hold for arbitrary $E \in \mathcal{B}$. But $\mathcal{B}$ is λ-dense in $\sigma(\Sigma \times \Sigma)$ and so the equality would be true for all $E \in \sigma(\Sigma \times \Sigma)$. This clearly gives a contradiction, because according to the result of Stegall ([3], 3J) the set $\{\int_E E(f|\mathcal{B})d\lambda : E \in \mathcal{B}\}$ is norm relatively compact.

Observe yet, that according to Proposition 1, a similar result holds for arbitrary $\mathcal{C} \supset \mathcal{B}$ being a completion of a countably generated $\tilde{\mathcal{C}}$ with respect to $\lambda|\tilde{\mathcal{C}}$.

If one does not want to use the result of Stegall, then the non existence of $E(f|\Sigma \times PW)$ can be proved. Namely, an easy calculation shows that if $E(f|\Sigma \times PW)$ existed it would be equal to $\chi - (\frac{1}{2})$, where $(\frac{1}{2}) \in l_\infty(W)$ is the sequence with all coordinates equal to 1/2. But according to ([3], Theorem 2B), the function $a \to \chi_a$ is not Pettis integrable with respect to $\bar{\mu}$, and so $\chi - (1/2)$ is not Pettis integrable on Σ as well.

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