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The Sums of Closed Subspaces in a Topological Linear Space

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In this paper we collect some results concerning the structure of closed subspaces in a topological linear space. We establish some relations between algebraical properties of this structure and the topology of the space.

Throughout the paper, let X be a real Hausdorff topological linear space with a topological dual X^*. Let $L(X)$ denote the set of all closed subspaces of X. By an (algebraical) sum $A + B$ of subspaces $A, B \in L(X)$ we mean the set $\{x + y \mid x \in A, y \in B\}$. Our starting point is the following question: When is the sum of two closed subspaces again a closed subspace? For example, it is well known that the sum in question is closed provided at least one of the summands is finite-dimensional. In this connection there is an interesting theorem of V. I. Gurarii and H. P. Rosenthal (see e.g. [10]).

Theorem 1. Let X be a Banach space and $E, F \in L(X)$. If any closed subspaces A and B of E and F, respectively, are finite-dimensional, then $E + F \in L(X)$.

We restrict now attention to spaces in which every sum of finitely many closed subspaces is a closed subspace.

Definition 1. X is said to be modular if $E + F \in L(X)$ for all $E, F \in L(X)$.

As G. W. Mackey proved ([8]), X is modular if and only if the set $L(X)$ equipped with the set inclusion forms a modular lattice. This assertion justifies the terminology used in definition 1. Important example of modular space is the locally convex direct sum of any system of one-dimensional spaces ($\bigoplus_{\alpha \in I} R_\alpha$, where $R_\alpha = R$ for every $\alpha \in I$) and the product of this system ($\prod_{\alpha \in I} R_\alpha$, where $R_\alpha = R$ for every $\alpha \in I$). Indeed, the space $\bigoplus_{\alpha \in I} R_\alpha$ has the strongest admissible locally convex topology. So every linear subspace of $\bigoplus_{\alpha \in I} R_\alpha$ is closed and the modularity follows immediately. According to a canonical duality between spaces $\bigoplus_{\alpha \in I} R_\alpha$ and $\prod_{\alpha \in I} R_\alpha$, the lattices $L(\bigoplus_{\alpha \in I} R_\alpha), L(\prod_{\alpha \in I} R_\alpha)$

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are dual isomorphic (dual isomorphism is given by the polars). Consequently, the product $\prod_{\alpha \in I} R_\alpha$ is modular, too. On the other hand, there are many examples of non-modular spaces. A classical theorem of G. W. Mackey ([8]) says that a normed space is modular if and only if it is finite-dimensional. In the first part of this paper we summarize some results extending this Mackey theorem ([4]). A space X has a Hahn-Banach extension property (abb. HBEP) if the following version of Hahn-Banach theorem holds: Every linear form continuous on a given closed subspace of X has a continuous extension over the entire space X. Let us remark that there are many non locally convex spaces with HBEP [2, 5]. Following Wilbur [12], we say that X is total if X admits a continuous norm. The following theorem characterizes bounded subsets in some modular spaces.

Theorem 2. Let X be a modular space satisfying the following conditions:

(1) X has a HBEP,

(2) there is a weaker locally convex metrizable topology on X,

(3) if $M \in L(X)$ is infinite-dimensional then the Mackey topology $\tau(M, M^*)$ is strictly stronger than the weak topology $\sigma(X, X^*)$.

Then every bounded subset of X is finite-dimensional (i.e., it is contained in some finite-dimensional space).

So, modularity of X implies in this case some topological properties such as quazicompleteness, semicompleteness and semireflexivity in a locally convex case (for precise definitions, see [11]).

We intend to prove Theorem 2 in a subsequent paper, the proofs of the following results may be found in [4].

Corollary. 1. Total modular locally convex space has only finite-dimensional bounded subsets.

Making use of the foregoing corollary, we obtain the following consequences giving a lucid characterizations of modular spaces in some typical situations.

Theorem 3. Total bornological space X is modular if and only if it is isomorphic to any locally convex direct sum $\bigoplus_{\alpha \in I} R_\alpha$, where $R_\alpha = R$ for every $\alpha \in I$.

Let us now consider total metrizable locally convex modular spaces. According to the Baire category theorem, all metrzable locally convex direct sums of real lines are finite-dimensional, so, using Theorem 3, we have the following corollary.

Corollary 2. The only modular total metrizable locally convex spaces are finite-dimensional spaces.

From this point of view the foregoing results can be interpreted as a generalization of Mackey theorem.

In the second part of this paper we shall deal with a modular F-space. By an F-space we mean a complete metrizable topological linear space. First we need to introduce some notions. A space X is called minimal if there is no Hausdorff linear topology on X strictly weaker than the original one, and it is called q-minimal (quotient-minimal, see [1]) if all its Hasudorff quotients are minimal. Trivial examples of minimal spaces are finite-dimensional spaces and their products. As known, a locally convex space X is (quotient-) minimal if and only if X is isomorphic to the product of real lines. (see [1, 9, 11 p. 191]). A sequence (x_n) in X is said to be M-basic sequence [6, 7] if there is a sequence (x_n^*) in the dual space $\overline{\text{sp}}\,\{x_n \mid n \in N\}^*$ such that $x_n^*(x_m) = \delta_{n,m}$ for every $n, m \in N$ and moreover $\bigcap_{n\in N} \operatorname{Ker} x_n^* = \{0\}$ (i.e., the sequence (x_n^*) is total on $\overline{\text{sp}}\,\{x_n \mid n \in N\}^*$). An M-basic sequence is called regular if there is a neighbourhood U of 0 such that $x_n \notin U$ for all $n \in N$. Deep theorem of N. J. Kalton [6] says that every nonminimal F-space contains a regular M-basic sequence. We need the following theorem of L. Drewnowski [1] which generalizes Theorem 1.

Theorem 4. Let X be an F-space and let $E, F \in L(X)$. If any isomorphic closed subspaces A and B of E and F, respectively, are q-minimal, then $E + F \in L(X)$.

The following interesting theorem of A. Martineu characterizes modular Frechet spaces.

Theorem 5. A Frechet space is modular if and only if it is minimal.

Let us present here the proof of the non-locally convex version of Theorem 5.

Theorem 6. An F-space is modular if and only if it is q-minimal.

Proof. Let X be a modular F-space. First we prove that X is minimal. If, on the contrary, X is not minimal then X contains a regular M-basic sequence (x_n). Put $u_n = x_{2n-1}$ and choose a sequence (v_n) such that for every $n \in N$ we have

$$\text{sp}\,\{u_n, v_n\} = \text{sp}\,\{x_{2n-1}, x_{2n}\} \quad \text{and} \quad \varrho(u_n - v_n, 0) < 1/n\,,$$

where ϱ is a metric inducing the topology of X.

Put $A = \overline{\text{sp}}\,\{u_n \mid n \in N\}$, $B = \overline{\text{sp}}\,\{v_n \mid n \in N\}$. Making use of the properties of M-basic sequences, it can be shown that $A \cap B = \{0\}$. Then we can define the mapping $p: A + B \to A$ by putting

$$p(x + y) = x \quad \text{for every} \quad x \in A\,, \quad y \in B\,.$$

The mapping p is not continuous, because $\lim_{n\to\infty} u_n - v_n = 0$, while the sequence (u_n) is bounded away from zero ((x_n) is regular). Since p is a closed mapping, we see, according to the closed graph theorem, that the space $A + B$ is not complete. Thus $A + B \notin L(X)$, which is a contradiction.

Because every Hausdorff quotient of modular F-space is again a modular F-space, the preceding considerations implies that X is q-minimal.

The reverse implication in Theorem 6 follows easily from Theorem 4 and the fact that every closed subspace of q-minimal space is again q-minimal [1, Prop. 3.1 (a)].

A crucial problem in the theory of basic sequences in F-spaces is the following problem [6]: Are there some non locally convex q-minimal F-spaces? If the answer is no, as some results indicates [6, 7], then by Theorem 6, the only infinite-dimensional modular F-space is the space of all sequences (with the product topology).

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