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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 30 (1989), No. 2, 121--124

Persistent URL: http://dml.cz/dmlcz/701803

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Received 15 March 1989

We sumarize recent results concerning the monotony and the additivity of the integral on generalized measure spaces. We also pose some open questions.

1. Introduction

Generalized measure spaces were introduced by Suppes [12] as a model of probability and integration for quantum theories. Let X be a nonempty set. A collection $\Sigma \subset \exp X$ is called a *σ-class* if it satisfies:

(i) $X \in \Sigma$,

(ii) $Y \in \Sigma \Rightarrow X - Y \in \Sigma$,

(iii) $Y_i \in \Sigma\ (i \in N)$, Y_i mutually disjoint $\Rightarrow \bigcup_{i \in N} Y_i \in \Sigma$.

By a measure we mean a σ-additive function m on Σ with values in $[0, \infty)$. Then the triple (X, Σ, m) is called a *generalized measure space.*

As we do not require Σ to be closed under the formation of all unions, generalized measure spaces suit well as models of probability which admit the description of noncompatible events. (Motivated by quantum theory, events are called noncompatible if they are not simultaneously observable. Corresponding to this, two elements $Y, Z \in \Sigma$ are called noncompatible if $Y \cup Z \notin \Sigma$.) The noncompatibility is e.g. encountered in every system in which the measurement of one quantity effects the other quantity [4]. The need of a model of probability and integration admitting the noncompatibility has arisen in both quantum theory and other fields of application, including sociology, learning systems and artificial intelligence [13].

Throughout this paper (X, Σ, m) will be a generalized measure space. A function $f: X \to R$ is called *measurable* if $f^{-1}(A) \in \Sigma$ for any Borel set $A \subset R$. We use the notation $B_f = \{f^{-1}(A): A \text{ a Borel subset of } R\}$. If f is measurable, B_f is a sub-σ-

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algebra of Σ and the restriction $m \mid B_f$ is an ordinary measure. We define $\int_X f\, dm = \int_X f\, d(m \mid B_f)$, where the right side means the usual Lebesgue integral. However, the development of the theory of integration on generalized measure spaces cannot follow the usual patern. For instance, the sum of two measurable functions f, g need not be measurable. Even if $f + g$ is measurable – we then call f, g *summable* – we find that basic properties of the integral cannot be proved trivially in generalized measure spaces. Particularly, the following two questions (posed by S. Gudder – [2, 3]) turned out to be interesting:

If f, g are measurable functions on a generalized measure space (X, Σ, m) and if $f \leqq g$, does this imply that $\int_X f\, dm \leqq \int_X g\, dm$ whenever these integrals exist?

If f, g are measurable summable functions on a generalized measure space (X, Σ, m), do we have

$$\int_X (f + g)\, dm = \int_X f\, dm + \int_X g\, dm \quad \text{whenever both sides}$$

are defined?

2. Results

The generalized integration theory requires new techniques in some places. As an example, let us consider the classical proof of the monotony of integration. Let f, g be measurable functions on a measure space (X, Σ, m), $f \leqq g$. Then

$$\int_X g\, dm - \int_X f\, dm = \int_X (g - f)\, dm \geqq 0 .$$

Note that we need the additivity for the latter equation to hold. If we suppose that (X, Σ, m) is only a generalized measure space, the additivity cannot be ensured. Moreover, the function $g - f$ need not be measurable at all and its integral need not exist. Hence, we have to use quite different procedures in proving the basic properties of the generalized integral.

However, several positive results were achieved. The monotony of the generalized integral was found to be valid (independently proved in [6], [5] and [11]). However, the additivity of the generalized integral does not hold in general. The first counterexample (for unbounded functions) was found in [1]. The authors conjectured there that the additivity holds for bounded functions. The conjecture is disproved in [7] by a counterexample for functions with bounded range which is dense in some interval. The first significant positive result was that of Zerbe and Gudder [14]: The additivity holds for two finitely valued functions. (It is interesting that this result cannot be extended to a greater number of functions. There is a counterexample for three finitely valued functions – see [7, 14].) While the approach of Zerbe and Gudder is based on highly non-trivial combinatorial reasoning, a new "plane-topological" method has brought a more general result:

Theorem 1. [10]: Let f, g be measurable summable functions on a generalized measure space (X, Σ, m). Let *Range* f be bounded and nowhere dense and let the closure of *Range* g be bounded and countable. Then $\int_X (f + g)\, dm = \int_X f\, dm + \int_X g\, dm$.

In both approaches of [14] and [10] the additivity follows from the presence of certain special sets in Σ. It remains an open question whether the countability condition on g can be omitted in Theorem 1. We also do not know any counterexample for two bounded functions such that one of them has nowhere dense range.

If m is supposed to be a countable convex combination of two-valued measures, the additivity can be proved under more general assumptions, as the following theorem shows:

Theorem 2. ([9], Th. 2.2.12; for special cases see also [7, 8]): Let f, g be measurable summable functions on a generalized measure space (X, Σ, m). Let m be a convex combination of measures m_i on Σ $(i \in N)$ such that all restrictions $m_i \mid B_f$, $m_i \mid B_g$ are two-valued. Suppose that

(1) at least one of the functions f, g is bounded from above or from below,

(2) the closure H of the set $\{(f(x), g(x)) \colon x \in X\} \subset R^2$ is nowhere dense,

(3) $R^2 - H$ is connected.

Then $\int_X (f + g)\, dm = \int_X f\, dm + \int_X g\, dm$.

Remark 3. Since the technique in generalized measure spaces is quite special, we interrupt here the review of results and reinforce the intuition of the reader by exhibiting an example. The example is also a little novelty in the area. It is the fact that none of the assumptions of Theorem 2 can be omitted. Indeed, there is an example in [8] showing that Theorem 2 loses its validity if we omit the assumption on the measure m. We show now that the assumption (3) is also necessary:

Let Q be the set of all rational numbers. Put $P = \{2^i \colon i \text{ an integer}\}$. On $X = P \times Q$ we define functions $f \colon (p, q) \mapsto p$, $g \colon (p, q) \mapsto q$. We take for Σ the σ-class generated by $B_f \cup B_g \cup B_{f+g}$. It can be easily verified that $\Sigma = B_f \cup \Sigma_1$, where Σ_1 is the σ-class generated by $B_g \cup B_{f+g}$, and $B_f \cap \Sigma_1 = \{\emptyset, X\}$. Let m_1 be the probability measure on Σ_1, concentrated in $(1, 0)$, and let m_2 be the probability measure on B_f, concentrated in $(2, 0)$. The measures m_1 and m_2 coincide on $B_f \cap \Sigma_1$ and so they have a common extension to a measure m on Σ. We have $\int_X f\, dm + \int_X g\, dm = 2 + 0 \neq \int_X (f + g)\, dm = 1$.

In the end we would like to mention another open problem. Denote by $\mathscr{A}(\Sigma)$ the σ-algebra of subsets of X generated by Σ. If the measure m admits an extension to a measure on $\mathscr{A}(\Sigma)$, then the integral on (X, Σ, m) has to have all "classical" properties. The counterexamples to the additivity show that such an extension is not possible in general. Nevertheless, the extension of m to $\mathscr{A}(\Sigma)$ may exist provided that Σ is the σ-class generated by $B_f \cup B_g \cup B_{f+g}$ and the functions f and g satisfy some appropriate conditions. Such an extension is possible under the assumptions of Theorem 2. However, it is not known whether it is possible under the assumptions of Theorem 1 or for f, g finitely valued. A positive answer to this extension problem may lead to an alternative proof of the monotony, the additivity and other properties of the generalized integral.

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