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# A Property of Doubly Stochastic Densities

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During the 17th Winter School on Abstract Analysis (Srní, 1989), Professor E. Behrends**) posed the following question
Given a probability density function $f(x, y)$ on the unit square satisfying the equalities

$$\int f(x, y)\, \mathrm{d}y = \int f(x, y)\, \mathrm{d}x = 1$$

is it possible to find a (Lebesgue) measure preserving transformation $T$ of the unit interval such that $f(x, Tx) > 0$ a.e.?

We shall show that the answer is affirmative.

For any measure preserving transformation $T$ of $[0, 1]$ denote by $\mu_T$ the (doubly stochastic) measure concentrated on the graph of $T$, i.e. the measure determined by the formula

$$\mu_T(A \times B) = m(B \cap T^{-1}(A))$$

where $m$ is Lebesgue measure on $[0, 1]$ and $A$, $B$ are Borel subsets of the unit interval. In general, a Borel probability measure $\mu$ on $[0, 1]^2$ is called doubly stochastic if $\mu(B \times [0, 1]) = \mu([0, 1] \times B) = m(B)$ for any Borel set $B$. The measure $\mathrm{d}\mu(x, y) = f(x, y)\, \mathrm{d}x\, \mathrm{d}y$ is clearly doubly stochastic and absolutely continuous with respect to $m \times m$.

Our solution of the problem relies on a result of V. N. Sudakov ([2], Prop. 42 and Thm. 8). Its convenient reformulation says that for any absolutely continuous doubly stochastic measure $\mu$ there exists a barycentric representation of $\mu$ over the measure $\mu_T$ with $T$ measure preserving and invertible (m.p.i.). More precisely, there is a probability measure $\nu$ on the group $\mathscr{G}_m$ of all (equivalence classes of) m.p.i. transformations of the unit interval such that

$$\mu(C) = \int \mu_T(C)\, \mathrm{d}\nu(T)$$

for any Borel subset $C$ of the unit square. Here $\mathscr{G}_m$ is endowed with its natural standard Borel structure determined by the functions $T \to m(B \cap T^{-1}(A))$.

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**) The problem was originally stated by Professor T. M. Rassias of Athens.

We also remark that the assertion $f(x, Tx) > 0$ a.e. is equivalent to the existence of a null set $N$ such that graph $T \subset \{(x, y): f(x, y) > 0\} \cup (N \times [0, 1])$.

Now the solution is contained in the following theorem

**Theorem 1.** Let $\mu$ be an arbitrary absolutely continuous doubly stochastic measure. If $\mu(C) = 1$ then there exist an m.p.i. transformation $T$ and a null subset $N$ of $[0, 1]$ such that graph $T \subset C \cup (N \times [0, 1])$.

**Proof.** We may assume that $C$ is Borel. Now by the Sudakov theorem we get

$$1 = \mu(C) = \int \mu_T(C)\, \mathrm{d}\nu(T)$$

so $\mu_T(C) = 1$ for $\nu$-a.e. $T$. This means that the intersection $C \cap$ graph $T$ projects onto a set of measure 1 in $[0, 1]$. In other words,

$$\text{graph } T \subset C \cup (N \times [0, 1])$$

for some null set $N$.

**Remark.** Actually, Sudakov's result gives more as the m.p.i. transformations in the barycentric representation of $\mu$ have pairwise disjoint graphs (they arise from a measurable partition of the unit square). Therefore we obtain continuum many $T$'s with disjoint graphs all satisfying the assertion $f(x, Tx) > 0$ a.e.

Now we present another result of the same kind which can be viewed as a topological variation of Theorem 1.

Consider a weak* continuous mapping $x \to \mu_x$ from $[0, 1]$ into the set of probability measures on $[0, 1]$. Denote by $\mu$ the associated probability measure on the unit square, i.e. for any Borel subset $C$ of the unit square let

$$\mu(C) = \int \mu_x(C_x)\, \mathrm{d}x$$

where $C_x = \{y: (x, y) \in C\}$. With this notation we have

**Theorem 2.** Let the topological support of each $\mu_x$ be connected (= a subinterval). If $C$ is such that $\mu(C) = 1$ then there exist a continuous transformation $T: [0, 1] \to \to [0, 1]$ and a null set $N$ in $[0, 1]$ such that graph $T \subset C \cup (N \times [0, 1])$.

**Proof.** The mapping $x \to \mu_x$ can be considered as a Feller transition probability. By Thm. 2 of [1] there exists a Borel function of two variables $\varphi_\omega(x)$ ($\omega$ and $x$ are from the unit interval) which is continuous with respect to $x$ and such that

$$\mu_x(A) = \int 1_A(\varphi_\omega(x))\, \mathrm{d}\omega$$

for any Borel subset $A$ of $[0, 1]$.

Now we have by Fubini's theorem

$$1 = \mu(C) = \int \mu_x(C_x)\, \mathrm{d}x = \iint 1_{C_x}(\varphi_\omega(x))\, \mathrm{d}\omega\, \mathrm{d}x = \int\left(\int 1_{C_x}(\varphi_\omega(x))\, \mathrm{d}x\right) \mathrm{d}\omega$$

so

$$1_{C_x}(\varphi_\omega(x))\, \mathrm{d}x = 1 \quad \omega\text{-a.e.}$$

This means $\varphi_\omega(x) \in C_x$, or equivalently $(x, \varphi_\omega(x)) \in C$, except for $x \in N_\omega$ with $m(N_\omega) = 0$. In other words,

$$\text{graph}\, \varphi_\omega \subset C \cup (N \times [0, 1]) \quad \text{for a.e. } \omega .$$

### References


[1] IWANIK A., Integral representations of stochastic kernels, in: Aspects of Positivity in Functional Analysis, R. Nagel, U. Schlotterbeck, M. P. H. Wolff (editors), Elsevier (North-Holland), 1986, 223—230.

[2] SUDAKOV V. N., Geometric problems in the theory of infinite dimensional probability distributions (in Russian), Trudy Mat. Inst. Steklov 141 (1976).