ON COMMUTATIVITY OF INTERPOLATION WITH INTERSECTION

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The purpose of this note is to present a partial answer to a problem of Peetre on commutativity of an interpolation method with intersection. We are interested, in particular, in the case of the real interpolation method.

First, we recall some notations from the interpolation theory used in [2] and [9].

A Banach couple $\bar{A} = \{A_0, A_1\}$ is a pair of Banach spaces A_0 and A_1 both continuously imbedded in some Hausdorff topological vector space (thus $A_0 + A_1$ is defined). F is an interpolation method if, for any Banach couple $\bar{A} = \{A_0, A_1\}$, $F(\bar{A})$ is a Banach space such that $A_0 \cap A_1 \subset F(\bar{A}) \subset A_0 + A_1$, and for any two Banach couples $\bar{A} = \{A_0, A_1\}$ and $\bar{B} = \{B_0, B_1\}$, every linear operator that maps A_0 boundedly into B_0 and A_1 into B_1 also maps $F(\bar{A})$ boundedly into $F(\bar{B})$.

There exist plenty of interpolation methods, but we will use the real interpolation method. For any Banach lattice of measurable functions Φ on $(\mathbb{R}_+, dt/t)$, $\mathbb{R}_+ = (0, \infty)$ containing $\min(1,t)$, the real interpolation method (or K_Φ method) $K_\Phi(\bar{A})$ is defined to consist of all $a \in A_0 + A_1$ such that $K(\cdot, a; \bar{A}) \in \Phi$ with the norm $\|a\|_{K_\Phi(\bar{A})} = \|K(\cdot, a; \bar{A})\|_\Phi$, where for $a \in A_0 + A_1$ and $t > 0$

$$K(t, a; \bar{A}) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1,\ a_0 \in A_0,\ a_1 \in A_1\}.$$

Observe that in particular if $\Phi = L^p_{t^{-\theta}}(\mathbb{R}_+, dt/t)$, $0 < \theta < 1$, $1 \le p \le \infty$, the space $K_\Phi(\bar{A})$ coincides with the familiar space $\bar{A}_{\theta,p}$ of Lions – Peetre.

Now, we should return to topic.

Let A_0, A_1 and A_2 be Banach spaces continuously imbedded in some Hausdorff topological vector space, and let F be an interpolation method. We consider the Peetre's question: when is it true that

$$(1) \qquad F(A_0,A_1\cap A_2) = F(A_0,A_1)\cap F(A_0,A_2)$$

up to equivalence of norm; it is obvious that we have inclusion $\subset$. There arises the question when in (1) inclusion $\subset$ can be replaces by equality.

This problem is not yet solved but there are some partial results. The purpose of this note is to present a partial answer to problem (1) by giving some general examples.

We note that if for $a\in(A_0+A_1)\cap(A_0+A_2)$ the following inequality

$$(2) \qquad K(t,a;A_0,A_1\cap A_2) \le C(K(t,a;A_0,A_1) + K(t,a;A_0,A_2))$$

holds then so does equality (1) for the real interpolation method $F = K_\Phi$.

1. Peetre in [7] proved that if $\{A_0,A_1\}$ is a ~~quasi-linearizable~~ couple, i.e., there exist linear operators $V_i(t): A_0+A_1 \to A_i$, $i=0,1$ (depending on $t>0$) such that

$$V_0(t)a+V_1(t)a = a \quad \text{and} \quad \|V_0(t)a\|_{A_0}+t\|V_1(t)a\|_{A_1} \le C_1 K(t,a;\bar{A})$$

$$\text{for} \quad a\in A_0+A_1,$$

and if moreover

$$\|V_1(t)a\|_{A_2} \le C_2\|a\|_{A_2} \qquad \text{for} \quad a\in A_2,$$

then inequality (2) holds.

The couples $\{C,C^1\}$, $\{L^p_{w_0}, L^p_{w_1}\}$, $\{L^p(\mathbb{R}^n), W^{k,p}(\mathbb{R}^n)\}$ are quasi-linearizable and the couple $\{L^{p_0}, L^{p_1}\}$, $p_0\ne p_1$ is not quasi-linearizable (see [6]).

2. It turns out that even for Hilbert spaces equality (1) need not hold, as it was shown in an example by Triebel [8].
Namely, we consider three spaces: $L^2 = L^2(0,1)$, Sobolev space $W^{1,2} = W^{1,2}(0,1)$ and weighted $L^2_w = L^2_w(0,1)$ with weight

$w(x) = \min(x,1-x)^{-1/2}$. Then for $\theta \in [1/2,1)$ we have

$$(L^2, W^{1,2} \cap L^2_w)_{\theta,2} = (L^2, W^{1,2}_o)_{\theta,2} = \begin{cases} W^{\theta,2}_o & \theta \neq 1/2 \\ W^{\frac{1}{2},2} \cap L^2_w, & \theta = 1/2 \end{cases}$$

and

$$(L^2, W^{1,2})_{\theta,2} \cap (L^2, L^2_w)_{\theta,2} = W^{\theta,2} \cap L^2_{w^\theta} = W^{\theta,2},$$

where $W^{\theta,2}_o$ denotes the closure of $C^\infty_o(0,1)$ in the space $W^{\theta,2}$. Hence equality (1) does not hold.

3. If $A_o = A_1 + A_2$ then inequality (2) holds with $C = 2$. Namely, if $0 < t < 1$ then from theorem 3 and 2 in [4] we have

$$2^{-1}K(t,a;A_1+A_2,A_1 \cap A_2) \le K(t,a;A_2,A_1) + K(t,a;A_1,A_2)$$

$$= K(t,a;A_1+A_2,A_1) + K(t,a;A_1+A_2,A_2)$$

and if $t \ge 1$ then obviously

$$2K(t,a;A_1+A_2,A_1 \cap A_2) = 2\|a\|_{A_1+A_2} = K(t,a;A_1+A_2,A_1) +$$

$$+ K(t,a;A_1+A_2,A_2)$$

Hence inequality (2) holds with $C = 2$.

4. J.Peetre posed in [7] the problem of equality (1) for $F = K_{\theta,p}$ if we replace arbitrary Banach spaces by symmetric spaces. We prove here that not only (1) but also inequality (2) is true even for Banach lattices of measurable functions.

Theorem 1 (see [5]). If A_o, A_1 and A_2 are Banach lattices on (Ω,Σ,μ) then inequality (2) holds with $C = 2$.

P r o o f. For each $\varepsilon > 0$ there exist decompositions $a = a_o + a_1 = a'_o + a_2$ such that

$$\|a_o\|_{A_o} + t\|a_1\|_{A_1} \le (1+\varepsilon)K(t,a;A_o,A_1) \quad \text{and}$$

$$\|a'_o\|_{A_o} + t\|a_2\|_{A_2} \le (1+\varepsilon)K(t,a;A_o,A_2).$$

Put $U = \{s \in \Omega : |a_1(s)| \le |a_2(s)| \ \mu\text{-a.e.}\}$ and define b_0, b_1 by

$$b_0(s) = \begin{cases} a_0(s), & s \in U \\ a_0'(s), & s \in \Omega \setminus U \end{cases}, \quad b_1(s) = \begin{cases} a_1(s) & s \in U \\ a_2(s), & s \in \Omega \setminus U. \end{cases}$$

Then $b_0 + b_1 = a$ and $|b_0| \le |a_0| + |a_0'|$, $|b_1| \le \min(|a_1|, |a_2|)$ μ-a.e. Hence

$$K(t,a;A_0,A_1 \cap A_2) \le \|b_0\|_{A_0} + t\|b_1\|_{A_1 \cap A_2}$$

$$\le \|a_0\|_{A_0} + \|a_0'\|_{A_0} + t \max(\|a_1\|_{A_1}, \|a_2\|_{A_2})$$

$$\le 2(1+\varepsilon)K(t,a;A_0,A_1) + 2(1+\varepsilon)K(t,a;A_0,A_2)$$

and the proof is finished.

5. The following result is an important application of the Theorem 1.

<u>Theorem 2</u>. If all spaces A_0, A_1 and A_2 can be obtained by the K-method from a fixed Banach couple $\bar{B} = \{B_0, B_1\}$, i.e., $A_i = K_{\Phi_i}(\bar{B})$, $i = 0,1,2$ then inequality (2) holds.

P r o o f. By the Brudnyǐ-Krugljak theorem ([3], Th.8.1) there exists a constant $\gamma = \gamma(\bar{B}) < 14$ such that

$$(3) \qquad K(t,a;K_{\Phi_0}(\bar{B}), K_{\Phi_1 \cap \Phi_2}(\bar{B})) \le \gamma K(t,K(\cdot,a;\bar{B});\tilde{\Phi}_0,\tilde{\Phi}_1 \cap \tilde{\Phi}_2),$$

where $\tilde{\Phi}_i = \{f : \tilde{f} \in \Phi_i\}$, $\|f\|_{\tilde{\Phi}_i} = \|\tilde{f}\|_{\Phi_i}$ and $\tilde{f} := \inf\{g : g \ge |f|$ a.e. and g concave$\}$.

The same argument as in the previous theorem shows that inequality

$$(4) \qquad K(t,b;\tilde{\Phi}_0,\tilde{\Phi}_1 \cap \tilde{\Phi}_2) \le 2(K(t,a;\tilde{\Phi}_0,\tilde{\Phi}_1) + K(t,a;\tilde{\Phi}_0,\tilde{\Phi}_2))$$

holds.

Since Φ_i are Banach lattices we have

$$K(t,K(\ ,a;\bar{B});\widetilde{\Phi}_o,\widetilde{\Phi}_i) = \inf\{\|\widetilde{x}_o\|_{\Phi_o} + t\|\widetilde{x}_i\|_{\Phi_i} : K(\cdot,a;\bar{B}) \leq x_o+x_i\}$$

$$\leq \inf\{\|K(\cdot,a_o;\bar{B})\|_{\Phi_o} + t\|K(\cdot,a_i;\bar{B})\|_{\Phi_i} : a = a_o+a_i\}$$

$$= K(t,a;K_{\Phi_o}(\bar{B}),K_{\Phi_i}(\bar{B})),\ i = 1,2.$$

Hence

$$K(t,a;A_o,A_1\cap A_2) = K(t,a;K_{\Phi_o}(\bar{B}),\ K_{\Phi_1\cap\Phi_2}(\bar{B}))$$

[by inequality (3)]

$$\leq \gamma K(t,K(\cdot,a;\bar{B});\widetilde{\Phi}_o,\widetilde{\Phi}_1\cap\widetilde{\Phi}_2)$$

[by inequality (4)]

$$\leq 2\gamma(K(t,K(\cdot,a;\bar{B});\widetilde{\Phi}_o,\widetilde{\Phi}_1) + K(t,K(\cdot,a;\bar{B});\widetilde{\Phi}_o,\widetilde{\Phi}_2)$$

[by the above inequalities]

$$\leq 2\gamma(K(t,a;K_{\Phi_o}(\bar{B}),K_{\Phi_1}(\bar{B})) + K(t,a;K_{\Phi_o}(\bar{B}),K_{\Phi_2}(\bar{B})))$$

$$= 2\gamma(K(t,a;A_o,A_1) + K(t,a;A_o,A_2))$$

and the inequality (2) holds.

Immediately from Theorem 2 follows that if $A_o = A_1+A_2$ or $A_o = A_1$, or $A_o = A_2$ then inequality (2) holds.

In the special case when Φ_i are weighted L^∞-spaces with some concave weights, Theorem 2 was proved by Asekritova [1] in her dissertation by a quite different approach.

The problem what is the necessary and sufficient condition for the validity of (1) is still open.

I am grateful to V.I.Ovchinnikov who made accessible for me paper [1] at the time when I was in Woronež in December 1984.

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