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REMARKS ON DENSITY TOPOLOGY AND ITS CATEGORY ANALOGUE

Władysław Wilczyński

The density topology was studied in [1] and [2] and its category analogue in [3] and [4]. In the first case the authors were dealing only with Lebesgue measurable sets, in the second with the sets having the Baire property. In both cases very essential role is played by the definition of the density point of a set. In this note we shall discuss some definitions of the density point which are equivalent for the Lebesgue measurable sets (sets having the Baire property), but not equivalent in the general case. The first part of the paper will be devoted to "measure", the second to "category". We shall suppose CH.

Let R denote the real line, $\mathcal{L}$ - the σ-algebra of Lebesgue measurable sets and m - the linear Lebesgue measure. If $A \in \mathcal{L}$, then 0 is a density point of A if and only if a/ $\lim_{h\to 0^+} (2h)^{-1} \cdot m(A \cap (-h,h)) = 1$. It is not difficult to see that this condition is equivalent to each of the following conditions: b/ the sequence of characteristic functions of the sets $(n\cdot A) \cap [-1,1]$, $n \in N$ converges in measure to a function which is identically 1 on $[-1,1]$ (here $n\cdot A = \{n\cdot x : x \in A\}$), which we shall denote

$$\chi_{((n\cdot A) \cap [-1,1])} \xrightarrow[n \to \infty]{\text{in measu}} 1;$$

it means that for each increasing sequence $\{n_k\}_{k\in N}$ of natural numbers there exists a subsequence $\{n_{k_p}\}_{p\in N}$ such that

$$\chi_{((n_{k_p}\cdot A) \cap [-1,1])} \xrightarrow[p \to \infty]{\text{a.e. on } [-1,1]} 1;$$

and c/ for each increasing sequence $\{t_n\}_{n\in N}$ of positive numbers tending to infinity there exists a subsequence $\{t_{n_k}\}_{k\in N}$ such that

$$\chi_{((t_{n_k}\cdot A) \cap [-1,1])} \xrightarrow[k \to \infty]{\text{a.e. on } [-1,1]} 1.$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

For a set $A \subset R$ which is not necessarily Lebesgue measurable we shall introduce the following definitions of the density point:

DEFINITION 1. 0 is a density point of A if and only if there exists a measurable set $B \subset A$ for which 0 is a density point (it is the same as to say that $\lim_{h \to 0^+} (2h)^{-1} \cdot m_*(A \cap (-h,h)) = 1$, where m_* stands for the inner Lebesgue measure.

DEFINITION 2. 0 is a density point of A if and only if

$$\chi_{((n \cdot A) \cap [-1,1])} \xrightarrow[n \to \infty]{\text{in measure}} 1.$$

DEFINITION 3. 0 is a density point of A if and only if for each increasing sequence $\{t_n\}_{n \in N}$ of positive numbers tending to infinity there exists a subsequence $\{t_{n_k}\}_{k \in N}$ such that

$$\chi_{((t_{n_k} \cdot A) \cap [-1,1])} \xrightarrow[k \to \infty]{\text{a.e. in } [-1,1]} 1.$$

Observe that in def. 1 and 3 it can happen that characteristic functions under considerations are not measurable, however the sequence of such functions can converge a.e. to a measurable function.

In the sequel we shall frequently use the following convention: x_0 is a density point of A (in the sense of def. 1,2 or 3) if and only if 0 is a density point of $A - x_0 = \{x - x_0 : x \in A\}$ in the sense of def. 1,2 or 3, respectively. We shall also say that 0 is a dispersion point of $A \subset R$ if and only if 0 is a density point of $R - A$.

PROPOSITION 1. If 0 is a density point of A in the sense of def. 1, then it is also a density point of A in the sense of def. 2 and def. 3.

P r o o f. Take a measurable set $B \subset A$ from def. 1, observe that

$$\chi_{((nB) \cap [-1,1])} \leq \chi_{((nA) \cap [-1,1])}$$

and use the equivalence of the above-mentioned conditions for B.

Observe that the fact that 0 is a dispersion point of A can be described using definitions similar to def. 2 and 3 in which 0 is used instead of 1 in the right-hand side of both formulas.

THEOREM 1. There exists a set $C \subset R$ such that 0 is a density point of C in the sense of def. 3 (and def. 2) but 0 is not a density point of C in the sense of def. 1.

We shall need the following lemma:

LEMMA 1. There exists a non-measurable set $A \subset [0,1]$ such that for each different positive numbers t', t'' $\operatorname{card}((t' \cdot A) \cap (t'' \cdot A)) \leq \aleph_0$.

Proof of the lemma: Let $\langle G_\alpha \rangle_{\alpha < \omega_1}$ be a transfinite sequence of all open subsets of $[0,1]$ for which $m(G_\alpha) < 1$. Let $\langle t_\alpha \rangle_{\alpha < \omega_1}$ be a transfinite sequence of all positive numbers. Choose $x_0 \in (0,1] - G_0$. It is possible. Suppose we have chosen points x_β for $\beta < \alpha < \omega_1$. Let $E_\alpha = \langle (t_\delta)^{-1} \cdot t_\gamma \cdot x_\beta : \delta < \alpha, \gamma < \alpha, \beta < \alpha \rangle$. Obviously $\operatorname{card}(E_\alpha) \leq \aleph_0$. Choose $x_\alpha \in (0,1] - (G_\alpha \cup E_\alpha)$. It is again possible. By the transfinite induction we obtain a set $A = \langle x_\alpha : \alpha < \omega_1 \rangle$. We shall prove that A has all required properties. Let t', t'' be different positive numbers. Then $t' = t_{\alpha_1}$, $t'' = t_{\alpha_2}$ for some $\alpha_1, \alpha_2 < \omega_1$. If $x \in (t_{\alpha_1} \cdot A) \cap (t_{\alpha_2} \cdot A)$, then $x = t_{\alpha_1} \cdot x_{\beta_1} = t_{\alpha_2} \cdot x_{\beta_2}$ for some $\beta_1, \beta_2 < \omega_1$. Obviously $\beta_1 \neq \beta_2$, suppose that $\beta_1 < \beta_2$. Then $x_{\beta_2} = (t_{\alpha_2})^{-1} \cdot t_{\alpha_1} \cdot x_{\beta_1}$. If $\beta_2 > \max(\alpha_1, \alpha_2)$, then $x_{\beta_2} \in E_{\beta_2}$ which is impossible. Hence $\beta_2 \leq \max(\alpha_1, \alpha_2)$ (and $\beta_1 \leq \max(\alpha_1, \alpha_2)$), so $(t_{\alpha_1} \cdot A) \cap (t_{\alpha_2} \cdot A)$ is at most denumerable. The case $\beta_1 > \beta_2$ is analogous. From the construction it follows immediately that $m^*(A) = 1$ (m^* is the outer Lebesgue measure). Suppose that A is measurable. Then for $t' = 1$, $t'' = 2$ we have $m((t' \cdot A) \cap (t'' \cdot A)) = 1$ which contradicts the denumerability of this set. Hence A is non-measurable.

Proof of the theorem: We shall show that 0 is a dispersion point of A in the sense of def. 3. Indeed, let $\langle t_n \rangle_{n \in N}$ be an arbitrary sequence of positive numbers tending to infinity. We shall show that required subsequence can be equal to the whole sequence. First observe that

$$\lim_n \sup ((t_n \cdot A) \cap [-1,1]) \subset \bigcup_{\substack{m,n \in N \\ m \neq n}} ((t_m \cdot A) \cap (t_n \cdot A)),$$

so

$$\operatorname{card} (\lim_n \sup ((t_n \cdot A) \cap [-1,1])) \leq \aleph_0,$$

hence

$$\chi_{((t_n \cdot A) \cap [-1,1])} \xrightarrow[n \to \infty]{\text{a.e. in } [-1,1]} 0.$$

Then 0 is a density point of $C = R - A$ in the sense of def. 3 (and obviously of def. 2). Simultaneously 0 is not a density point of C in the sense of def. 1, because for each measurable $B \subset C$ we have $m(B \cap [0,1]) = 0$ (for $B \cap [0,1] \cap A = \emptyset$) and hence 0 is not a density point of B in the sense of def. 1.

PROPOSITION 2. If 0 is a density point of A in the sense of def. 3, then it is also a density point of A in the sense of def. 2.

P r o o f. Obvious.

THEOREM 2. There exists a set $C \subset R$ such that 0 is a density point of C in the sense of def. 2 but it is not a density point of C in the sense of def. 3.

P r o o f. Let A be a set from lemma 1. Let $\langle r_k \rangle_{k \in N}$ be a decreasing sequence tending to zero and such that for each $k,l \in N$, $k \neq l$, a number $r_l \cdot (r_k)^{-1}$ is not rational. Put

$$B = \bigcup_{k=1} (r_k \cdot A).$$

Observe that

$$\limsup_n (n \cdot B) \subset \bigcup_{\substack{m,n \in N \\ m \neq n}} ((mB) \cap (nB)) = \bigcup_{\substack{m,n \in N \\ m \neq n}} \bigcup_{\substack{k,l \in N \\ k \neq l}} ((mr_k \cdot A) \cap (nr_l \cdot A))$$

and that the last set is at most denumerable for $mr_k \neq nr_l$ for each $k,l \in N$, $k \neq l$. Hence 0 is a dispersion point of B in the sense of def. 2, so a density point of $C = R - B$ in the sense of def. 2. Simultaneously if $t_n = (r_n)^{-1}$, $n \in N$ we observe that $\langle t_n \rangle_{n \in N}$ converges to infinity and for each subsequence $\langle t_{n_k} \rangle_{k \in N}$ we have

$$\limsup_k (t_{n_k} \cdot B) \supset \limsup_k (t_{n_k} \cdot r_{n_k} \cdot A),$$

because $B \supset r_{n_k} \cdot A$. But $t_{n_k} \cdot r_{n_k} \cdot A = A$, so

$$\limsup_k (t_{n_k} \cdot B) \supset A$$

and 0 is not a dispersion point of A in the sense of def. 3 (recall that A is not of measure 0). So the set $C = R - B$ fulfills all requirements.

It seems that the definition 2 is less natural than the remaining definitions, so in the sequel we shall speak only about 1-density points and 3-density points (it means about density points in the sense of def. 1 and 3, respectively). Let $\Phi_1(A) = \{x \in R : x$ is a 1-density point of $A\}$ and $\Phi_3(A) = \{x \in R : x$ is a 3-density point of $A\}$ for arbitrary $A \subset R$. Put $\mathcal{T}_{d_1} = \{A \subset R : A \subset \Phi_1(A)\}$ and $\mathcal{T}_{d_3} = \{A \subset R : A \subset \Phi_3(A)\}$. Obviously $\mathcal{T}_{d_1} \subset \mathcal{T}_{d_3}$. The family $\mathcal{T}_{d_1}$ is the well-known density topology (see [1], p. 500). We shall show that $\mathcal{T}_{d_3}$ is also a topology and that $\mathcal{T}_{d_1} \neq \mathcal{T}_{d_3}$.

PROPOSITION 3. For each sets $A, B \subset R$

a/ $\Phi_3(A \cap B) = \Phi_3(A) \cap \Phi_3(B)$

b/ if $A \subset B$, then $\Phi_3(A) \subset \Phi_3(B)$.

P r o o f. a/ can be proved exactly as in [4]; b/ is obvious.

THEOREM 3. $\mathcal{T}_{d_3}$ is a topology.

P r o o f. It follows immediately from the proposition 3.

Now we shall show some properties of the operation Φ_3.

THEOREM 4. There exists a set $C \subset (0,1)$ such that $C \triangle \Phi_3(C)$ is not of Lebesgue measure zero.

We shall need the following lemma in which by the dilatation f with a centre $x_0 \in (0,1)$ and a coefficient $t > 0$ we shall mean the following transformation

$$f(x) = x_0 + (x - x_0) \cdot t.$$

LEMMA 2. There exists a set $A \notin \mathcal{L}$ such that for each dilatations f', f'' having the same centre and different coefficients we have $\operatorname{card}(f'(A) \cap f''(A)) \leq \aleph_0$.

P r o o f. Let $\{G_\alpha\}_{\alpha < \omega_1}$ be a sequence of all open subsets of $(0,1)$, $m(G_\alpha) < 1$. Let $\{f_\alpha\}_{\alpha < \omega_1}$ be a transfinite one-to-one sequence of all dilatations with centres in $(0,1)$ and positive coefficients. Choose $x_0 \in (0,1) - G_0$. It is possible. Suppose that we have chosen points x_β for $\beta < \alpha < \omega_1$. Let $E_\alpha =$

$= \{ f_\delta^{-1}(f_\gamma(x_\beta)) : \delta < \alpha, \gamma < \alpha, \beta < \alpha, f_\delta$ and f_γ have the same centre as f_α (the set E_α can be empty). Obviously card $(E_\alpha) \leq \aleph_0$. Choose $x_\alpha \in (0,1) - (G_\alpha \cup E_\alpha)$. It is again possible. By the transfinite induction we obtain a set $A = \{x_\alpha : \alpha < \omega_1\}$. Let f', f'' be different dilatations with the same centre. Then $f' = f_{\alpha_1}$, $f'' = f_{\alpha_2}$ for some $\alpha_1, \alpha_2 < \omega_1$. If $x \in f_{\alpha_1}(A) \cap f_{\alpha_2}(A)$, then $x = f_{\alpha_1}(x_{\beta_1}) = f_{\alpha_2}(x_{\beta_2})$. Again we observe that $\beta_1 \neq \beta_2$ and essentially by the same argument as in lemma 1 we prove that card$(f'(A) \cap f''(A)) \leq \aleph_0$. The non-measurability of A is proved exactly as in lemma 1.

P r o o f o f t h e t h e o r e m: As in theorem 1 we prove that every point $x \in (0,1)$ is a dispersion point of A, so every point $x \in (0,1)$ is a density point of $C = (0,1) - A$. Hence $\Phi_3(C) = (0,1)$ and $C \triangle \Phi_3(C)$ is not measurable.

COROLLARY: $C \in \mathcal{T}_{d_3} - \mathcal{T}_{d_4}$.

In the above theorem we obtain a measurable set $\Phi_3(C)$. However, it is not the rule.

THEOREM 5. There exists a set $C \subset (0,1)$ for which $\Phi_3(C)$ is not measurable.

P r o o f. We shall start again with the same construction of similar type using the transfinite induction.

Let $E \subset (0,1)$ be an abitrary non-measurable set. Let $\{(y_\alpha, t_\alpha)\}_{\alpha < \omega_1}$ be a transfinite sequence in which each pair (y,t), where $y \in E$, $t > 0$ occurs exactly once. Let f_α denote the dilatation with the centre y_α and a coefficient t_α. Let $\{(z_\alpha, G_\alpha)\}_{\alpha < \omega_1}$ be a transfinite sequence in which each pair (z,G) where $z \in (0,1) - E$, $G \subset (0,1)$ is an open set, $m(G) < 1$, occurs exactly once. Choose arbitrary x_0. Choose $u_0 \in (0,1) - G_0$ and put $u_0^{(n)} = z_0 + 2^{-n+1} \cdot (u_0 - z_0)$ for $n \in N$ so $(u_0^{(1)} = u_0)$. Suppose that we have chosen x_β for $\beta < \alpha < \omega_1$ and $u_\beta^{(n)}$ for $n \in N$, $\beta < \alpha < \omega_1$. Let $E_\alpha = \{ f_\delta^{-1}(f_\gamma(x_\beta)) : \delta < \alpha, \gamma < \alpha, \beta < \alpha$, f_δ and f_γ have the same centre as $f_\alpha \} \cup \{ f_\delta^{-1}(f_\gamma(u_\beta^{(n)})) : \delta < \alpha$, $\gamma < \alpha, \beta < \alpha, n \in N$, f_δ and f_γ have the same centre as $f_\alpha\}$. Choose $x_\alpha \in (0,1) - E_\alpha$. Choose $u_\alpha \in (0,1) - (G_\alpha \cup \{z_\alpha\})$ in such a way that for each $n \in N$ $u_\alpha^{(n)} = z_\alpha + 2^{-n+1} \cdot (u_\alpha - z_\alpha)$ does not belong to E_α. It is possible because the set $(0,1) - (G_\alpha \cup \{z_\alpha\})$

is of the power of the continuum and the set E_α is countable. By the transfinite induction we obtain a set $A = \{x_\alpha : \alpha < \omega_1\} \cup \cup \{u_\alpha^{(n)} : n \in N, \alpha < \omega_1\}$. If f', f'' are different dilatations with the same centre $y \in E$, then $\mathrm{card}\,(f'(A) \cap f''(A)) \le \aleph_0$. Indeed, if $f' = f_{\alpha_1}$, $f'' = f_{\alpha_2}$ and $x \in f_{\alpha_1}(A) \cap f_{\alpha_2}(A)$, then $x = f_{\alpha_1}(x_{\beta_1})$ or $x = f_{\alpha_1}(u_{\beta_1}^{(n)})$ and $x = f_{\alpha_2}(x_{\beta_2})$ or $x = f_{\alpha_2}(u_{\beta_2}^{(n)})$ It cannot be $\beta_1 = \beta_2$ because the dilatations are different. Suppose that $x = f_{\alpha_1}(x_{\beta_1})$ and $\beta_1 > \beta_2$ then as before we prove that $\beta_1 \le \max(\alpha_1, \alpha_2)$. If $x = f_{\alpha_1}(u_{\beta_1}^{(n)})$ and $\beta_1 > \beta_2$, then in the case $\beta_1 > \max(\alpha_1, \alpha_2)$ we have $u_{\beta_1}^{(n)} \in E_{\beta_1}$ which is impossible. Hence $\mathrm{card}\,(f'(A) \cap f''(A)) \le \aleph_0$. So, each point of E is a 3--dispersion point of A.

Simultaneously from the construction of $\{u_\alpha^{(n)} : \alpha < \omega_1, n \in N\}$ it follows immediately that to every point $z \in (0,1) - E$ there corresponds a set $A_z \subset A$ $(A_z = \{u_\alpha^{(n)} : n \in N, z_\alpha = z\}$ which has outer measure equal to 1 and which fulfills the following property: $f(A_z) \supset A_z$ if f is a dilatation with the centre z and with a coefficient of the form 2^n, $n \in N$. Hence z is not a 3-dispersion point of A_z (a sequence $\{2^n\}_{n\in N}$ suffices), and obviously z is not a 3-dispersion point of A. Then $\Phi_3((0,1) - A) = (0,1) - E$ a nonmeasurable set.

Now let $\mathcal{S}$ denote the σ-algebra of sets having the Baire property and $\mathcal{J}$ the σ-ideal of sets of the first category. Recall basic facts from [3] and [4]. If $A \in \mathcal{S}$, then 0 is an $\mathcal{J}$-density point of A if and only if the sequence $\{\chi_{((n\cdot A)\cap[-1,1]}\}_{n\in N}$ converges to 1 with respect to $\mathcal{J}$ (for each increasing sequence $\{n_k\}_{k\in N}$ of natural numbers there exists a subsequence $\{n_{k_p}\}_{p\in N}$ such that

$$\chi_{((n_{k_p}\cdot A)\cap[-1,1])} \xrightarrow[p\to\infty]{\mathcal{J}\text{-a.e. on } [-1,1]} 1,$$

where $\mathcal{J}$-a.e. means except on a set belonging to $\mathcal{J}$). One can use a sequence $\{t_n\}_{n\in N}$ of positive numbers tending to infinity. The facts that x_0 is an $\mathcal{J}$-density point and that x_0 is an $\mathcal{J}$-dispersion point of A are described as before. If $\Phi(A) = \{x \in R : : x$ is an $\mathcal{J}$-density point of $A\}$ for $A \in \mathcal{S}$, then Φ is so--called lower density and $\mathcal{T}_{\mathcal{J}} = \{A \in \mathcal{S} : A \subset \Phi(A)\}$ is a topology.

For a set $A \subset R$ which has not necessarily the Baire property we shall introduce the following definitions of the $\mathcal{J}$-density point:

DEFINITION 1'. 0 is an $\mathcal{J}$-density point of A if and only if there exists a set $B \subset A$ having the Baire property for which 0

is an $\mathcal{J}$-density point

DEFINITION 2'. 0 is an $\mathcal{J}$-density point of A if and only if $\langle \chi_{((n\cdot A)\cap[-1,1])}\rangle_{n\in N}$ converges to 1 with respect to $\mathcal{J}$.

DEFINITION 3'. 0 is an $\mathcal{J}$-density point of A if and only if for each increasing sequence $\langle t_n\rangle_{n\in N}$ of positive numbers tending to infinity there exists a subsequence $\langle t_{n_k}\rangle_{k\in N}$ such that

$$\chi_{((t_{n_k}\cdot A)\cap[-1,1])} \xrightarrow[k\to\infty]{\text{I. a.e. on } [-1,1]} 1.$$

Essentially in the same way (using only in all constructions a transfinite sequence $\langle P_\alpha\rangle_{\alpha<\omega_1}$ of all F_σ sets which are not residual in $(0,1)$ in the place of open sets having measure less than 1) one can obtain the following results:

PROPOSITION 1'. If 0 is an $\mathcal{J}$-density point of A in the sense of def. 1', then it is also an $\mathcal{J}$-density point of A in the sense of def. 2' and 3'.

THEOREM 1'. There exists a set $C\subset R$ such that 0 is an $\mathcal{J}$--density point of C in the sense of def. 3' (and def. 2') but 0 is not an $\mathcal{J}$-density point of C in the sense of def. 1'.

PROPOSITION 2'. If 0 is an $\mathcal{J}$-density point of A in the sense of def. 3', then it is also an $\mathcal{J}$-density point of A in the sense of def. 2'.

THEOREM 2'. There exists a set $C\subset R$ such that 0 is an $\mathcal{J}$--density point of C in the sense of def. 2' but it is not an $\mathcal{J}$--density point of C in the sense of def. 3'.

If we denote $\Phi_1'(A)=\{x\in R: x$ is an $\mathcal{J}$-density point of A in the sense of def. 1'$\}$ and $\Phi_3'(A)=\{x\in R: x$ is an $\mathcal{J}$-density point of A in the sense of def. 3'$\}$, then both families $\mathcal{T}_{d_1}'=\{A\subset R: A\subset\Phi_1'(A)\}$ and $\mathcal{T}_{d_3}'=\{A\subset R: A\subset\Phi_3'(A)\}$ are topologies (see th. 3' below). The family $\mathcal{T}_{d_1}'$ is a T_3-topology (see [3] and [4]). Again $\mathcal{T}_{d_1}'\subsetneqq\mathcal{T}_{d_3}'$ (see corollary below).

PROPOSITION 3'. For each sets $A,B\subset R$

a/ $\bar{\Phi}_3'(A \cap B) = \bar{\Phi}_3'(A) \cap \bar{\Phi}_3'(B)$.
b/ if $A \subset B$, then $\bar{\Phi}_3'(A) \subset \bar{\Phi}_3'(B)$.

THEOREM 3′. $\mathcal{T}_{d_3}'$ is a topology.

The properties of $\bar{\Phi}_3'$ are similar to that of $\bar{\Phi}_3$.

THEOREM 4′. There exists a set $C \subset (0,1)$ such that $C \triangle \bar{\Phi}_3'(C) \notin \mathcal{J}$ (even $C \triangle \bar{\Phi}_3'(C) \notin S$).

COROLLARY: $C \in \mathcal{T}_{d_3}' - \mathcal{T}_{d_1}'$.

THEOREM 5′. There exists a set $C \subset (0,1)$ for which $\bar{\Phi}_3'(C)$ has not the Baire property.

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