Dikran N. Dikranjan; Eraldo Giuli

Epimorphims and cowellpoweredness of epireflective subcategories of Top

In: Zdeněk Frolík (ed.): Proceedings of the 12th Winter School on Abstract Analysis, Section of Topology. Circolo Matematico di Palermo, Palermo, 1984. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 6. pp. 121--136.

Persistent URL: http://dml.cz/dmlcz/701833

EPIMORPHIMS AND COWELLPOWEREDNESS OF EPIREFLECTIVE SUBCATEGORIES OF TOP

D.Dikranjan and E.Giuli *

Abstract. A functor F_A: Top ⟶ Top induced by a given epireflective subcategory **A** of the category **Top** of topological spaces is used to characterize epimorphisms in some familiar epireflective subcategories of **Top** and to solve for these subcategories, the problem of the cowell -poweredness. Furthermore an ordinal number $EO_A(X)$, for each X∈Top, is introduced and it is computed in several examples.
As an application it is shown that there is no epireflective subcategory of **Top** which is properly contained in the subcategory Top_2 of all Hausdorff spaces and whose extremal epireflective hull is Top_2.

1. In 1975 Salbany ([14]) introduced a closure operation $[\,]_A : 2^X \to 2^X$ defined on subsets of a topological space X by a class **A** of topological spaces. In 1980 Giuli ([6]) used that closure operation to study epireflections in epireflective subcategories of **Top**.He pointed out that epimorphisms in an epireflective subcategory **A** of **Top** are precisely the continuous maps which are dense with respect to $[\,]_A$.
Recently Dikranjan and Giuli ([4]) characterized $[\,]_A$ for some familiar epireflective subcategories **A** of **Top**. They showed that, as in the classical case of Hausdorff spaces, the closure operation $[\,]_A$ characterizes the spaces X of **A** in terms of the **A**-closure of the diagonal Δ_X for A=Top_0, FT_2, $Top_{2\frac{1}{2}}$, P(0-dim), (see 1.1 below).

In this paper we will use the previous closure operation to define, for each epireflective subcategory **A** of **Top**, a functor F_A:Top ⟶ Top. Then some sufficient conditions for the cowellpoweredness of **A** are given and they are used to answer the question of the cowellpoweredness of some epireflective subcategories of **Top**. Furthermore an ordinal number $EO_A(X)$ (called epimorphic order of X with

* Talk given by the second named author. The paper is in its final form and will not be published anywhere else.

This paper is in final form and no version of it will be submitted for publication elsewhere.

respect to A is introduced for each $X \in Top$ and in several examples it is computed.Iterations of the functor F_A and the relation with the A-reflection functor are also studied.
We will use the previous closure operation in a forthcoming paper for a new approach to the study of A-minimal and A-closed spaces ([5]).

1.1. The following subcategories of Top are symbolized as follows

Top_i = the subcategory of topological spaces satisfying the T_i-axiom (i=0,1,2)

FT_2 = The subcategory of functionally Hausdorff spaces, i.e., spaces X such that for any two different points x_1, x_2 there exists a continuous map $f:X \to R$ with $f(x_1) \neq f(x_2)$.

Top_3 = The subcategory of regular Hausdorff spaces.

$P(Top_3)$ = The subcategory consisting of spaces whose topology is finer than a regular Hausdorff topology.

$Top_{2\frac{1}{2}}$ = The subcategory of Urysohn spaces, i.e., spaces such that for any two different points there exist disjoint closed nbds.

$Top_{3\frac{1}{2}}$ = The subcategory of completely regular Hausdorff spaces.

0-dim = The subcategory of 0-dimensional spaces, i.e., Hausdorff spaces with a base of clopen sets.

P(0-dim)= The subcategory of spaces whose topology is finer than a 0-dimensional topology, i.e., spaces in which every point is the intersection of the clopen sets containing it([12]).

We recall that a full and isomorphism-closed subcategory A of Top is said to be epireflective (respectively bireflective, extramally epireflective) in Top if for each topological space X there exist $r_A(X)$ belonging to A and an epimorphism (respectively bimorphism, extremal epimorphism) $r_A:X \to r_A(X)$ such that, for each $A \in A$ and continuous map $f:X \to A$ there exists a (unique) continuous map $f':r_A(X) \to A$ satisfying the condition $r_A \circ f' = f$. r_A is called the A-reflection of X.
A is epireflective in Top iff it is closed under the formation of products and subspaces (= extremal subobjects). It is extremally epireflective iff it is epireflective and contains finer topologies. It is bireflective iff it is epireflective and contains (all) indiscrete spaces.
Every class B of topological spaces admits an epireflective hull E(B) (i.e., a smallest epireflective subcategory containing A), an

extremal epireflective hull P(B) and a bireflective hull I(B).
All categories listed in 1.1. are epireflective in **Top**. $\mathbf{Top}_i$, for $i = 0,1,2,2\frac{1}{2}$, and FT_2 are extremally epireflective in **Top**. For all categories **A** listed in 1.1. the subcategory $\tilde{\mathbf{A}} = \{X \in \mathbf{Top} : r_0(X) \in \mathbf{A}\}$ (where r_0 is the $\mathbf{Top}_0$-reflection) is bireflective in **Top**.
$\mathbf{Top}_{3\frac{1}{2}}$, FT_2 and $\widetilde{\mathbf{Top}}_{3\frac{1}{2}}$(subcategory of completely regular spaces) are respectively the epireflective hull, the extremal epireflective hull and the bireflective hull of the real line $\mathbb{R}$ in **Top** .
For general results on epireflective subcategories of **Top** see [7,8].
The categorical terminology is that of [10] .

In what follows **A** will denote an epireflective subcategory of **Top**. For each pair of continuous maps $(f,g:X \longrightarrow Y)$, Eq(f,g) will denote the equalizer in **Top** of f and g (i.e., $Eq(f,g) = \{x \in X : f(x) = g(x)\}$).

1.2. - **Definitions.** (a) A subset F of a topological space X is said to be closed with respect to **A** (in short **A-closed**) in X if there exist $A \in \mathbf{A}$ and continuous maps $f,g : X \longrightarrow A$ such that $Eq(f,g) = F$.
b) We will define **A-closure** of a subset M of X as follows:

$$[M]^X_{\mathbf{A}} = \bigcap\{F \subset X : M \subset F \text{ and } F \text{ is } \mathbf{A}\text{-closed}\}$$

When no confusion is possible we write $[M]_{\mathbf{A}}$ or simply $[M]$ instead of $[M]^X_{\mathbf{A}}$.
c) If $x \notin M$ and $f,g:X \longrightarrow A$, $A \in \mathbf{A}$, are continuous maps such that $M \subset Eq(f,g)$ and $f(x) \neq g(x)$ then, (f,g) is said to be an **A-separating pair for (x,M).**

By definition $x \notin [M]_{\mathbf{A}}$ iff there exists an **A**-separating pair for (x,M). The family of all **A**-closed sets of a topological space X trivially contains X and, by the productivity of **A** , it is closed under the formation of intersections (i.e., it is a Moore family). Thus the **A**-closure is a closure operation in the sense of Birkhoff ([2]).
Furthermore $[\emptyset]_{\mathbf{A}} = \emptyset$ for all epireflective subcategories **A** different from the trivial subcategory **Sgl** consisting of topological spaces whose underliyng sets have at most one point.
Even if $[M]_{\mathbf{A}} \cup [N]_{\mathbf{A}} \subset [M \cup N]_{\mathbf{A}}$ for each $M,N \subset X$, the epireflective hull of an infinite strongly rigid space (the continuous self-maps are precisely the constant maps and the identity map ([10])) provides an example of a non-additive closure operation ([3,4]).

2. The following lemma is very useful in the sequel.

2.1. Lemma. (a) For each $X \in \mathbf{Top}$ and $M \subset X$, the following holds:

$$[M]^X_A = (r_A)^{-1}\left([r_A(M)]^{r_A(X)}_A\right).$$

Thus A-closure is additive (thus a Kuratowski operation) for each $X \in \mathbf{Top}$ iff it is so for each $A \in \mathbf{A}$.

(b) For each $X \in P(A)$ and $M \subset X$, the following hold

$$[M]^X_{P(A)} = [M]^{r_A(X)}_{P(A)} = [M]^{r_A(X)}_A = [M]^X_A.$$

Thus P(A)-closure is a Kuratowski operation iff A-closure is.

Proof. (a) By 1.2. (x) of [4] $r_A([M]^X_A) \subset [r_A(M)]^{r_A(X)}_A$, so $[M]^X_A \subset (r_A)^{-1}([r_A(M)]^{r_A(X)}_A)$. On the other hand, if $x \notin [M]^X_A$ and $(f,g: X \longrightarrow A)$ is an A-separating pair for (x,M), then $(f', g': r_A(X) \to A)$ where $f' \circ r_A = f$ and $g' \circ r_A = g$, is an ~~A-separating~~ pair for $(r_A(X), r_A(M))$, so $x \notin (r_A)^{-1}([r_A(M)]^{r_A(X)}_A)$.

b) For each $X \in P(A)$, $r_A: X \to r_A(X)$ is the identity on the underlying sets then, it follows from (a) that $[M]^X_{P(A)} = [M]^{r_A(X)}_{P(A)}$. Furthermore $[M]^{r_A(X)}_{P(A)} \subset [M]^{r_A(X)}_A$ follows from the inclusion $A \subset P(A)$.

To show that $[M]^{r_A(X)}_A \subset [M]^X_{P(A)}$ take $x \notin [M]^X_{P(A)}$ and a P(A)-separating pair $(f,g: X \longrightarrow Y)$ for (x,M). Then $(f',g': r_A(X) \longrightarrow r_A(Y))$, where $r_A \circ f = f' \circ r_A$ and $r_A \circ g = g' \circ r_A$, is an A-separating pair for (x,M) in $r_A(X)$, so $x \notin [M]^{r_A(X)}_A$. For the last equality note that $r_A: X \to r_A(X)$ is the identity on the underlying sets then (a) gives $[M]^X_A = [M]^{r_A(X)}_A$ for every $M \subset X$.

For each $(X,\tau) \in \mathbf{Top}$, τ_A will denote the topology generated in X by the A-closure, i.e., the coarsest topology on X for which all A-closed sets are closed. $F_A: \mathbf{Top} \longrightarrow \mathbf{Top}$ will denote the functor which assigns to $(X, \tau) \in \mathbf{Top}$ the space (X, τ_A). For each continuous map $f: (X, \tau) \longrightarrow (Y, \sigma)$ in Top the continuity of $f = F_A(f): (X, \tau_A) \longrightarrow (Y, \sigma_A)$ follows from 1.2 (x) of [4].

By 2.1 of [4] for every $(X, \tau) \in \mathbf{Top}$, τ_A is the initial topology on X induced by the map $X \xrightarrow{r_A} F_A(r_A X)$, where r_A is the A-reflection of X. This is why (X, τ_A) is indiscrete iff $r_A X$ is a singleton. On the other hand, if $A \neq \mathbf{Sgl}$, then for each $(X,\tau) \in \mathbf{Top}$, $(X, \tau_A) \in \mathbf{Top}_1$ iff $r_A: X \to r_A X$ is injective. In particular if A is extremally epireflective, then $(X,\tau_A)\in Top_1$ iff $(X,\tau)\in A$. Conditions ensuring $(X,\tau_A)\in Top_2$ are discussed in 2.8.

Till the end of this section, we study the properties of the functor F_A. Set $A_0=\{X\in Top: F_A(X)=X\}$. Clearly $(X,\tau)\in A_0$ iff $r_A(X,\tau)\in A_0\cap A$ and X has the initial topology with respect to $r_A: X\to r_AX$.

In the following theorem we give explicitly τ_A for various categories A including those listed in 1.1. First recall the notion of θ-closure introduced by Velichko ([17]). For $(X,\tau)\in Top$ and $M\subset X$,

$$Cl_\theta M=\{x\in X: \text{ for each nbd } V \text{ of } x, \bar{V}\cap M\neq\emptyset\}.$$

Analogously one can introduce θ-interior $Int_\theta M=\{x\in X:$ there exists a nbd V of x, $\bar{V}\subset M\}$. A subset M of X is said to be θ-closed (θ-open) if $M=Cl_\theta M$ ($M=Int_\theta M$). The θ-closure is additive but not idempotent in general. The idempotent hull of Cl_θ is $[\ ]_{Top_{2\frac{1}{2}}}$ since for each $(X,\tau)\in Top_{2\frac{1}{2}}$ and $M\subset X$, $Cl_\theta M\subset[M]_{Top_{2\frac{1}{2}}}$ and M is θ-closed iff M is $Top_{2\frac{1}{2}}$-closed (see 2.5(**b**) of [4]). θ-closure was also studied by Schröder [15].

2.2. Theorem. (a) If A is bireflective (resp. $A=Top_1$) then τ_A is the discrete topology for every $(X,\tau)\in Top$ (resp. $(X,\tau)\in A$).

(b) If $A=Top_i$, $i=2,3,3\frac{1}{2}$, or $A=0\text{-dim}$, then $\tau_A=\tau$ for each $(X,\tau)\in A$.

(c) If $A=P(B)$ then for each $(X,\tau)\in A$, $\tau_A=\sigma_B$, where (X,σ) is the B-reflection of (X,τ). Thus the functors F_A and F_B coincide.

(d) For $B=Top_3$, $Top_{3\frac{1}{2}}$ and 0-dim and $A=P(B)$, F_A coincides on A with the B-reflection.

(e) For $A=Top_0$ and $(X,\tau)\in Top_0$, τ_A is the topology on X having, as open base, all locally closed subsets of (X,τ) (finite intersections of open and closed sets in (X,τ)). Thus $\tau_A\geq\tau$ and $(X,\tau_A)\in 0\text{-dim}$.

(f) For $A=Top_{2\frac{1}{2}}$ and $(X,\tau)\in A$, $U\in\tau_A$ iff U is θ-open. In particular $\tau=\tau_A$ iff $(X,\tau)\in Top_3$.

Proof. (a): By 1.10 (a) of [4] in this case the A-closure coincides with the identity operator.

(b): By 2.8 (i) of [4] in this case the A-closure coincides with the ordinary closure.

(c): It follows from 2.1 (b). (d): It follows from (b) and (c).

(e): As pointed out in 2.9 of [4] in this case the Top_0-closure coincides with the well-known front-closure ([1],[12]),

$$fr\ cl\ M=\{x\in X: \text{ for each nbd } V \text{ of } x, V\cap\overline{\{x\}}\cap M\neq\emptyset\}.$$

Thus $U\subset X$ is τ_A-open iff for each $x\in U$ there exists a nbd V such that $V\cap(X\setminus U)\cap\overline{\{x\}}=\emptyset$, i.e. $\overline{\{x\}}\cap V\subset U$. Clearly any $V\cap\overline{\{x\}}$ is clopen in (X,τ_A), so $(X,\tau_A)\in 0\text{-dim}$.

(f) Obviously Cl_θ is additive, thus $[\ ]_{Top_{2\frac{1}{2}}}$ being its idempotent hull will be a Kuratowski operator (in fact, $Cl_\theta([M]\cup[N]) = Cl_\theta([M])\cup Cl_\theta([N]) \subset [[M]]\cup[[N]] = [M]\cup[N]$, so $[M]\cup[N]$ is θ-closed, thus $Top_{2\frac{1}{2}}$-closed). The last assertion is proved in 2.4 of [4].

2.3 Remarks (a). The A-closure is additive in all subcategories A of Top listed in 1.1. We do not know any example of non additive A-closure operation different from the case A=epireflective hull of a class of strongly rigid spaces.

(b) By the explicit form of τ_{Top_0} it can be seen easily that for $(X,\tau)\in Top_0$, τ_{Top_0} is discrete iff for each $x\in X$ there exists a τ-nbd V such that $\{x\} = \overline{\{x\}}\cap V$.

The subcategory of such spaces of Top_0 will be denoted by T_D.

(c) The functor F_{Top_0} preserves embeddings and finite products (more precisely, for each family $\{(X_i,\tau_i)\}_{i\in I}$ in Top_0 with $(X,\tau) = \prod_{i\in I}(X_i,\tau_i)$, $\tau_{Top_0} = \prod_{i\in I}(\tau_i)_{Top_0}$ holds iff all but a finite number of the spaces X_i are singletons).

In general the functor F_A is submultiplicative, i.e., for each family $\{(X_i,\tau_i)\}_{i\in I}$ in Top, $(\prod_{i\in I}\tau_i)_A \geq \prod_{i\in I}(\tau_i)_A$. The following examples show that in general F_A does not preserve neiter embeddings nor finite products.

2.4. Examples (a) Let (H,τ') be the space given in 1.3 of [4]. Then $F\cup\{(0,0)\}$ is discrete in (H,τ'), while $F\cup\{(0,0)\}$ is not discrete as a subspace of $(H, \tau'_{Top_{2\frac{1}{2}}})$ which is compact.

(b) Let $A=E\{(X,\tau)\}$, where (X,τ) is an infinite strongly rigid space. Then τ_A is the cofinite topology on X, so Δ_X is not closed in $(X\times X, \tau_A\times\tau_A)$. On the other hand Δ_X is the equalizer of the projections, so Δ_X is closed in $(X^2, (\tau^2)_A)$. Thus $(\tau\times\tau)_A > \tau_A\times\tau_A$.

2.5 Proposition. If F_A is finitely multiplicative, then for each $(X,\tau)\in A$, $(X,\tau_A)\in Top_2$.

Proof. Consider Δ_X in $(X\times X, \tau_A\times\tau_A)$; since Δ_X is always A-closed in $(X\times X, \tau\times\tau)$ and $(\tau\times\tau)_A = \tau_A\times\tau_A$ this implies that Δ_X is closed in $(X\times X, \tau_A\times\tau_A)$, so $(X,\tau_A)\in Top_2$.

In the following Section we show that there exists $(X,\tau)\in Top_{2\frac{1}{2}}$ with $(X, \tau_{Top_{2\frac{1}{2}}})\notin Top_{2\frac{1}{2}}$. (Hence $F_{Top_{2\frac{1}{2}}}$ is not finitely multiplicative).

Till the end of this section we study conditions which ensure $\tau_A\leq\tau$ or (X,τ_A) discrete.

For $(X,\tau)\in Top$ denote by $I(X,\tau)$ the set of all isolated points of (X,τ) .

2.6 Lemma. For any epireflective subcategory **A** of **Top** and each $(X,\tau)\in\mathbf{A}$

$$(*) \qquad I(X,\tau)\subset I(X,\tau_{\mathbf{A}}).$$

Moreover, (*) holds for each $(X,\tau)\in\mathbf{Top}$ iff **A** is bireflective or $\mathbf{A}=\mathbf{Top}_0$.

Proof. Consider first the case when **A** is neither bireflective nor $\mathbf{Top}_0$. Then $\mathbf{A}\subset\mathbf{Top}_1$, so for every $(X,\tau)\in\mathbf{A}$ $(X,\tau_{\mathbf{A}})\in\mathbf{Top}_1$ holds. Therefore each isolated point of (X,τ) is τ-clopen, thus also $\tau_{\mathbf{A}}$-clopen by 1.2 (vi) of [4] . This proves (*). Remark that (*) does not hold for Sierpinski's two-points space (S,τ) (two points 0,1 with $\{0\}$ unique proper open set) since $I(S,\tau)\neq\emptyset$ and $I(S,\tau_{\mathbf{A}})=\emptyset$ (the space $(S,\tau_{\mathbf{A}})$ is indiscrete since the reflection of (S,τ) in **A** is a singleton because of $\mathbf{A}\subset\mathbf{Top}_1$).

It remains to show that (*) holds for every $(X,\tau)\in\mathbf{Top}$ if **A** is bireflective or $\mathbf{A}=\mathbf{Top}_0$. This is obvious in the first case since $\tau_{\mathbf{A}}$ is always discrete according to 2.2 (a). Assume $\mathbf{A}=\mathbf{Top}_0$ and take an arbitrary $(X,\tau)\in\mathbf{Top}$. Then for each $x\in I(X,\tau)$ the characteristic (continuous) map $f:X\to S$ of the open set $\{x\}$ and the constant at 1 form an **A**-separating pair for $(x,X\setminus\{x\})$ so $x\in I(X,\tau_{\mathbf{A}})$.

In the following proposition we show that the converse inclusion of (*) for any space $(X,\tau)\in\mathbf{A}$ implies $\mathbf{A}\subset\mathbf{Top}_2$.

2.7 Proposition. For each epireflective subcategory **A** of **Top** the following conditions are equivalent:

(a) $\mathbf{A}\in\mathbf{Top}_2$;

(b) for each $(X,\tau)\in\mathbf{A}$ $\tau_{\mathbf{A}}\leq\tau$;

(c) for each $(X,\tau)\in\mathbf{Top}$ $\tau_{\mathbf{A}}\leq\tau$;

(d) for each $(X,\tau)\in\mathbf{A}$ $I(X,\tau)=I(X,\tau_{\mathbf{A}})$;

(e) every $(X,\tau)\in\mathbf{A}$ is discrete whenever $(X,\tau_{\mathbf{A}})$ is discrete.

Proof. The equivalence (a)$\Leftrightarrow$(b) was given in 1.10 (b) from [4] The equivalence (b)$\Leftrightarrow$(c) follows from 2.1 (a). Clearly (b) implies (d) and (d) implies (e). To finish the proof we have to show (e)$\Rightarrow$(a).

We can assume without loss of generality that **A** is extremally epireflective. In fact, if $\mathbf{B}=P(\mathbf{A})$ then because of 2.2 (c) the functors $F_{\mathbf{A}}$ and $F_{\mathbf{B}}$ coincide. To show that each $(X,\tau)\in\mathbf{B}$ satisfies (e) consider the reflection (X,σ) of (X,τ) in **A**. Then by 2.2 (c) $\tau_{\mathbf{B}}=\sigma_{\mathbf{A}}$. Now if $\tau_{\mathbf{B}}$ is discrete then by (e) (X,σ) is discrete, thus (X,τ) is discrete too. So we can assume that **A** is extremally epireflective, i.e., A=B.

If A is bireflective then A=Top and (e) is not verified since τ_{Top} is always discrete. Therefore $A \subset Top_0$. Now $A=Top_0$ contradicts (e) since there exists a non-discrete space $(X, \tau) \in T_D$, then τ_{Top_0} is discrete.

We have shown that (e) implies $A \subset Top_1$. Assume there exists a space $(X, \tau) \in A$ such that $(X, \tau) \notin Top_2$. Then there exist two distinct points x and y in X such that for any nbd V of x and any nbd U of y in (X, τ)

(**) $$V \cap U \neq \emptyset.$$

Now set $Y=\{p\} \cup X \setminus \{x,y\}$ and consider the following topology σ on Y. All points different from p are isolated, for nbds system of p take all intersections (**) added the point p. Clearly σ is non discrete because of (**). Consider the maps f_x and f_y of Y into X defined by, $f_x(u)=f_y(u)=u$ if $u \neq p$ and $f_x(p)=x$, $f_y(p)=y$. The continuity of f_x and f_y follows directly from the definition of σ. On the other hand both maps are injective, hence $(Y, \sigma) \in A$ because $X \in A$ and A is extremally epireflective. Now the space (Y, σ) does not satisfy (e) since σ_A is discrete. In fact by 2.6 $I(Y, \sigma_A) \supset I(Y, \sigma)=Y \setminus \{p\}$ and (f_x, f_y) is an A-separating pair for $(p, Y \setminus \{p\})$, so $\{p\}$ is σ_A-open.

3. It is well known that Top_2 is a cowellpowered category, i.e., the class of all Top_2-epimorphisms (i.e. dense continuous maps) with domain a fixed Hausdorff space has a representative set ([7]).
In 1975 Herrlich [9] first produced an example of a non cowellpowered epireflective subcategory of Top: the epireflective hull of a proper class of strongly rigid spaces such that the continuous maps between them are precisely the identities or the constant maps.

In 1983 Schröder showed that $Top_{2\frac{1}{2}}$ is not cowellpowered ([16]). He produced for each ordinal number β a Urysohn space Y_β of cardinality $\aleph_0 \cdot card(\beta)$ and an embedding $e_\beta : Q \longrightarrow Y_\beta$, where Q is the space of rational numbers with the usual topology, such that e_β is a $Top_{2\frac{1}{2}}$-epimorphism.

In what follows we shall show that all remaining categories listed in 1.1 are cowellpowered. The following proposition given in [4] and [6] will be used.

3.1 Proposition. $f:X \longrightarrow Y$ is an A-epimorphism iff f(X) is A-dense in Y, i.e., $[f(X)]_A=Y$.

3.2 Lemma. Let A and B be epireflective subcategories of Top and let

$F:A \longrightarrow B$ be a functor satisfying the following conditions:

(1) F preserves epimorphisms, i.e., for each A-epimorphism $f:X \to Y$ the map $F(f):F(X) \longrightarrow F(Y)$ is a B-epimorphism;

(2) F is a concrete functor, i.e., if $U:\mathbf{Top} \to \mathbf{Set}$ is the forgetful functor then $U \circ F = U$.

Then **A** is cowellpowered whenever **B** is cowellpowered.

Proof. Trivial.

3.3 Corollary. Let **B** be a cowellpowered epireflective subcategory of **Top**, then so is P(**B**).

Proof. For **A**=P(**B**) and $F=r_B$-the **B**-reflection- apply 3.2. Clearly F satisfies (2), on the other hand, by 2.1.(b), $f:X \to Y$ is an epimorphism in **A** iff $f=F(f):F(X) \longrightarrow F(Y)$ is an epimorphism in **B**. Thus F satisfies also (1).

3.4 Corollary. If **A** is an epireflective subcategory of **Top** such that for each $(X,\tau) \in \mathbf{A}$ $(X,\tau_A) \in \mathbf{Top}_2$, then **A** is cowellpovered.

Proof. For $\mathbf{B}=\mathbf{Top}_2$ and $F=F_A$ we apply 3.2. Obviously (2) holds; on the other hand for each epimorphism $f:X \to Y$ in **A** f(X) is **A**-dense in Y by virtue of 3.1. Therefore f(X) is dense in F(Y) hence $f:F(X) \to F(Y)$ is an epimorphism in $\mathbf{B}=\mathbf{Top}_2$ and $\mathbf{Top}_2$ is cowellpowered.

For all subcategories of **Top** listed in 1.1 except $\mathbf{Top}_{2\frac{1}{2}}$, τ_A is Hausdorff so all they are cowellpowered.

3.5 Corollary. If **A** is an epireflective subcategory of **Top** such that F_A is finitely multiplicative than **A** is cowellpowered.

Proof. By virtue of 2.5, **A** satisfies the condition in 3.4, so **A** is cowellpowered.

Some familiar extremally epireflective subcategories of **Top** are the extremal epireflective hull of a proper epireflective subcategory (e.g. $FT_2=P(\mathbf{Top}_{3\frac{1}{2}})$). $\mathbf{Top}_2$ does not have that property as the following proposition shows.

3.6 Proposition. If **A** is an extremally epireflective subcategory of **Top** and for every $(X,\tau) \in \mathbf{A}$, $\tau_A = \tau$, then there does not exist a proper epireflective subcategory $\mathbf{B} \subset \mathbf{A}$ such that P(**B**)=**A**.

Proof. Since $\tau_A = \tau$ for each $(X,\tau) \in \mathbf{A}$, by virtue of proposition 2.7, $\mathbf{A} \subset \mathbf{Top}_2$. Assume there exists an epireflective subcategory **B** of **Top** such that **A**=P(**B**).

By 2.2 (c), for each $(X,\tau) \in \mathbf{A}$ with **B**-reflection $r_B(X,\tau)=(X,\sigma)$, $\tau_A = \sigma_B$ holds. Since $\mathbf{B} \subset \mathbf{A} \subset \mathbf{Top}_2$, $\sigma_B \le \sigma$, thus we get $\tau = \tau_A = \sigma_B \le \sigma$.

On the other hand always $\tau \geq \sigma$ holds, so for each $(X,\tau) \in A$, $r_B(X,\tau)=(X,\tau)$. Therefore B=A.

3.7 Question. Does there exist such a B as in 3.6 for $A=Top_{2\frac{1}{2}}$? By virtue of 3.3 such a B will not be cowellpowered.

4. In this section we consider iterations of the functor F_A:Top→Top defined in section 2. Let A be epireflective subcategory of Top;then for each ordinal number α we define a topology τ_{A^α} on X in the following way: $\tau_{A^0}=\tau$ and $\tau_{A^{\alpha+1}}=(\tau_{A^\alpha})_A$ for any α; if α is a limit ordinal $\tau_{A^\alpha}=\inf_{\beta<\alpha}\tau_{A^\beta}$. It is easy to check that setting $F_{A^\alpha}(X,\tau)=(X,\tau_{A^\alpha})$ we get a functor F_{A^α}:Top→Top. By virtue of 2.7 if $A\subset Top_2$,for each $(X,\tau)\in$ Top,the topologies τ_{A^α} form a decreasing chain, so there will exist an ordinal number α such that $\tau_{A^{\alpha+1}}=\tau_{A^\alpha}$.

4.1 Definition. Let $A\subset Top_2$ and $(X,\tau)\in$ Top; the smallet ordinal α, such that $\tau_{A^{\alpha+1}}=\tau_{A^\alpha}$ will be called epimorphic order of (X,τ) with respect to A and will be denoted by $EO_A(X,\tau)$.

In particular $EO_A(X)=0$ iff $X\in A_0$, otherwise $EO_A(X)=1+EO_A(F_A(X))$ with easy check .

Epimorphic order can be defined in a similar way also for categories A such that $\tau\leq\tau_A$ for each $(X,\tau)\in$ Top .

4.2 Examples. Let (X,τ) be an infinite strongly rigid space and $A=E\{(X,\tau)\}$; then τ_A is the cofinite topology on X, so $r_A(X,\tau_A)$ is a singleton, therefore (X,τ_{A^2}) is indiscrete, so $EO_A(X,\tau)=2$.

(b) Let $B\subset B_0$ and A=P(B), then $EO_A(X,\tau)=0$ iff $r_A(X,\tau)\in B$ and X has the initial topology with respect to $X\longrightarrow r_A(X,\tau)$, otherwise $EO_A(X,\tau)=1$.

(c) Let (Y_β,τ_β) be the Urishon space constructed in [16] for an ordinal β satisfying $1<\beta\leq\omega+1$; then $EO_{Top2\frac{1}{2}}(Y_\beta,\tau_\beta)=2$ while $EO_{Top2\frac{1}{2}}(Y_1,\tau_1)=1$. Moreover $F^2_{Top_2}(Y_\beta,\tau_\beta)\in$0-dim for these ordinals.

(d) If A is bireflective and X∈Top then $EO_A(X)=0$ iff X is discrete, otherwise $EO_A(X)=1$.

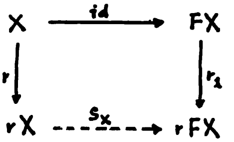
(c) For $A=Top_1$ $EO_A(X)=0$ iff $r_{Top_1}(X)$ is discrete and X has the initial topology with respect to $X\to r_{Top_1}(X)$, otherwise $EO_A(X)=1$.

(f) For $A=Top_0$ and X ∈ Top, $EO_A(X)=0$ iff $r_{Top_0}(X)$ is discrete and X has the initial topology with respect to $X\to r_{Top_0}(X)$; $EO_A(X)=1$ iff $r_{Top_0}(X)$ is non discrete and belongs to T_D, $EO_A(X)=2$ iff $r_{Top_0}(X)\notin T_D$.

We have no examples of epimorphic order greater than 2.

In order to calculate easier the epimorphic order we have to know better the interrelation between the functors F_A and r_A. In what follows we omit the index A for brevity, A is always contained in Top_2 and $[\ ]_A$ is a K.op.

For each $X \in Top$ consider the diagram



By the definition of r there exists a unique continuous map $S_x : rX \longrightarrow rFX$ which makes commutative the diagram.

4.3 Lemma. The map $S_x : rX \longrightarrow rFX$ defined above is continuous when we consider on rX the topology generated by the A-closure, i.e. $S_x : FrX \longrightarrow rFX$ is continuous .

Proof. We have to show that for each closed set M in rFX, $S_x^{-1}(M)$ is A-closed in rX. By the continuity of r_1, $r_1^{-1}(M)$ is closed in FX. By 2.1 (a)

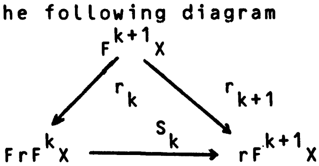
$$\left[r_1^{-1} M\right]^X = r^{-1}\left(\left[r(r_1^{-1}M)\right]^{rX}\right) = r_1^{-1}M \quad ;$$

on the other hand $r_1 = S_x \circ r$, so $r(r_1^{-1} M) = s_x^{-1}(M)$, thus $r_1^{-1}M = r^{-1}\left(\left[S_x^{-1}M\right]^{rX}\right)$.

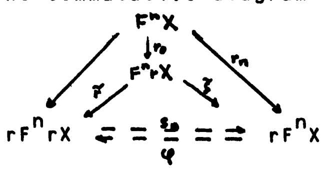
Applying r we get $S_x^{-1}M = \left[S_x^{-1}M\right]$ which proves the continuity of $S_x : FrX \longrightarrow rFX$.

4.4 Proposition. For each natural number n and each $X \in Top$, $rF^n rX$ is naturally isomorphic to $rF^n X$.

Proof. For any natural $k < n$ the above lemma applied to the space $Y = F^k X$ provides a natural continuous map $S_k : FrF^k X \longrightarrow rF^{k+1}X$ which makes commutative the following diagram



where r_k and r_{k+1} are the corresponding reflections. Applying the functor F^{n-k-1} we get the commutative diagram



where $\tilde{r}$ is the A-reflection. By the definition of the reflection there exists a unique continuous map $S_n : rF^n rX \longrightarrow rF^n X$ such that $S_n \circ r = S$. Let us see that S_n is an isomorphism. Again by the properties of the reflection there exists a unique continuous map $\varphi : rF^n X \longrightarrow rF^n rX$ such that $\varphi r_n = \tilde{r} r_o$. Consider now the composition $\psi = \varphi \circ S_n \circ \tilde{r} \circ r_o$; by the definition of S_n and φ we get $\psi = \varphi \circ \tilde{S} \circ r_o = \varphi \circ r_n = \tilde{r} r_o$. Thus the restriction of $\varphi \circ S_n$ on $\tilde{r} r_o(F^n X)$ is the identity.
Since $\tilde{r} r_o$ is an epimorphism this gives $\varphi \circ S_n = id$ on $rF^n rX$. In the same way one proves that $S_n \circ \varphi$ is the identity on $rF^n X$.

4.5 Remark. Consider the semigroup Σ of all functors Top $\longrightarrow$ Top generated by r and F. By the definition of r, $r = r^2$ holds. On the other hand 4.4 shows that, for any n, there is an equivalence between $rF^n \circ r$ and rF^n. Let Σ_1 be the quotient of Σ with respect to the equivalence of functors. Then the functors F^m and $F^n rF^t$ with m,n and t non-negative integers (F^o is the identity functor) represent all elements of Σ_1 (*). The multiplication is given by

$$(F^n \circ rF^t)F^m = F^n \circ rF^{t+m}, \quad F^m \circ (F^n \circ rF^t) = F^{m+n} \circ rF^t, \quad (F^n \circ rF^t)(F^{n'} \circ rF^{t'}) = F^n \circ rF^{t+n'+t'} .$$

It was mentioned in section 2 that for any $X \in$ Top $FrX \longrightarrow FrX$ is initial. Proposition 4.4 enables us to show it for any natural n.

4.6 Corollary. For any $X \in$ Top and for any positive integer n, $F^n X \longrightarrow F^n rX$ it initial.

Proof; By the definition of $F^n X$, $F^n X \xrightarrow{f} FrF^{n-1} X$ is initial. By 4.4, $rF^{n-1} X$ is naturally isomorphic to $rF^{n-1} rX$.
Consider the commutative diagram

$$\begin{array}{ccc} F^{n-1} X & \xrightarrow{\quad f \quad} & rF^{n-1} X \\ \downarrow r & & \downarrow S_{n-1} \\ F^{n-1} rX & \xrightarrow{\quad r_1 \quad} & rF^{n-1} rX \end{array}$$

where S_{n-1} is the natural isomorphism given in 4.4, r and r_1 are reflections.
Applying the functor F we get the commutative diagram

$$\begin{array}{ccc} F^n X & \xrightarrow{\quad f \quad} & FrF^{n-1} X \\ \downarrow r & & \downarrow S_{n-1} \\ F^n rX & \xrightarrow{\quad r_1 \quad} & FrF^{n-1} rX \end{array}$$

(*) Σ_1 is finite for all categories listed in 1.1 except may be $Top_{2\frac{1}{2}}$ (See 4.2) with the same underlying sets and maps. Now $r_1 \circ r = S_{n-1} \circ f$ is initial, therefore r is initial too.

4.7 Remark. (a) The assertion of the above corollary is no valid for $n=0$ (see (4.12 (b)).

(b) We do not know whether 4.6 is true for infinite ordinals. A positive answer would imply the validity of the following corollary for arbitrary non-zero ordinals.

4.8 Corollary. Let n be a positive integer and $X \in Top$ with $rX \in A_o$. Then $EO_A(X)=n$ iff $EO_A(rX)=n$.

Proof. By 4.6 $EO_A(X) \leq EO_A(rX)$ since $F^{n+1} rX = F^n rX$ would imply $F^{n+1}X = F^n X$. Since $X \to rX$ is surjective, different topologies on rX give rise to different initial topologies on X, i.e., $F^{n+1}X=F^nX$ would imply $F^{n+1}rX=F^nrX$, thus $EO_{A(rX)} \leq EO_A(X)$.

It may happen $rX \in A_o$, i.e., $EO_A(rX)=0$ and $EO_A(X)=1$ if $X \to rX$ is not initial. The above corollary permits easier calculation of the epimorphic order.

4.9 Example. Let (Y_β, τ_β) denotes the Urishon space constructed for the ordinal β in [15] . If $\beta > \omega+1$ one can see that $Z=F_{Top2\frac{1}{2}}(Y_\beta, \tau_\beta)$ is not even Hausdorff. However for every $\beta > \omega+1$ the Hausdorff reflection of Z is already Urisohn, i.e. $r_{Top2\frac{1}{2}} Z = r_{Top2} Z$. Moreover there exist a continuous bijection $F_{Top2\frac{1}{2}}(Y_{\omega+1}, \tau_{\omega+1}) \xrightarrow{\varphi} rZ$ such that $rZ \xrightarrow{\varphi^{-1}} F^2_{Top2\frac{1}{2}}(Y_{\omega+1}, \tau_{\omega+1})$ is continuous and not open. Since $EO_{Top2\frac{1}{2}}(Y_{\omega+1}, \tau_{\omega+1})=2$ this implies $EO_{Top2\frac{1}{2}}(rZ)=1$. By corollari 4.8 $EO_{Top2\frac{1}{2}}(Z)=1$, so by the definitions of epimorphic order $EO_{Top2\frac{1}{2}}(Y_\beta, \tau_\beta)=2$ for $\beta > \omega+1$.

The above example justifies the following definition.

4.10 Definition . Let β be an ordinal number, denote by $A^{(\beta)}$ the category of all spaces $X \in A$ such that $F_A^\gamma(X) \in A$ for each $\gamma \leq \beta$. Set $A^{(\infty)} = \bigcap_\beta A^{(\beta)}$, i.e., $A^{(\infty)}$ is the category of all spaces $X \in A$ such that $F_A^\gamma(X) \in A$ for each $\gamma \leq EO_A(X)$.

For example in 4.9 $Y_\beta \notin Top^{(1)}_{2\frac{1}{2}}$; for A as in 4.2 (b) $A^{(\infty)}=A$.

4.11 Theorem. Let A be an epireflective subcategory of Top, then:

(a) A_o is bireflective in Top.

(b) $A \cap A_o$ is bireflective in A, thus $A \cap A_o \subset A^{(\infty)} \subset P(A \cap A_o)$.

(c) If A is extremally epireflective then, for each ordinal β,

$A^{(\beta)}$ and $A^{(\infty)}$ are extremally epireflective; in particular $A^{(\infty)}=P(A\cap A_0)$.

Proof. (a) For any $X \in$ Top define $r_{A_0}(X) = F_{A^\alpha}(X)$, where $\alpha = EO_A(X)$. Now for every $Y \in A_0$ and any map $f:X \to Y$ applying F_{A^α} we get $f=F_{A^\alpha}(f):F_{A^\alpha}(X) \to F_{A^\alpha}(Y)=Y$. Thus r_{A_0} is a bireflection of Top in A_0.

(b) Follows from (a).

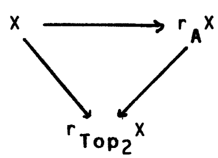
(c) Let $Y \in A^{(\beta)}$, then $F^\beta(Y) \in A$. For any subspace X of Y applying to the embedding $i:X \to Y$ the functor F^β we get $i=F^\beta(X) \to F^\beta(Y)$. Since A is extremally epireflective this implies $F^\beta(X) \in A$. For any family $\{X_i\}$ of spaces in $A^{(\beta)}$, $F^\beta(X_i) \in A$, therefore $F^\beta(\prod_i X_i)$, having a topology finer than that of $\prod_i F^\beta(X_i)$, belongs to A. Therefore $A^{(\beta)}$ is extremally epireflective.

The rest is obvious.

4.12 Remark. Analogous theorem can be proved for categories A which satisfy $\tau \le \tau_A$ for each $(X,\tau) \in$ Top. In such a case A_0 is a coreflective subcategory of A and the coreflection is given by $F^\alpha(X) \to X$ where $\alpha = EO_A(X)$.

(b) In 4.9 $Z \to r_{Top_{2\frac{1}{2}}}(Z)$ is not initial (this shows that in general $FX \to rFX$ is not initial).

(c) Since A_0 is a bireflective subcategory of Top, Top$=P(A_0)$ holds. On the other hand always $A_0 \neq$ Top. In fact, assume $A \subset A_0$, then by 2.7 $A \subset Top_2$. Since $X \in A_0$ iff $r_A X \in A_0$ and $X \to r_A X$ is initial, it suffices to find $X \in$ Top such that $X \to r_A X$ is not initial. Now $A \subset Top_2$ provides the following commutative diagram



this is why a space X such that $X \to rX$ is not initial, will do (take, for example, the space Z from (b)).

The following theorem characterizes the categories A satisfying $A^{(\infty)}=A$.

4.13 Theorem. For an extremally epireflective subcategory A of Top the following conditions are equivalent:

(a) there exists an epireflective subcategory B of Top such that $B \subset B_0$ and $A=P(B)$.

(b) $A_0^{(\infty)}=A$.

Proof. (a) $\Rightarrow$ (b) is obvious since, for any $(X,\tau)\in A$, (X,τ_A) $B\subset A$. On the other hand (b) $\Rightarrow$ (a) follows from 4.11 with $B=A\cap A_0$.

4.14 Remarks. (a) By 4.2 (b) both conditions in 4.13 imply $EO_A(X)\leq 1$ for any $X\in Top$. We do not know whether the converse is also true. Observe that if $EO_A(X)\leq 1$ for every $X\in A$, than by 4.8 $EO_A(X)\leq 1$ for every $X\in Top$.

(b) In general for any extremally epireflective subcategory A of Top, $A^{(\infty)}=P(A\cap A_0)$ according to 4.11 (c), thus for $X\in A$, $EO_{A^{(\infty)}}(X)=1$ iff $X\notin A_0$. On the other hand it may happen $EO_A(X)>EO_{A^{(\infty)}}(X)$ (take for example $X=Y_\omega$ as in 4.2 (c); then for $A=Top_{2\frac{1}{2}}$, $A\cap A_0=Top_3$, therefore $A^{(\infty)}=P(Top_3)$ and $X\in A$, $EO_A(X)=2>EO_{A^{(\infty)}}(X)=1$).

Reference

[1] S.Baron, Note on epi in T_0, Canad. Math. Bull. 11 (1968) 503-504.

[2] G.Birkhoff, Lattice Theory (Amer.Math.Soc.Colloq.Publ.25, New York, 1948).

[3] F.Cagliari and N.Cicchese, Epireflective subcategories and epiclosure, Riv.Mat.Univ. Parma, (4) 8 (1982) 1-8.

[4] D.Dikranjan and E.Giuli, Closure operators induced by topological epireflections, Coll.Math.Soc.Janos Bolyai, Topology,Eger (1983), to appear.

[5] D.Dikranjan and E. Giuli, Uryson spaces:old and new, preprint.

[6] E.Giuli, Bases of topological epireflections, Topology Appl.11 (1980) 285-294.

[7] H.Herrlich, Topologische Reflectionen und Coreflexionen, Lecture Notes in Math. 78 (Springer, Berlin-New York, 1968).

[8] H.Herrlich, Categorical Topology, General Topology Appl.1(1971) 1-15.

[9] H.Herrlich, Epireflective subcategories of TOP need not be cowellpowered, Comment. Math.Univ.Carolinae 16 (1975) 713-715.

[10] H.Herrlich and G.E. Strecker, Category Theory (Heldermann-Verlag, Berlin, 1979).

[11] V.Kannan and M.Rajagopalan, Constructions and Applications of Rigid Spaces I, Adv. in Math. 29 (1978) 83-130.

[12] L.Nel and R.G.Wilson, Epireflections in the category of T_0-spaces, Fund.Math. 75 (1972) 69-74.

[13] P.J.Nyikos, Epireflective subcategories of Hausdorff spaces, in: Categorical Topology, Lecture Notes in Math. 540 (Springer,Berlin New York, 1976) 452-481.

[14] S.Salbany, Reflective subcategories and closure operators, in: Categorical Topology, Lecture Notes in Math. 540 (Springer,Berlin New York, 1976) 548-565.

[15] J.Schröder, Epi und extremer mono in $\underline{T}_{2a}$, Arch.Math. XXV (1974) 561-565.

[16] J.Schröder, The category of Urysohn spaces is not cowellpowered, Topology Appl. 16 (1983) 237-242.

[17] H.Velichko, H-closed topological spaces. Mat. Sb.(N.S.) 70(112) (1966), 98-112.

D.Dikranjan
Scuola Normale Superiore
Piazza dei Cavalieri,7
Pisa (Italy)

E.Giuli
Dipartimento di Matematica
Università
L'Aquila (Italy)